COINCIDENCE PROBABILITIES

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1. Introduction. It was shown in [14] that if $P(t) = (P_{ij}(t))$ is the transition probability matrix of a birth and death process, then the determinants

$$(1) \qquad P\left(t; \begin{matrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{matrix}\right) = \begin{vmatrix} P_{i_1 j_1}(t) \cdots P_{i_1 j_n}(t) \\ \vdots \qquad\qquad \vdots \\ P_{i_n j_1}(t) \cdots P_{i_n j_n}(t) \end{vmatrix}$$

where $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ are strictly positive when $t > 0$. In this paper it is shown that these determinants have an interesting probabilistic significance.

(A) *Suppose that n labelled particles start out in states* $i_1, \cdots, i_n$ *and execute the process simultaneously and independently. Then the determinant* (1) *is equal to the probability that at time t the particles will be found in states* $j_1, \cdots, j_n$ *respectively without any two of them ever having been coincident* (simultaneously in the same state) *in the intervening time.*

From this statement it follows that the determinant is non-negative, and as will be seen strict positivity can be deduced from natural hypotheses, for example if $P_{i_\alpha j_\alpha}(t) > 0$ for $\alpha = 1, \cdots, n$ and every $t > 0$.

The truth of the above statement rests chiefly on the facts that the process is *one-dimensional*—its state space is linearly ordered, and that the path functions of the process are *everywhere "continuous"*. Of course the path functions are discontinuous in the ordinary sense but the discontinuities are only of magnitude one. Thus when a transition occurs the diffusing particle moves from a given state only into one of the two neighboring states, and even if the particle goes off to infinity in a finite time it either remains there or else it returns in a continuous way and does not suddenly reappear in one of the finite states. These two properties of one-dimensionality and "continuity" have the effect that when several particles execute the process simultaneously and independently, a change in the order of the particles cannot occur unless a coincidence first takes place. (The states are all stable so that with probability one a transition involves only one of the particles.)

It is also important for our results that the processes involved have the strong Markoff property of Hunt [10], [11], (see also [19]). However it is a consequence of theorems of Chung [3] that any continuous time

Received December 18, 1958. This work was supported in part by an Offiice of Naval Research Contract at Stanford University.

parameter Markoff chain whose states are all stable has the strong Markoff property.

There exist processes of birth-and-death type whose path functions may have discontinuities at infinity. Such processes have been described in some detail by Feller. Although the above result (A) does not apply to these processes they fall within a more general class of processes which we discuss next.

We consider a stationary Markoff process whose state space is a set of integers and whose states are all stable. Let $(P_{ij}(t))$ be the transition probability matrix. Then

(B) *Suppose that n labelled particles start in states $i_1, \cdots, i_n$ and execute the process simultaneouly and independently. For each permutation σ of $1, \cdots, n$ let A_σ denote the event that at time t the particles are in states $j_{\sigma(1)}, \cdots, j_{\sigma(n)}$ respectively, without any two of them ever having been coincident in the intervening time. Then*

$$P\left(t\,;\begin{matrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{matrix}\right) = \sum_{\sigma} (\operatorname{sign} \sigma) \Pr\{A_\sigma\}$$

where the sum runs over all permutations of $1, \cdots, n$ and sign $\sigma = 1$ or -1 *according as σ is an even or an odd permutation.*

The first stated result is seen to be a special case of this one. For if the path functions are "continuous" and $i_1 < \cdots < i_n, j_1 < \cdots < j_n$ then $\Pr\{A_\sigma\}$ is zero except when σ is the identity permutation. There is one other case in which the general formula permits an interesting simplification, namely when the process is a local cyclic process. By this we mean that the states may be viewed as $N+1$ points $0, 1, \cdots, N$ on a circle and transitions occur only between neighboring states, 1 and N being neighbors of zero and $N-1$ and 0 neighbors of N. We take $0 \leq i_1 < \cdots < i_n \leq N$ and $0 \leq j_1 < \cdots < j_n \leq N$ and then $\Pr\{A_\sigma\}$ is zero unless σ is a cyclic permutation. Since the cyclic permutations of an odd number of objects are all even permutations we have in this situation

$$P\left(t\,;\begin{matrix} i_1, \cdots, i_n \\ j_1, \cdots, j_n \end{matrix}\right) = \sum_{\text{cyclic } \sigma} \Pr\{A_\sigma\}, \qquad n \text{ odd}. \tag{3}$$

This determinant is therefore non-negative.

Analogous results hold for one dimensional diffusion processes. Let $P(t, x, E)$ be the transition probability function of a stationary process whose state space is an interval on the extended real line. It will be assumed that the process has the strong Markoff property and that its path functions are *continuous* everywhere. Given two Borel sets E, F the inequality $E < F$ will denote that $x < y$ for every $x \in E, y \in F$. We take n states $x_1 < x_2 < \cdots < x_n$ and n Borel sets $E_1 < E_2 < \cdots < E_n$

and form the determinant

$$(4) \qquad P\left(t\,;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) = \begin{vmatrix} P(t, x_1, E_1) \cdots P(t, x_1, E_n) \\ \vdots \\ P(t, x_n, E_1) \cdots P(t, x_n, E_n) \end{vmatrix}$$

(C) *Suppose that* n *labelled particles start in states* $x_1, \cdots, x_n$ *and execute the process simultaneously and independently. Then the determinant* (4) *is equal to the probability that at time* t *the particles will be found in the sets* $E_1, \cdots, E_n$ *respectively without any two of them ever having been coincident in the intervening time.*

Next consider a stationary strong Markoff process whose state space is a metric space and whose path functions are continuous on the right. We take n states $x_1, \cdots, x_n$ and n Borel sets $E_1, \cdots, E_n$ and again form the determinant (4).

(D) *Suppose that* n *labelled particles start in the states* $x_1, \cdots, x_n$ *and execute the process simultaneously and independently. For each permutation* σ *of* $1, 2, \cdots, n$ *let* A_σ *denote the event that at time* t *the particles are in the states* $E_{\sigma_1}, \cdots, E_{\sigma_n}$ *respectively without any two of them ever having been coincident in the intervening time. Then*

$$(5) \qquad P\left(t\,;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) = \sum_{\sigma} (\operatorname{sign} \sigma) \Pr\{A_\sigma\}$$

where the sum runs over all permutations σ.

The last result contains all of the preceding ones as special cases. It has another interesting special case, namely when the state space is a circle and the path functions are continuous.

There is a mapping $\theta \rightarrow e^{i\theta} = x$ of the closed interval $0 \leq \theta \leq 2\pi$ onto the circle. Given n Boral sets $E_1, \cdots, E_n$ on the circle we say $E_1 < \cdots < E_n$ if there are n Borel sets $E'_1 < \cdots < E'_n$ in the interval $(0, 2\pi]$ or $[0, 2\pi)$ which are mapped onto $E_1, \cdots, E_n$ respectively by the above mapping. Specializing the sets to be one point sets gives the meaning for $x_1 < \cdots < x_n$ when $x_1, \cdots, x_n$ are n points on the circle.

Now let $P(t, x, E)$ be the transition probability function of a strong Markoff process on the circle with *continuous* path functions. Because of the continuity of paths a change in the cyclic order of several diffusing particles on the circle cannot occur unless a coincidence first takes place. Thus the terms in (5) corresponding to non-cyclic permutations σ will all be zero. Finally we take advantage of the fact that the cyclic permutations of an *odd* number of objects are all even permutations, and obtain the following.

(E) *Suppose* $x_1 < \cdots < x_n$, $E_1 < \cdots < E_n$ *and* n *labelled parti-*

cles start at $x_1, \cdots, x_n$ respectively and execute the process simultaneously and independently. If n is odd and A_σ is defined as before then

$$P\left(t\,;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) = \sum_{\text{cyclic } \sigma} \Pr\{A_\sigma\} \tag{6}$$

where the sum runs over all cyclic permutations.

Similar but more complicated results are valid in still more general situations. For example we restrict our discussion to stationary processes although both the methods and the results can be extended to non-stationary processes. A generalization of another type which has interesting applications is obtained when the n particles execute *different processes.*

Let $P_\alpha(t, x, E), \alpha = 1, \cdots, n$ be transition probability functions of n strong Markoff process on the real line with continuous path functions. Choose n states $x_1 < \cdots < x_n$ and n Borel sets $E_1 < \cdots < E_n$ and form the determinant

$$\det P_\alpha(t, x_\alpha, E_\beta)\ . \tag{7}$$

If n labelled particles start in states $x_1, \cdots, x_n$ respectively, and execute the processes simultaneously and independently, the ith particle executing the ith process, then the determinant (7) is the probability that at time t the particles will be found in the sets $E_1, \cdots, E_n$ respectively, without any two of them ever having been coincident in the intervening time.

The formal proofs of formulas (5) and (6) and of the interpretation of $P\left(t,\ \begin{matrix} x_1, x_2 \cdots, x_n \\ E_1, E_2 \cdots, E_n \end{matrix}\right)$ are elaborated in § 5. For this purpose the relevant preliminaries and definitions concerning Markoff processes are summarized in § 4.

In § 6 we offer some observations on the problem of determining when the strong Markoff property applies to direct products of processes. In this connection we direct attention to those aspects of this problem relevant to our analysis of the main theorem of § 5.

Section 2 contains a brief heuristic proof of (C) in the situation of two particles. This is inserted in order to motivate the formal proof of § 5. Section 3 discusses the connections of the concept of total positivity, to statements (A) - (E).

Total positivity is significant in relation to the theory of vibrations of mechanical systems [8], the method of inversion of convolution transforms [9], and the techniques of mathematical economics [13]. In this paper total positivity is shown to be also important in describing the structure of one dimensional strong Markoff processes whose path functions are continuous. In a vague sense the most general totally positive kernel can be built from convolutions of stochastic processes whose path functions are continuous. In principle, the representation desired is similar to the representation formula which applies to Pólya frequency functions discovered by Schoenberg [20]. A detailed discussion of this idea will be published separately. In this connection we mention that Loewner has completely analyzed the generation of totally positive matrices from infinitesimal elements [18].

In § 7 we investigate conditions which insure that the determinant (4) is strictly positive. We find that this is the case if $P(t, x, E) > 0$ whenever $t > 0$, E is any open set and $P(t, x, E)$ represents the transition probability function of a strong Markoff process on the real line with continuous path functions.

The following converse proposition is of interest. Suppose the transition function $P(t, x, E)$ of a Markoff process has the property that all determinants of the form (4) are non-negative. Does there exist a realization of the process such that almost all path functions are continuous? This is true with some mild further restrictions. In § 8 with the aid of a theorem of Ray [19] we are able to establish a partial converse based on a restriction about the local character of $P(t, x, E)$. It will be recognized that most cases of Markoff processes obey this requirement.

In § 9 we characterize the most general one dimensional spatially homogeneous process whose transition kernel is totally positive.

The final section presents a series of examples of totally positive kernels derived from Markoff processes with continuous path functions.

2. A heuristic argument. In this section we give a non-rigorous outline of the method of proof for the case of two particles. Let $P(t, x, E)$ be the transition probability function of a stationary Markoff process on the real line. Suppose that two distinguishable particles start at x_1 and $x_2 > x_1$ and let $E_1 < E_2$ be two Borel sets. The determinant

$$P\left(t\,;\begin{matrix} x_1 & x_2 \\ E_1 & E_2 \end{matrix}\right) = P(t, x_1, E_1)P(t, x_2, E_2) - P(t, x_1, E_2)P(t, x_2, E_1)$$

is equal to $\Pr\{A_1'\} - \Pr\{A_2'\}$ where A_1' is the event that at time t the first particle is in E_1, the second in E_2 and A_2' is the event that at time t the first particle is in E_2, the second in E_1. Each event A_i', regarded as a collection of paths, may be split up into two disjoint sets $A_i + A_i''$ where A_i consists of all the paths in A_i' for which no coincidence occurs before time t and A_i'' consists of the paths in A_i' with at least one coincidence before time t. We assume the paths are sufficiently smooth so that for each path in A_1'' and A_2'' there is a first coincidence time. This will certainly be the case if all paths are continuous on the right. Choose a path in A_1'' and at the time of first coincidence interchange the labels of the two particles. This converts the given path into a path in A_2'' and the resulting map of A_1'' into A_2'' is clearly one-to-one and onto. Because of the Markoff property and because the particles act independently it is plausible that this map is measure preserving so that

$$\Pr\{A_1''\} = \Pr\{A_2''\}$$

and granting this it follows that

$$\begin{aligned} P\left(t;\ \begin{matrix} x_1, & x_2 \\ E_1 & E_2 \end{matrix}\right) &= \Pr\{A_1'\} - \Pr\{A_2'\} \\ &= \Pr\{A_1\} - \Pr\{A_2\}\ , \end{aligned}$$

which is the general form of the result. If the path functions are all continuous then $\Pr\{A_2\} = 0$ and the formula becomes

$$P\left(t;\ \begin{matrix} x_1, & x_2 \\ E_1 & E_2 \end{matrix}\right) = \Pr\{A_1\}\ .$$

3. **Total positivity.** A matrix is called (strictly) *totally positive* if all of its minors of all orders are (strictly positive) non-negative. Such matrices and their continuous analogues the totally positive kernels occur in a variety of applications and have been studied by numerous authors. A lucid outline of the theory together with an extensive bibliography has been given by Schoenberg [21], Krein and Gantmacher [8]. Our results indicate the existence of large natural classes of semi-groups of totally positive matrices and totally positive kernels. One simply takes the transition probability function of a one dimensional diffusion process with continuous path functions. A number of interesting examples are given in § 10.

Conversely the total positivity of the transition function may be used to draw conclusions regarding continuity of the path functions. A program along these lines has already been carried out by the authors for the case of birth and death processes [12]. (see also § 8.)

Our attention was first drawn to total positivity in connection with diffusion processes by unpublished results of C. Loewner who showed that the fundamental solution of

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x}$$

on a finite interval with smooth a and b and classical boundary conditions, is totally positive.

4. **Definitions.** As indicated in the introduction we are chiefly concerned with processes on the integers, the real line, or the circle. In

order to deal with all cases at once it is convenient to discuss certain results for a more general process whose state space is a metric space X.

Let X be a metric space, $\mathfrak{B}$ the Borel field generated by the open sets of X, and $\mathfrak{B}'$ the Borel ring generated by the finite intervals on $0 \leq t < \infty$. Suppose there is given a set Ω called the sample space and an X-valued function $x(t, \omega), 0 \leq t < \infty, \omega \in \Omega$. Let $\mathfrak{M}$ be the Borel field of subsets of Ω generated by the sets of the form $\{\omega ; x(t, \omega) \in E\}$ where $t \geq 0$ and $E \in \mathfrak{B}$. Suppose that for each $x \in X$ there is given a probability measure P_x on X such that $P_x\{\omega ; x(0, \omega) = x\} = 1$. Then the function $x(t, \omega)$ is called a stochastic process on X with sample space Ω and distributions $\{P_x\}$.

The stochastic process is said to have right continuous path functions if for every fixed ω the function $x(\cdot, \omega)$ is right continuous on $0 \leq t < \infty$.

Let $\mathscr{M}_t$ denote the Borel field generated by all sets $\{\omega; x(s, \omega) \in E\}$ where $E \in \mathfrak{B}$ and $0 \leq s \leq t$. Conditional probabilities relative to $\mathfrak{M}_t$ will be denoted by $P_x\{\cdots | x(s), s \leq t\}$. The stochastic process is called a stationary Markoff process if for every fixed t

$$
\begin{aligned}
P_x\{x(t_i + t, \omega) \in E_i, i = 1, \cdots, n \mid x(s), s \leq t\} \\
= P_{x(t,\omega)}\{x(t_i, \omega) \in E_i, i = 1, \cdots, n\}
\end{aligned}
$$

with probability one when $0 < t_1 < \cdots < t_n$ and $E_1, \cdots, E_n \in \mathfrak{B}$.

We will be concerned only with stationary Markoff processes in X with right continuous path functions. It will always be assumed that the function

$$P(t, x, E) = P_x\{x(t, \omega) \in E\}$$

is measurable relative to $\mathfrak{B}' \otimes \mathfrak{B}$. This function satisfies the Chapman-Kolmogoroff equation:

$$P(t + s, x, E) = \int_X P(t, x, dy) P(s, y, E) .$$

Let F be a closed set in X. The time of first hitting F is defined as

$$\tau_F(\omega) = \inf \{t ; x(t, \omega) \in F\}$$

where the inf of the void set is taken to be $+\infty$. The place of first hitting F is defined, if $\tau_F(\omega) < \infty$, as

$$\xi_F(\omega) = x(\tau_F(\omega), \omega) .$$

The Markoff process will be called a *strong Markoff process* if for any closed set F we have the first passage relation

$$P_x\{x(t,\omega)\in E\} = P_x\{x(t,\omega)\in E, \tau_F(\omega)>t\} + \int_0^t\int_F P_y\{x(t-s,\omega)\in E\}P_x\left\{\begin{matrix}\tau_F(\omega)\in ds\\ \xi_F(\omega)\in dy\end{matrix}\right\} .$$

In this relation it is implicitly assumed that the sets $\{\omega; \tau(\omega)<t\}$ and $\{\omega; \tau(\omega)<t, \xi(\omega)\in H\}$ where H is a closed subset of F, are $\mathfrak{W}_t$ measurable for each t. A discussion of the validity of these assumptions made in § 6. It is there shown that under very slight conditions on the transition function the assumption holds.

It seems reasonable to believe that the direct product of a finite number of strong Markoff processes is again a strong Markoff process. At the present time we are not able to prove that this is generally true, although in the proof of the main theorem we assume this result. On the other hand proofs can be given which cover the vast majority of the special cases of interest. As noted above it follows from theorems of Chung that the strong Markoff property is preserved under direct products for processes with countably many states all of which are stable. This includes the birth and death case. In § 6 we give a proof for direct products of a one dimensional diffusion process whose transition probability function $P(t, x, E)$ is jointly continuous in t and x. This covers the case when $P(t, x, E)$ comes from a diffusion equation

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x}$$

with $a(x)$, $b(x)$ continuous and $a(x)>0$. References to other theorems of this kind are given in § 6.

Let X_i, $i=1,\cdots,n$ be metric spaces and for each i let $x_i(t,\omega_i)$ be a stationary Markoff process in X_i with sample space Ω_i and distributions $\{P^{(i)}_{x_i}\}$. We form the product space $\overline{X}=X_1\otimes\cdots\otimes X_n$ in which the generic point is an n-tuple $\bar{x}=(x_1,\cdots,x_n)$ with $x_i\in X_i$. The space $\overline{X}$ with the distance $\rho(\bar{x},\bar{y})=\Sigma\rho(x_i,y_i)$ is a metric space. The vector valued function $\bar{x}(t,\bar{\omega})=(x_1(t,\omega_1),\cdots,x_n(t,\omega_n))$ is a stationary Markoff process in $\overline{X}$ whose sample space is the direct product $\overline{\Omega}$ of the Ω_i and whose distributions are the direct product measures

$$\overline{P}_{\bar{x}} = \prod_i P^{(i)}_{x_i}, \qquad \bar{x}=(x_1,\cdots,x_n) .$$

$\bar{x}(t,\bar{\omega})$ is called the *direct product* of the given processes.

5. The main theorem. Let X be a metric space, and $x(t,\omega)$ a stationary strong Markoff process in X with right continuous sample functions, sample space Ω and distributions $\{P_x\}$. We form the direct products $\overline{X}$, $\overline{\Omega}$ of n copies of X and Ω respectively and the direct product

$\bar{x}(t, \bar{\omega})$ of n copies of the given process. We say this direct product process represents "n labelled particles executing the $x(t, \omega)$ process simultaneously and independently", and this is the sense in which that phrase is to be interpreted in statements (A) - (E) of the introduction. *We assume $\bar{x}(t, \bar{\omega})$ is a strong Markoff process* (see § 6).

The associated distributions are

$$P_{\bar{x}} = \prod_{i=1}^{n} P_{x_i}, \qquad \bar{x} = (x_1, \cdots, x_n) .$$

The set F of coincident states consists of the points $\bar{x} = (x_1, \cdots, x_n)$ with at least two of the x_i equal to one another. A permutation λ of the n letters $1, 2, \cdots, n$ is called a transposition if there are two letters $i < j$ such that $\lambda(i) = j$, $\lambda(j) = i$, and $\lambda(r) = r$ if $i \neq r \neq j$. In this case we use the notation $\lambda = (i, j)$. A coincident state $\bar{x} = (x_1, \cdots, x_n)$ is said to belong to the transposition $\lambda = (i, j)$, $i < j$ if $x_1, \cdots, x_{j-1}$ are all different but $x_i = x_j$. Thus every coincident state belongs to a unique transposition, and for a given λ the set of all coincident states belonging to λ will be denoted by $F(\lambda)$. The group of all $n!$ permutations of $1, 2, \cdots, n$ will be denoted by S and the set of all transpositions by Λ.

Given n Borel sets $E_1, \cdots, E_n$ in X and a permutation $\sigma \in S$, the direct product set

$$E_\sigma = E_{\sigma(1)} \otimes \cdots \otimes E_{\sigma(n)}$$

is a Borel set in $\bar{X}$. Let $A'_\sigma = \{\bar{\omega} ; \bar{x}(t, \bar{\omega}) \in E_\sigma\}$ where $t > 0$ is fixed. Then if $\bar{x} = (x_1, \cdots, x_n)$

$$P\left(t ; \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) = \sum_{\sigma \in S} (\operatorname{sign} \sigma) P_{\bar{x}}\{A'_\sigma\}$$

by definition of the determinant and of $P_{\bar{x}}$.

The time $\tau(\bar{\omega})$ of first coincidence is defined as the time of first hitting F:

$$\tau(\bar{\omega}) = \tau_F(\bar{\omega}) = \inf \{t ; x(t, \bar{\omega}) \in F\} .$$

The place of first coincidence is $\xi(\bar{\omega}) = \bar{x}(\tau(\bar{\omega}), \bar{\omega})$. Our main result can now be stated very simply as follows.

THEOREM 1. *The sets*

$$A_\sigma = \{\omega ; \omega \in A'_\sigma, \tau(\bar{\omega}) > t\}$$

are all measurable and

$$P\left(t ; \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) = \sum_{\sigma \in S} (\operatorname{sign} \sigma) P_{\bar{x}}\{A_\sigma\} .$$

Proof. Since τ is measurable the sets A_σ are also measurable. For each σ we apply the strong Markoff property to obtain

$$P_{\bar{x}}\{A'_\sigma\} = P_{\bar{x}}\{A_\sigma\} + \int_0^t d\Phi(s)\int_F P_{\bar{y}}\{\bar{x}(t-s,\bar{\omega}) \in \bar{E}_\sigma\}\mu(dy)$$

where

$$\Phi(s) = P_{\bar{x}}\{\tau(\bar{\omega}) \le s\} ,$$
$$\mu(M) = P_{\bar{x}}\{\xi(\bar{\omega}) \in M \mid \tau(\bar{\omega}) = s\} .$$

Now F is the union of the disjoint Borel sets $F(\lambda)$, $\lambda \in \Lambda$, and if $\bar{y} \in F(\lambda)$ then $P_{\bar{y}}\{\bar{x}(t-s,\omega) \in \bar{E}_\sigma\} = P_{\bar{y}}\{\bar{x}(t-s,\bar{\omega}) \in \bar{E}_{\lambda\sigma}\}$. Hence

$$\begin{aligned}
&\sum_{\sigma\in S}(\text{sign}\,\sigma)\,[P_{\bar{x}}\{A'_\sigma\} - P_{\bar{x}}\{A_\sigma\}] \\
&= \sum_{\sigma\in S}\sum_{\lambda\in\Lambda}(\text{sign}\,\sigma)\int_0^t d\Phi(s)\int_{F(\lambda)} P_{\bar{y}}\{\bar{x}(t-s,\bar{\omega}) \in \bar{E}_\sigma\}\mu(dy) \\
&= \sum_{\sigma\in S}\sum_{\lambda\in\Lambda} -(\text{sign}\,\lambda\sigma)\int_0^t d\Phi(s)\int_{F(\lambda)} P_{\bar{y}}\{x(t-s,\bar{\omega}) \in \bar{E}_{\lambda\sigma}\}\mu(dy) \\
&= -\sum_{\sigma\in S}(\text{sign}\,\sigma)\,[P_{\bar{x}}\{A'_\sigma\} - P_{\bar{x}}\{A_\sigma\}] .
\end{aligned}$$

This quantity is therefore zero and

$$\begin{aligned}
P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) &= \sum_{\sigma\in S}(\text{sign}\,\sigma)P_{\bar{x}}\{A'_\sigma\} \\
&= \sum_{\sigma\in S}(\text{sign}\,\sigma)P_{\bar{x}}\{A_\sigma\} .
\end{aligned}$$

The various assertions (A) - (D) of the introduction can be obtained by specializing the above theorem in the appropriate way.

6. Strong Markoff property for direct products. For the vast majority of one-dimensional diffusion processes which are met in applications one finds that the transition probability function $P(t,x,E)$ is jointly continuous in t and x. It will be shown that the direct product of n-copies of such a process has the strong Markoff property. The proof imitates the proof of a theorem of Dynkin and Jushkevich [7].

THEOREM 2. *Let $x(t,\omega)$ be a stationary Markoff process on the real line with continuous path functions and transition probability function $P(t,x,E)$ which is jointly continuous in t, x. Then the direct product $\bar{x}(t,\bar{\omega})$ of n copies of this process is a strong Markoff process.*

Proof. Let F be a closed set in the n-dimensional space, $\tau(\bar{\omega})$ the time of first hitting F for the direct product process, and $\xi(\bar{\omega})$ the place of first hitting F. The fact that $\tau(\omega)$ and $\xi(\bar{\omega})$ are measurable functions is a trivial consequence of the continuity of the path functions. With a given integer $m \geq 1$ let $\tau_m(\bar{\omega}) = k/m$, where k is the integer such that

$$\frac{k-1}{m} < \tau(\bar{\omega}) \leq \frac{k}{m},$$

and let $\xi_m(\bar{\omega}) = \bar{x}(\tau_m(\bar{\omega}), \bar{\omega})$. Then for any Borel set $\bar{E}$

$$P_{\bar{x}}(\bar{x}(t, \omega) \in \bar{E}\} = P_{\bar{x}}\{\bar{x}(t, \bar{\omega}) \in \bar{E}, \tau_m(\bar{\omega}) > t\} + \sum_{1 \leq k \leq mt} P_{\bar{x}}\left\{\bar{x}(t, \bar{\omega}) \in \bar{E},\ \tau_m(\bar{\omega}) = \frac{k}{m}\right\}.$$

Let

$$A_k(\bar{\omega}) = \begin{cases} 1 & \text{if} \quad \tau_m(\bar{\omega}) = \dfrac{k}{m}, \\ 0 & \text{if} \quad \tau_m(\bar{\omega}) \neq \dfrac{k}{m}, \end{cases}$$

and

$$f(\bar{y}) = \begin{cases} 1 & \text{if} \quad \bar{y} \in \bar{E}, \\ 0 & \text{if} \quad \bar{y} \notin \bar{E}. \end{cases}$$

Then

$$\begin{aligned} P_{\bar{x}}\left\{\bar{x}(t, \bar{\omega}) \in \bar{E},\ \tau_m(\bar{\omega}) = \frac{k}{m}\right\} &= E_{\bar{x}}\{A_k(\bar{\omega}) f(x(t, \omega))\} \\ &= E_{\bar{x}}\left\{E\left\{A_k(\bar{\omega}) f(\bar{x}(t, \bar{\omega})) \mid x(s),\ s \leq \frac{k}{m}\right\}\right\} \\ &= E_{\bar{x}}\left\{A_k(\bar{\omega}) E\left\{f(\bar{x}(t, \bar{\omega})) \mid x(s),\ s \leq \frac{k}{m}\right\}\right\} \\ &= E_{\bar{x}}\left\{A_k(\bar{\omega}) P_{\bar{x}(k/m, \bar{\omega})}\left\{\bar{x}\left(t - \frac{k}{m}, \bar{\omega}\right) \in \bar{E}\right\}\right\} \\ &= \int_{\bar{X}} P_{\bar{y}}\left\{\bar{x}\left(t - \frac{k}{m}, \bar{\omega}\right) \in \bar{E}\right\} P_{\bar{x}}\left\{\xi_m(\bar{\omega}) \in dy,\ \tau_m(\bar{\omega}) = \frac{k}{m}\right\} \end{aligned}$$

and hence we have the first passage relation for τ_m:

$$P_{\bar{x}}\{\bar{x}(t, \bar{\omega}) \in \bar{E}\} = P_{\bar{x}}\{\bar{x}(t, \bar{\omega}) \in \bar{E}, \tau_m(\bar{\omega}) > t\} + \int_0^t \int_{\bar{X}} P_{\bar{y}}\{\bar{x}(t - s, \bar{\omega}) \in \bar{E}\} P_{\bar{x}} \begin{Bmatrix} \tau_m(\bar{\omega}) \in ds \\ \xi_m(\bar{\omega}) \in dy \end{Bmatrix}.$$

For every $\bar{\omega}$ we have $\tau_m(\bar{\omega}) \geq \tau_{m+1}(\bar{\omega}) \downarrow \tau(\bar{\omega})$ and by continuity of path functions $\xi_m(\bar{\omega}) \to \xi(\bar{\omega})$ as $m \to \infty$. Hence $\tau_m(\bar{\omega})$, $\xi_m(\bar{\omega})$ converge in measure to $\tau(\bar{\omega})$, $\xi(\bar{\omega})$. Since $P_{\bar{y}}\{\bar{x}(t-s, \bar{\omega}) \in \bar{E}\}$ is jointly continuous in $\bar{y}$ and s and is bounded we may let $m \to \infty$ in the above formula and obtain the first passage relation for $\tau(\bar{\omega})$. This completes the proof.

The referee has brought to our attention the following stronger theorem of Blumenthal, [1, Theorem 1.1], which is slightly reworded here.

THEOREM. *If the process has right continuous path functions and if for every bounded continuous function f the function $\int f(y)P(t, x, dy)$ is continuous in x for each $t > 0$, then the process has the strong Markoff property.*

In this theorem the state space X is any metric space. Naturally this theorem requires more involved arguments than the above Theorem 2. Finally we mention that a very thorough discussion of the Markoff chain case has been given by Chung [4].

7. Strict total positivity. Let X be the non-negative integers and $x(t, \omega)$ a stationary strong Markoff process on X with all states stable and "continuous" path functions. If $P(t) = (P_{ij}(t))$ is the transition probability matrix of the process then it follows from assertion (A) that this matrix is totally positive. Let us call the process *a* strict process if $P_{ij}(t) > 0$ for every i, j and all $t > 0$. We will prove

THEOREM 3. *If the process is strict then its transition probability matrix is strictly totally positive for every $t > 0$.*

Proof. The proof is similar to the proof of a related theorem in [14], namely Theorem 20 on page 543. It is seen from the proof of that theorem that it is sufficient for our purposes to prove that if $i_1 < i_2 < \cdots < i_n$ then

$$P\left(t;\, \begin{matrix} i_1, \cdots, i_n \\ i_1, \cdots, i_n \end{matrix}\right) > 0$$

for every $t > 0$, that is the principal subdeterminants are strictly positive. However since

$$P\left(2t;\, \begin{matrix} i_1, \cdots, i_n \\ i_1, \cdots, i_n \end{matrix}\right) \geq \left[P\left(t;\, \begin{matrix} i_1, \cdots, i_n \\ i_1, \cdots, i_n \end{matrix}\right)\right]^2$$

it is enough to show that these determinants are strictly positive for sufficiently small $t > 0$. Because the path functions are right continuous, if $\{r_k\}$ is an ordering of the positive rationals, the set

$$\bigcup_{m=1}^{\infty} \bigcap_{r_k \le 1/m} \{\omega\,; x(r_k, \omega) = i \mid x(0, \omega) = i\}$$

has probability one. Hence for some $m = m(i) > 0$ there is a positive probability R_i that a path starting at i remains at i for at least up to time $1/m(i)$. Now if $0 < t < \max_{1 \le k \le n} 1/m(i_k)$ then we have

$$P\left(t\,; \begin{matrix} i_1, \cdots, i_n \\ i_1, \cdots, i_n \end{matrix}\right) \ge \prod_{k=1}^{n} R_{i_k} > 0$$

and this proves the theorem.

Now let $x(t, \omega)$ be a stationary strong Markoff process on the real line with continuous path functions satisfying the hypothesis of Theorem 1. Let $P(t, x, E)$ be the transition probability function of the process. The process will be called *strict* if $P(t, x, E) > 0$ whenever $t > 0$ and E is any non-void open set. We will prove

THEOREM 4. *If the process is strict then its transition probability function is strictly totally positive in the sense that if $t>0$, $x_1 < \cdots < x_n$ and $E_1 < \cdots < E_n$ are non-void open sets then*

$$P\left(t\,; \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) > 0\ .$$

We begin with two lemmas in which the hypotheses of the theorem are assumed.

DEFINITION. If a, b are two points on the real line then

$$\begin{aligned} \tau_a(\omega) &= \inf\{t\,; x(t, \omega) = a\}\ , \\ M(t, x, a) &= P_x\{\tau_a(\omega) \le t\}\ , \\ M(t, x, a, b) &= P_x\{\tau_a(\omega) \le t, \tau_b(\omega) > t\}\ . \end{aligned}$$

LEMMA. *If $a < x < b$ then $M(t, x, a, b) > 0$ and $M(t, x, b, a) > 0$ for every $t > 0$.*

Proof. Assume that $M(t, x, b, a) = 0$ for some $t = t_0 > 0$ and hence for every $t < t_0$. Then if $J = [b, \infty)$ we have for every $t < t_0$

$$P(t, x, J) = \int_0^t P(t - s, a, J) d_s M(s, x, a)$$

and in virtue of the continuity of paths

$$P(t, a, J) = \int_0^t P(t - s, x, J) d_s M(s, a, x) .$$

Now because of the continuity of paths we can choose t_1 so $0 < t_1 < t_0$ and

$$M(t, a, x) M(t, x, a) \leq 1/2 \qquad \text{for } 0 \leq t \leq t_1 .$$

Since $P(s, a, J) \leq 1$ for all $s \leq t_1$ it follows from the integral equations that

$$P(s, a, J) \leq 1/2 \qquad \text{for } s \leq t_1 ,$$

and by an iteration argument we obtain $P(t_1, a, J) = 0$ which contradicts the hypothesis. Hence $M(t, x, b, a) > 0$ for $t > 0$. Similarly $M(t, x, a, b) > 0$ for $t > 0$.

DEFINITION. Given an open interval $V = (a, b)$ let

$$R(t, x, V) = P_x\{\tau_a(\omega) > t, \tau_b(\omega) > t\} .$$

LEMMA 2. *If* $x \in V = (a, b)$ then $R(t, x, V) > 0$ *for all* $t > 0$.

Proof. Assume that for some $x \in V$ and $t' > 0$ we have $R(t', x, V) = 0$. Then $R(t, x, V) = 0$ for all $t > t'$. Because of continuity of paths $t_0 = \inf\{t ; R(t, x, V) = 0\}$ is positive. Now choose any $y \in V, y \neq x$. To fix the ideas we assume $x < y < b$. If $\varepsilon > 0$ is so small that $M(t', x, y, a) - M(\varepsilon, x, y, a) > 0$ then the inequality

$$0 = R(t', x, V) \geq \int_\epsilon^{t'} R(t' - \tau, y, V) d_\tau M(\tau, x, y, a)$$

shows that $R(t' - \epsilon, y, V) = 0$. Consequently if $t_1 = \inf\{t ; R(t, y, V) = 0\}$ then

$$0 < t_1 \leq t_0 - \varepsilon < t_0 .$$

But we can now repeat the argument and show that $t_0 < t_1$. This contradiction proves the lemma.

Proof of the Theorem. Let $x_1 < \cdots < x_n$ and $E_1 < \cdots < E_n$ be nonvoid open sets. The *index* of the determinant

$$P\left(t ; \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right)$$

is defined to be the number k of values of i for which x_i is not in E_i. Thus the index of an nth order determinant of this kind is an integer between 0 and n inclusive.

In each set E_i choose a non-void open interval U_i such that $x_i \in U_i$ if $x_i \in E_i$ but $\overline{U_i}$ contains no x_j if $x_i \notin E_i$. Because of the probabilistic interpretation

$$P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) \geq P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ U_1, \cdots, U_n \end{matrix}\right).$$

These two determinants have the same index k. If $k = 0$, then from the probabilistic interpretation and the second lemma above

$$P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ U_1, \cdots, U_n \end{matrix}\right) \geq \prod_{i=1}^{n} R(t, x_i, U_i) > 0 .$$

Thus the subdeterminants with index zero are positive. Now suppose the index is $k > 0$. We can find n open intervals $U'_1, \cdots, U'_n$ whose closures are mutually disjoint such that $x_i \in U'_i$ for every i and $U'_i = U_i$ if $x_i \in U_i$. We can choose n points $x'_1, \cdots, x'_n$ such that $x'_i \in U_i$ for every i and $x'_i = x_i$ if $x_i \in U_i$. Now in the collection $U_1, \cdots, U_n, U'_1, \cdots, U'_n$ there are exactly $m = n + k$ distinct intervals and they are disjoint. Denote them by $V_1 < \cdots < V_m$. Similary in $x_1, \cdots, x_n, x'_1, \cdots, x'_n$ there are exactly $n + k$ distinct points. Denote them by $y_1 < \cdots < y_m$ and then $y_i \in V_i$ for each i. Let $B(t)$ be the m-square matrix with elements

$$b_{ij}(t) = P(t, y_i, V_j) .$$

The determinant $P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ U_1, \cdots, U_n \end{matrix}\right)$ is a minor of $B(t)$. Moreover $B(t)$ is totally positive, all of its elements are strictly positive, and its principal minors have index zero and are therefore strictly positive. Hence by Lemma 14 of [**14**] all minors of $B(t)$ of index one are strictly positive. This proves that $P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) > 0$ if the index of this determinant is ≤ 1. We now assume that for some integer $r, 1 \leq r < n,$ all the determinants of the type $P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right)$ with index $\leq r$ are strictly positive.

Let $1 \leq i_1 < \cdots < i_n \leq m, 1 \leq j_1 < \cdots < j_n \leq m$ and

$$\sum_{\nu=1}^{n} | i_\nu - j_\nu | = r + 1 .$$

Then

$$P\left(t+s;\ \begin{matrix} y_{i_1}, \cdots, y_{i_n} \\ V_{j_1}, \cdots, V_{j_n} \end{matrix}\right)$$

$$\geq \sum_{1 \leq \alpha_1 < \cdots < \alpha_n \leq m} \int_{v_1 \in V_{\alpha_1}} \cdots \int_{v_n \in V_{\alpha_n}} P\left(t;\ \begin{matrix} y_{i_1}, \cdots, y_{i_n} \\ dv_1, \cdots, dv_n \end{matrix}\right) P\left(s;\ \begin{matrix} v_1, \cdots, v_n \\ V_{j_1}, \cdots, V_{j_n} \end{matrix}\right)$$

and in this sum there is at least one term with

$$\sum_{\nu=1}^{n} | i_\nu - \alpha_\nu | \leq r , \qquad \sum_{\nu=1}^{n} | \alpha_\nu - j_\nu | \leq r .$$

For this term the integrand $P\left(s;\ \begin{matrix} v_1, \cdots, v_n \\ V_{j_1}, \cdots, V_{j_n} \end{matrix}\right)$ is positive for every $v_1, \cdots, v_n$ in the range of integration because $v_\nu \in V_{j_\nu}$ for at least $n-r$ values of ν. Also for this term the integrator $P\left(t;\ \begin{matrix} y_{i_1}, \cdots, y_{i_n} \\ dv_1, \cdots, dv_n \end{matrix}\right)$ has positive measure on the range of integration because $y_{i_\nu} \in V_{\alpha_\nu}$ for at least $n-r$ values of ν. Hence the special term and also the entire sum is strictly positive. This proves that $P\left(t;\ \begin{matrix} x_1, \cdots, x_n \\ E_1, \cdots, E_n \end{matrix}\right) > 0$ if the index of this determinant is $\leq r+1$, and the theorem follows by induction on the index.

8. Local character of $P(t, x, E)$ and continuity of path functions. Let $P(t, x, E)$ be the transition probability function of a stationary Markoff process on the real line. Given $\delta > 0$ we define

$$\begin{aligned} V(x, \delta) &= [x+\delta, \infty) , \\ U(x, \delta) &= (-\infty, x-\delta] , \\ I'(x, \delta) &= U(x, \delta) \cup V(x, \delta) . \end{aligned}$$

The transition probabilities are called of *local character* if $P(t, x, I'(x, \delta)) = o(t)$ for each x and $\delta > 0$. They are called *uniformly of local character* if for each $\delta > 0$ and each compact set F on the real line the relation $P(t, x, I'(x, \delta)) = o(t)$ holds uniformly for $x \in F$. We will prove that if the transition probabilities are positive of order two (see Theorem 5) and if for some $\alpha > 0$ we have $P(t, x, I'(x, \delta)) = o(t^\alpha)$ for each x and each $\delta > 0$ then the transition probabilities are uniformly of local character, and in fact for every $\beta > 0$ the relation $P(t, x, I'(x, \delta)) = o(t^\beta)$ holds uniformly on compact sets. This is of interest in connection with a theorem of Ray [19] to the effect that if the transition probabilities are uniformly of local character and if $P(t, x, X) = 1$ where X is the real line (not the extended real line) then the process has path functions continuous except possibly at $+\infty$ and $-\infty$.

THEOREM 5. *Let $P(t, x, E)$ be stationary transition probabilities on the real line such that $P(t, x, E) \to 1$ as $t \to 0+$ if x is an interior point of E. If $P(t, x, E)$ is positive of order two (i. e. the second order determinants of* (4) *are non-negative) and if there is an $\alpha > 0$ such that for every x and every $\delta > 0$ we have $P(t, x, I'(x, \delta)) = o(t^\alpha)$ then for every compact set F on the real line and every $\beta > 0$, $\delta > 0$ there is a constant $M = M(F, \delta, \beta)$ such that*

$$P(t, x, I'(x, \delta)) \leq Mt^\beta$$

for every $x \in F$.

Proof. Given a point x on the real line and $\delta > 0$ let $y = x + \delta/2$ and $N = (y - \delta/4, y + \delta/4)$. Then because of the second order positivity

$$P(t, x, V(x, \delta))P(t, y, N) \leq P(t, x, N)P(t, y, V(x, \delta)) .$$

Both factors of the right member of this inequality are $O(t^\alpha)$ while $P(t, y, N) \to 1$ as $t \to 0$. Hence $P(t, x, V(x, \delta)) = O(t^{2\alpha})$.

This is valid for arbitrary x and δ, so the argument can be iterated, and for any integer $n \geq 1$ we have

$$P(t, x, V(x, \delta)) = O(t^{2^n\alpha}) .$$

The O symbol so far may depend on x and certainly depends on δ. A similar argument applies to $P(t, x, U(x, \delta))$ and combining them we have

$$P(t, x, I'(x, \delta)) = O(t^\beta)$$

for any $\beta > 0$.

Now suppose $x < y < z$, let $E = (z, \infty)$ and let W be an open interval containing y but whose closure does not contain z. Then

$$\begin{aligned} P(t, x, E)P(t, y, W) &\leq P(t, x, W)P(t, y, E) \\ &\leq P(t, y, E) \end{aligned}$$

There is a positive $t_0 = t_0(y, E)$ such that $P(t, y, W) \geq 1/2$ for $t \leq t_0$ and therefore

$$P(t, x, E) \leq 2P(t, y, E) \qquad \text{if } t \leq t_0 .$$

Similarly if $z < y < x$ and $E = (-\infty, z)$ then there is a positive $t_1 = t_1(y, E)$ such that

$$P(t, x, E) \leq 2P(t, y, E) \qquad \text{if } t \leq t_1 .$$

Now let $F = [a, b]$ be a finite interval and $\delta > 0$. Choose a finite number of points $y_1, \cdots, y_m$ such that every open subinterval of $(a - \delta,$

$b+\delta)$ of length $(1/2)\delta$ contains at least one of the points y_i. Given $x \in F$ there are indices α, β such that

$$x - \frac{1}{2}\delta < y_\alpha < x < y_\beta < x + \frac{1}{2}\delta .$$

Since $U(x, \delta) \subseteq U(y_\alpha, \delta/4)$ and $V(x, \delta) \subseteq V(y_\beta, \delta/4)$ we have

$$P(t, x, U(x, \delta)) \leq 2P\left(t, y_\alpha, U\left(y_\alpha, \frac{\delta}{4}\right)\right) ,$$

$$P(t, x, V(x, \delta)) \leq 2P\left(t, y_\beta, V\left(y_\beta, \frac{\delta}{4}\right)\right)$$

for sufficiently small t. In fact these inequalities are valid if t is less than the least of the numbers $t_0(y_i, V(y_i, \delta/4))$, $t_1(y_i, U(y_i, \delta/4))$, $i=1, 2, \cdots, m$. Since each of the finite collection of functions $P(t, y_i, V(y_i, \delta/4))$, $P(t, y_i, U(y_i, \delta/4))$, $i = 1, 2, \cdots, m$ is $o(t^\beta)$ for any $\beta > 0$, it follows at once that for fixed $\delta > 0, \beta > 0$ $P(t, x, I'(x, \delta)) = O(t^\beta)$ uniformly for $x \in F$.

9. Homogeneous processes. A process on the real line will be called a homogeneous process if it is a stationary strong Markoff process with right continuous path functions and its transition probability function satisfies the homogeneity relation

$$P(t, x+h, E) = P(t, x, E-h)$$

where $E - h = \{y ; y + h \in E\}$. This class of processes includes all the processes with stationary independent increments and is slightly more general. If X denotes the real line then for any homogeneous process the function

$$P(t, x, X) = P(t, 0, X) = \alpha(t)$$

is independent of x. From the Chapman-Kolmogoroff equation $\alpha(t+s)= \alpha(t)\alpha(s)$ and then because of monotonicity $\alpha(t) = e^{-\beta t}$ where $0 \leq \beta \leq +\infty$. The case $\beta = 0$ gives the processes with stationary independent increments. The general homogeneous process is obtained by taking a process with stationary independent increments and stopping it after a random time T with $\Pr\{T > t\} = e^{-\beta t}$. The trivial case $\beta = +\infty$ is excluded in the remainder of this section.

There are two special kinds of homogeneous processes of particular interest from our point of view. First the *essentially determined* ones for which, if E is any open set

$$P(t, x, E) = \begin{cases} e^{-\beta t} & \text{if} \quad x + vt \in E \\ 0 & \text{otherwise} \end{cases}$$

where v is a real constant and $0 \leq \beta < \infty$. And second, those *derived from the Wiener process*, for which

$$P(t, x, E) = \frac{e^{-\beta t}}{\sqrt{2\pi\sigma t}} \int_E \exp\left[-\frac{(x + vt - y)^2}{2\sigma t}\right] dy$$

where v is a real and σ a positive constant and $0 \leq \beta < \infty$. These two types are interesting because they have continuous path functions and the transition probability functions are therefore totally positive. For those derived from the Wiener process it is strictly totally positive, while for the essentially determined ones it is not. The main result in this section is the following.

THEOREM 6. *If the transition probability function of a homogeneous process is totally positive then the process is either an essentially determined one or else one derived from the Wiener process.*

Together with the results of § 5 this theorem shows that for homogeneous processes total positivity is equivalent to continuity of the path functions. At the close of this section we show by a different method that for homogeneous processes positivity of order two is already equivalent to continuity of the path functions. This assertion is probably true not only for homogeneous processes but for arbitrary one dimensinal strong Markoff processes with right continuous path functions. Although we are not yet able to prove the result in this generality, we do have a proof for the case of birth and death processes, which is published separately [12].

Proof. Let $P(t, x, E)$ be the transition probability function of a totally positive homogeneous process and let $P(t, x, (-\infty, \infty)) = e^{-\beta t}$. We form the function

$$P_\varepsilon(t, x, E) = \int_{-\infty}^{\infty} e^{\beta t} P(t, y, E) q_\varepsilon(t, y - x) dy$$

where $\varepsilon > 0$ and $q_\varepsilon(t, x) = (2\pi\varepsilon t)^{-1/2} \exp[-(x^2/2\varepsilon t)]$. Then P_ε is a homogeneous strictly totally positive kernel for $t > 0$, it satisfies the Chapman Kolmogoroff equation, and is analytic in its dependence on x. There is therefore a density function $p_\varepsilon(t, x)$ such that

$$P_\varepsilon(t, x, E) = \int_E p_\varepsilon(t, y - x) dy .$$

For fixed ε, p_ε is measurable in t, x and is analytic in x for fixed ε, t. From the formula

$$P_\varepsilon(t, y - x) = \lim_{h \to 0+} \frac{1}{h} P_\varepsilon(t, x, (y, y + h))$$

we deduce that if $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ then det $p_\varepsilon(t, x_i - y_j) \geq 0$ for $t > 0$. Thus for fixed t and ε the function $p_\varepsilon(t, x)$ is a Pólya frequency function $\left(\text{we have} \int_{-\infty}^{\infty} p_\varepsilon(t, x)dx = 1\right)$ in the sense of Schoenberg [20] and the Laplace transform

$$\frac{1}{\psi(s, t)} = \int_{-\infty}^{\infty} e^{-xs} p_\varepsilon(t, x)dx$$

converges in a strip $-a < \operatorname{Re}[s] < a$ with $a > 0$, and has there a representation

$$\psi(s, t) = e^{-\gamma s^2 + \delta s} \prod_{\nu=1}^{\infty} (1 + \delta_\nu s) e^{-\delta_\nu s}$$

where $\gamma \geq 0, \delta, \delta_\nu$ are real, $0 < \gamma + \sum \delta_\nu^2 < \infty$. The constants $\gamma, \delta, \delta_\nu$ will of course depend on t. From the Chapman-Kolmogoroff equation we have $\psi(s, t) = [\psi(s, t/n)]^n$ where n is any positive integer. Consequently any zero of $\psi(s, t)$ must be of order at least n and n being arbitrary there can be no zeros. Hence

$$\psi(s, t) = e^{\delta s - \gamma s^2}, \qquad \gamma > 0 .$$

Again using the Chapman Kolmogoroff equation in the form $\psi(s, t+\tau) = \psi(s, t)\psi(s, \tau)$ we deduce that $\delta = at, \gamma = b^2 t$ where a, b are real and independent of t. Now if $t > 0$ is fixed $F(x) = e^{\beta t} P(t, x, (0, \infty))$ is nondecreasing, $F(-\infty) = 0, F(+\infty) = 1$, that is F is a distribution function, and the above result shows that the convolution of F with the normal density $q_\epsilon(t, x)$ is a distribution of normal type. By a well known theorem [17], F is also of normal type and we have

$$\int_{-\infty}^{\infty} e^{-sx} dF(x) = e^{-ats} + (b^2 - \varepsilon) ts^2$$

with $b^2 - \varepsilon \geq 0$. If $b^2 - \varepsilon = 0$ the given homogeneous process is an essentially determined one while if $b^2 - \varepsilon > 0$ it is one derived from the Wiener process.

Another approach to the problem of determining when homogeneous processes or equivalentely infinitely divisible processes are totally positive is based on the Levy Khintchine representation. We consider an infinitely divisible process $x(t)$ properly centered with no fixed points of discontinuities whose characteristic function $\varphi(t, s)$ has an expression

$$(1) \qquad \log \varphi(t, s) = t\left[i\gamma s + \int_{-\infty}^{\infty} \left(e^{isx} - 1 - \frac{isx}{1 + x^2}\right) dG(x)\right]$$

$$= t\psi(s)$$

with the aid of (1) we are able to establish

$$\lim_{t\to 0}\frac{1}{t}\Pr\{|x(t)-x(0)|\geq\lambda\} = \int_{|x|\geq\lambda>0} dG(x) \tag{2}$$

when λ and $-\lambda$ are continuity points of G. This limit relation is essentially known but for lack of any available specific reference we sketch a proof.

The proof consists of defining

$$H(t,\lambda) = \begin{cases} \dfrac{\Pr\{x(t)-x(0)\leq\lambda\}-1;}{t} & \text{for} \quad \lambda>0 \\ \dfrac{\Pr\{x(t)-x(0)\leq\lambda\}}{t} & \text{for} \quad \lambda<0 \end{cases}$$

and forming the Fourier Stieljes transform of H which reduces to $(\varphi(t, s)-1)/t$. This clearly converges pointwise as $t\to 0$ to $\psi(s)$. Invoking the Levy convergence criteria following comparison with (1) establishes (2).

An alternative proof of (2) can be based on verifying the validity of (2) first for the case of a finite composition of independent Poisson processes and afterwards passing to a limit to obtain the general infinitely divisible process.

The truth of (2) also follows by exploiting the properties of the infinitely divisible process $U_{\lambda,t}$ which counts the number of jumps of magnitude exceeding λ that the process $x(t)$ executes in time t. (See [5] page 424).

Because of (2) and Theorem 5, we see that $x(t)$ is totally positive of order 2 if and only if $\int_{|x|\geq\lambda} dG(x) = 0$ for all $\lambda>0$. Hence the only totally positive infinitely divisible process is the Wiener process except for a drift factor.

10. Examples. In this section we present some examples of totally positive semigroups of matrices and kernels. These matrices and kernels are fundamental solutions of parabolic differential equations (or differential difference equations).

In generating examples of totally positive kernels it is useful to note that if $P(t, x, E)$ represents a totally positive kernel and $P(t, x, E)$ possesses a continuous density $p(t, x, y)$ with respect to a σ-finite measure μ then $p(t, x, y)$ is totally positive in the sense that

$$\det p(t, x_i, y_j)\geq 0$$

where $x_1<x_2<\cdots<x_n$ and $y_1<y_2<\cdots<y_n$. The proof consists of

selecting $E_1 < E_2 < \cdots < E_n$ where E_i is a sufficiently small open set enclosing y_i and computing

$$\lim\left[\frac{1}{\mu(E_1)\mu(E_2)\cdots\mu(E_n)}\det P(t, x_i, E_j)\right]$$

the limit taken as $\mu(E_i)$ tends to 0 for all i.

Ex. (i) The analytic properties of birth and death matrices have already been investigated in detail by the authors [14]. In Theorem 20 of that paper it is shown that with every solvable Stieltjes moment problem there is associated one or more strictly totally positive semigroups of matrices. A few examples of interest are recorded:

(a) Let $L_n^\alpha(x)$ be the usual Laguerre polynomials $\left(\text{normalized so that } L_n^\alpha(0) = \binom{n+\alpha}{n}\right)$, and let $P(t)$ be the infinite matrix with elements

$$P_{nm}(t) = \int_0^\infty e^{-xt} L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-x} dx .$$

Then $P(t)$ is strictly totally positive for $t > 0, \alpha > -1$.

(b) Let $c_n(x, a)$ be the Poisson-Charlier polynomials [15] and $P(t)$ the matrix with elements

$$P_{nm}(t) = \sum_{k=0}^\infty e^{-kt} c_n(k, a) c_m(k, a) \frac{a^k}{k!} .$$

Then $P(t)$ is strictly totally positive for $t > 0, a > 0$.

Ex (ii) The Wiener process on the real line is a strong Markoff process with continuous path functions. The direct product of n copies of this process is the n-dimensional Wiener process which is known to be a strong Markoff process. Therefore the kernel

$$P(t, x, E) = \frac{1}{\sqrt{4\pi t}} \int_E \exp\left[\frac{-(x-y)^2}{4t}\right] dy$$

is totally positive for $t > 0$ (strictly, since $P(t, x, E) > 0$ when E is an open set).

Ex. (iii) If $Y(t) = (Y_1(t), \cdots, Y_k(t))$ is the k-dimensional Wiener process and $X(t)$ is its radial part, *i.e.*,

$$X(t) = \left[\sum_1^k Y_i^2(t)\right]^{1/2}$$

then $X(t)$ is a process on $0 \leq x < \infty$ with continuous path functions. These processes have been studied by Levy [16], Spitzer [22] and others. The corresponding diffusion equation and transition function are

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2\gamma}{x}\frac{\partial u}{\partial x} ,$$

$$P(t, x, E) = \int_E p(t, x, y) d\mu(y) ,$$

where

$$\gamma = \frac{k-1}{2} ,$$

$$p(t, x, y) = \int_0^\infty e^{-\alpha^2 t}\, T(\alpha y) T(\alpha y) d\mu(\alpha) ,$$

$$T(x) = \Gamma\left(\gamma + \frac{1}{2}\right)\left(\frac{x}{2}\right)^{1/2-\gamma} J_{\gamma-1/2}(x) ,$$

$$\mu(x) = \frac{x^{2\gamma+1}}{2^{\gamma+1/2}\Gamma(\gamma+3/2)} ,$$

where J stands for the usual Bessel function.

These formulas make sense for arbitrary $\gamma \geq 0$ and have been studied by Bochner [2]. The density may be written in the form

$$p(t, x, y) = (2t)^{-(\gamma+1/2)} \exp\left(\frac{-x^2}{4t}\right) \exp\left(\frac{-y^2}{4t}\right) T\left(\frac{ixy}{2t}\right) .$$

Now $T(ix/2t)$ is a power series with positive coefficients, in fact

$$T\left(\frac{ix}{2t}\right) = \sum_{k=0}^{\infty} a_k x^{2k} = \int_{0-}^{\infty} x^s d\sigma(s)$$

where

$$a_k = \frac{\Gamma(\gamma+1/2)}{k!\,\Gamma(\gamma+k+1/2)(4t)^{2k}}$$

and $\sigma(s)$ is an increasing step function whose jumps occur at the even integers. Let $0 \leq x_1 < x_2 < \cdots < x_n$ and $0 \leq y_1 < y_2 < \cdots < y_n$. If $0 \leq s_1 < s_2 < \cdots < s_n$ then the Vandermonde determinant

$$\varDelta\begin{pmatrix} x_1, \cdots, x_n \\ s_1, \cdots, s_n \end{pmatrix} = \det\{(x_\alpha)^{s_\beta}\}$$

is known to be non-negative, positive if $x_1 > 0$. From the formula

$$\det T\left(\frac{ix_\alpha y_\beta}{2t}\right) = \iint_{0 \leq s_1 < s_2 \cdots < s_n < \infty} \varDelta\begin{pmatrix} x_1 \cdots x_n \\ s_1 \cdots s_n \end{pmatrix} \varDelta\begin{pmatrix} y_1 \cdots y_n \\ s_1 \cdots s_n \end{pmatrix} d\sigma(s_1) d\sigma(s_2) \cdots d\sigma(s_n)$$

it readily follows that $T(ixy/2t)$ and hence also $p(t, x, y)$ is strictly totally positive.

Ex. (iv) If we consider Brownian motion on the circle the transition density function has the form

$$p(t, \theta, \psi) = 1 + 2 \sum_{n=1}^{\infty} e^{-4\pi^2 nt} \cos 2\pi n(\theta - \psi)$$

where θ and ψ traverse the unit interval. This formula may be derived as the fundamental solution of the heat equation on the circle. In this case the hypothesis of Theorem 1 are fulfilled and we deduce that all odd order determinants of $p(t, \theta, \psi)$ are non-negative (actually strictly positive); viz

If $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_{2n+1} \leq 1$ and $0 \leq \psi_1 < \psi_2 < \cdots < \psi_{2n+1} \leq 1$ then det $p(t, \theta_i, \psi_j) > 0$.

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