PROBLEMS IN SPECTRAL OPERATORS

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Introduction. An important problem in the theory of spectral operators in Banach spaces initiated by N. Dunford [5; 6] is that of deciding whether the linear operators of the types encountered in analysis are spectral. Various conditions for spectral operators have been given in [5], but further research is needed in order to apply them to specific cases. J. Schwarz [11] has shown that a class of operators arising from, not necessarily self adjoint, integro-differential boundary-value problems consists of spectral operators. The present investigation originated in a problem on stationary sequences in Banach spaces which led to the study of unitary operators, namely linear isometries of the space onto itself, from this point of view. Accordingly, attention was focused on the class of unitary operators, and the limitations imposed on the operators under study were designed to include it.

Section 1 contains a summary of definitions and results from [5; 6]. A distinction, significant only in non-reflexive spaces, is made between spectral and merely prespectral operators according to the topology in which σ-additivity of the resolutions of the identity is required. As shown in § 2, a resolution of the identity of a prespectral operator uniquely determines the resolutions of the identity of its spectral restrictions. A simple example shows how this can be used to prove that certain operators are not spectral.

Known results are combined in § 3 to yield a necessary condition for spectral operators of scalar type, which involves only the norms of rational functions of the operators. If the space is reflexive and the spectrum an R-set [1, p. 397], the condition is also sufficient. Using the results of § 2 this condition is localised to "cyclic" subspaces generated by single elements. A much more general approach to localization, via the notion of vector measures associated with the operator, is expounded in [3]. It is felt though that the present considerations retain their interest owing to the explicit conditions given. The method of [3] also implies the results of § 2 on restrictions for the case of a reflexive space. Section 3 ends with some characterizations of finite dimensional cyclic subspaces.

The above results are specialized in § 4 to unitary operators which, if the space is reflexive, satisfy all the subsidiary conditions. As a corollary it follows that in a reflexive space a unitary operator is spectral

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if and only if every stationary sequence it generates is spectral.

The final section contains examples of non-spectral unitary operators. It is shown that a unitary operator U in the space of continuous functions defined on a compact Hausdorff space is not spectral provided the homeomorphism determined by U is non-periodic. Using the boundness of the norms of the values of a resolution of the identity, examples are given of non spectral unitary operators in the spaces l_p, $1 \leq p \leq \infty$, $p \neq 2$. The two methods used are combined to show that if in particular the permutation of the basis, determined by a unitary operator in the last mentioned spaces has an infinite "cycle", the operator is not spectral. Examples of non spectral unitary operators in the spaces L_p, $p \neq 2$, (c) and (c_0) follow as corollaries.

1. **Preliminaries.** Let $\mathfrak{X}$ denote a complex B-space and $\mathfrak{B}$ the Boolean algebra of Borel subsets of the complex plane p. A *spectral measure* in $\mathfrak{X}$ is a homomorphism E of $\mathfrak{B}$ onto a Boolean algebra of projections of $\mathfrak{X}$ such that: $E(p) = I =$ identity operator, $E(\phi) = 0$, and $||E(\sigma)|| \leq M < \infty$, M independent of $\sigma \in \mathfrak{B}$. The Boolean operations on commuting projections A, B are defined, as usual, by

$$A \cap B = AB , \qquad A \cup B = A + B - AB .$$

A spectral measure E in $\mathfrak{X}$ is said to be *of class Γ* in case Γ is a total linear manifold in $\mathfrak{X}^*$ and $x^*E(\ )x$ is σ-additive on $\mathfrak{B}$ for $x \in \mathfrak{X}$, $x^* \in \Gamma$.

Let $B(\mathfrak{X})$ be the B-algebra of bounded linear operators of $\mathfrak{X}$ into itself. If $T \in B(\mathfrak{X})$ and $\mathfrak{Y}$ is a (closed) subspace of $\mathfrak{X}$, we denote by $T \mid Y$ the restriction of T to $\mathfrak{Y}$, and by $\sigma(T)$ and $\rho(T)$ respectively the spectrum and resolvent set of T. Thus, if $\mathfrak{Y}$ is, invariant under T, $\sigma(T \mid Y)$ denotes the spectrum of T considered as an operator in $\mathfrak{Y}$. For $\zeta \in \rho(T)$, $(\zeta - T)^{-1}$ is abbreviated to $T(\zeta)$.

An operator $T \in B(\mathfrak{X})$ is called a *prespectral operator* (of class Γ) in case there exists a spectral measure E of some class Γ such that

$$TE(\sigma) = E(\sigma)T , \qquad \sigma(T \mid E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma} , \quad \sigma \in \mathfrak{B} .$$

E is then called a *resolution of the identity* for T.

An operator in $\mathfrak{B}(\mathfrak{X})$ is called a *spectral operator* if it is prespectral of class $\mathfrak{X}^*$. In this case, E is σ-additive on $\mathfrak{B}$ in the strong operator topology, and the boundness of its range is a consequence of the other requirements [6, p. 325]. A spectral operator T has a unique resolution of the identity E[6, Th. 6]. If $A \in B(\mathfrak{X})$ commutes with T, then it commutes with E[6, Th. 5].

It may also easily be shown that if the bounded subsets of $\mathfrak{X}$ are weakly sequentially conditionally compact, in particular if $\mathfrak{X}$ is reflexive, then every prespectral operator in $\mathfrak{X}$ is spectral.

Let $T \in B(\mathfrak{X})$, $x \in \mathfrak{X}$. By an abuse of language, an $\mathfrak{X}$-valued function f defined and analytic on an open set $D(f) \subseteq p$ is called an *analytic extension* of $T(\zeta)x$ if

$$(\zeta - T)f(\zeta) = x\,, \qquad \zeta \in D(f)\,.$$

$f(\zeta) = T(\zeta)x$ on $D(f) \cap \rho(T)$ for otherwise $(\zeta - T)(f(\zeta) - T(\zeta)x) = x - x = 0$ would imply $\zeta \in \sigma(T)$. Further we have

1.1. THEOREM. *If T is a prespectral operator, and f, g are analytic extensions of $T(\zeta)x$, then $f(\zeta) = g(\zeta)$ for $\zeta \in D(f) \cap D(g)$.* ([6, Th. 2], The further assumption $D(f) \supseteq \rho(T)$, which is made there, is not used in the proof).

Hence there exists a maximal open set which may serve as a domain of definition of an analytic extension of $T(\zeta)x$. This set is called the *resolvent set* of x, and is denoted by $\rho(x)$ (or $\rho_T(x)$, when more then one operator is involved in the discussion). Its complement $\sigma(x)$ (or $\sigma_T(x)$) is called the *spectrum* of x. The maximal analytic extension itself is denoted by $x(\zeta)$ (or $x_T(\zeta)$).

The main use of the concepts above is through the following characterization of spectral subspaces [6, Th. 4]:

1.2. THEOREM. *Let T be a prespectral operator in $\mathfrak{X}$ with a resolution of the identity E, and let $\sigma \subseteq p$ be closed. Then*

$$E(\sigma)\mathfrak{X} = \{x \,|\, \sigma(x) \subseteq \sigma\}\,.$$

Let E be a spectral measure which vanishes on the complement of a compact set σ, and let f be a complex valued function continuous on σ. Then the Riemann integral $\int_\sigma f(\zeta)E(d\zeta)$ exists in the uniform operator topology [6, Th. 7]. An operator S is said to be of *scalar type* if it is spectral and satisfies

1.3. $$S = \int \zeta E(d\zeta) \left(= \int_{\sigma(S)} \zeta E(d\zeta) \right),$$

where E is the resolution of the identity of S [6, Def. 1].

The reader is referred to [4] for the definition and properties of $f(T)$, where $T \in B(\mathfrak{X})$ and f belongs to a certain class of locally holomorphic functions. In the sequel, f will in general be a rational function with poles in $\rho(T)$. If S is of scalar type with the resolution of the identity E, then we have the functional calculus

1.4. $$f(S) = \int f(\zeta)E(d\zeta)\,.$$

We refer to [6] for the general case of a spectral operator.

Finally we shall need the concept of the *cyclic subspace* $[x]$ generated by an element $x \in \mathfrak{X}$. By this is meant the subspace spanned by $\{T(\zeta)x \mid \zeta \in \rho(T)\}$ [5, Def. 1.4]. It has the following properties [5, Lemma 1.5]:

1.5. LEMMA.
1.5.1. $x \in [x]$.
1.5.2. $f(T)[x] \subseteq [x]$.
1.5.3. *If* $y \in [x]$, *then* $[y] \subseteq [x]$.

2. Restrictions of prespectral operators. The following is a generalization of the uniqueness theorem for spectral operators mentioned in § 1.

2.1. THEOREM. *Let T be a prespectral operator in the B-space $\mathfrak{X}$, and let E be a resolution of the identity for T. Let $\mathfrak{Y}$ be a subspace of $\mathfrak{X}$ invariant under T. Then if $T \mid \mathfrak{Y}$ is spectral, its resolution of the identity equals the restriction $E \mid \mathfrak{Y}$ of E to $\mathfrak{Y}$.*

Proof. Let $y \in \mathfrak{Y}$. The function $y_{T|\mathfrak{Y}}(\zeta)$ is an analytic extension of $T(\zeta)y$ with domain $\rho_{T|\mathfrak{Y}}(y)$. Thus $\rho_{T|\mathfrak{Y}}(y) \subseteq \rho_T(y)$, or

$$\sigma_T(y) \subseteq \sigma_{T|\mathfrak{Y}}(y) \ . \tag{2.1.1}$$

Let F denote a resolution of the identity for $T \mid \mathfrak{Y}$. If σ is a closed subset of the complex plane, we have by 1.2

$$\sigma_{T|\mathfrak{Y}}(F(\sigma)y) \subseteq \sigma \ .$$

Therefore, by (2.1.1),

$$\sigma_T(F(\sigma)y) \subseteq \sigma,$$

and again by 1.2

$$E(\sigma)F(\sigma)y = F(\sigma)y \ . \tag{2.1.2}$$

If τ is a closed set disjoint from σ, we get, operating with $E(\sigma)$ on $E(\tau)F(\tau)y = F(\tau)y$,

$$E(\sigma)F(\tau)y = 0 \ . \tag{2.1.3}$$

(2.1.3) and the σ-additivity of F in the strong operator topology show that $E(\sigma)F(\sigma')y = 0$ (σ' denotes the complement of σ with respect to p). This together with (2.1.2) gives

$$E(\sigma)y = F(\sigma)y \ , \qquad \sigma \text{ closed.}$$

The properties of E and F now yield the same equality for every Borel set.

The theorem above shows that invariance of $\mathfrak{Y}$ under E (i.e., under every value of E) is a necessary condition in order that $T \mid \mathfrak{Y}$ be spectral. This condition is by no means automatically fulfilled, and this fact can be used to show that an operator is not spectral:

2.2. EXAMPLE. Let Ω be a compact topological space. We consider the B-space $C(\Omega)$ of all complex valued functions f continuous on Ω with $\|f\| = \max_{\omega \in \Omega} |f(\omega)|$. Let $\mu \in C(\Omega)$, and let S be the operator of multiplication by μ. Heuristically, S cannot in general be spectral because projections which "ought" to belong to the resolution of the identity are not members of $B(C(\Omega))$. This is made precise as follows. Let T be the extension of the multiplication to the space $\mathfrak{X} = M(\Omega)$ of complex valued functions f bounded on Ω with $\|f\| = \sup |f(\omega)|$. T is prespectral with a resolution of the identity: $E(\sigma)$ is the multiplication by $\chi_\sigma(\mu(\ ))$, where χ_σ is the characteristic function of σ. The σ-additivity may be verified with respect to the total linear manifold generated by the functionals x_ω^*, $\omega \in \Omega$, defined by $x_\omega^* x = x(\omega)$, $x \in M(\Omega)$. To see that $\sigma(T \mid E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}$, observe that if $\zeta \in \bar{\sigma}'$, $(T \mid E(\sigma)\mathfrak{X})(\zeta)$ is the multiplication by $\chi_\sigma(\mu(\ ))(\zeta - \mu)^{-1}$ (here $0/0 = 0$). We omit the details. Now suppose, for instance, that μ is not constant on a connected component of Ω, and that ω_1, ω_2 are two points in the component such that $\mu(\omega_1) \neq \mu(\omega_2)$. Then taking $\sigma = \{\mu(\omega_1)\}$ we see that $E(\sigma)$ does not leave $C(\Omega)$ invariant. Hence $S = T \mid C(\Omega)$ is not spectral.

The next theorem is a partial converse of Theorem 2.1. We need two lemmas.

2.3. LEMMA. *Let T be a prespectral operator in the B-space $\mathfrak{X}$, and let $A \in B(\mathfrak{X})$ commute with T. If $x \in \mathfrak{X}$, then $\sigma(Ax) \subseteq \sigma(x)$ and $(Ax)(\zeta) = Ax(\zeta)$, $\zeta \in \rho(x)$.*

Proof. For $\zeta \in \rho(x)$, $(\zeta - T)Ax(\zeta) = A(\zeta - T)x(\zeta) = Ax$. The conclusion follows by the definition of $\sigma(Ax)$ and 1.1.

2.4. LEMMA. *If T is prespectral in $\mathfrak{X}$, $x \in \mathfrak{X}$ and τ is a connected component of $\rho(x)$ such that $\tau \cap \rho(T) \neq \phi$, then $x(\zeta) \in [x]$, $\zeta \in \tau$.*

Proof. Since $\rho(x)$ is open in the complex plane, τ has the same property and is therefore a region. Let $x^* \in \mathfrak{X}$ vanish on $[x]$. For $\zeta \in \rho(T)$, $x(\zeta) = T(\zeta)x \in [x]$; thus $f(\zeta) = x^*x(\zeta)$ vanishes on the open subset $\tau \cap \rho(T)$ of τ. Being regular, f vanishes identically on τ. A well known corollary of the Hahn-Banach extension theorem yields the conclusion.

It may also be shown that $\{\zeta \in \rho(x) | x(\zeta) \in [x]\}$ is open and closed in $\rho(x)$. If $\rho(T)$ is dense in the plane, then $x(\zeta) \in [x]$ for every $\zeta \in \rho(x)$ [5, Lemma 1.5.3]. Cf. however Example 2.6 below.

2.5. THEOREM. *Let T be a prespectral operator in $\mathfrak{X}$ with a resolution of the identity E. Let $\mathfrak{Y}$ be a subspace of $\mathfrak{X}$ invariant under $T(\zeta)$, $\zeta \in \rho(T)$, and under E. Then $T | \mathfrak{Y}$ is prespectral with a resolution of the identity $E | \mathfrak{Y}$. If T is spectral or spectral of type m (v. [6, p. 336]), $T | Y$ has the same property.*

Proof. Since $T = \dfrac{1}{2\pi i}\displaystyle\int_C T(\zeta)d\zeta$, where C is a circle containing $\sigma(T)$ in its interior and the integral is in Riemann's sense and in the uniform operator topology, $\mathfrak{Y}$ is invariant under T, and $T | \mathfrak{Y}$ is well defined. If T is spectral, we may assume invariance under T instead of under $T(\zeta)$, $\zeta \in \rho(T)$, using [6, Lemma 3].

All the assertions of the theorem are easily verified, except: For every $\sigma \in \mathfrak{B}$, $\sigma((T | \mathfrak{Y}) | (E | Y)(\sigma)\mathfrak{Y}) = \sigma(T | E(\sigma)\mathfrak{Y}) \subseteq \bar{\sigma}$. We have to show that if $\zeta \in \bar{\sigma}'$, then $\zeta - T$ induces a one-to-one mapping of $E(\sigma)\mathfrak{Y}$ onto itself. Since $\sigma(T | E(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}$, there is no $z \neq 0$ in $E(\sigma)\mathfrak{X}$ and hence in $E(\sigma)\mathfrak{Y}$ such that $(\zeta - T)z = 0$. It remains to show that the range of $(\zeta - T) | E(\sigma)\mathfrak{Y}$ is $E(\sigma)\mathfrak{Y}$. Let $z \in E(\sigma)\mathfrak{Y}$. Then $E(\sigma)z = z$, hence $E(\bar{\sigma})z = z$, and therefore by 1.2 $\sigma(z) \subseteq \bar{\sigma}$. Therefore $\zeta \in \rho(z)$, and since $(\zeta - T)z(\zeta) = z$ it suffices to show that $z(\zeta) \in E(\sigma)\mathfrak{Y}$. Let π be an open half plane with ζ on its boundary. From 1.2 it follows that $\sigma(E(\pi')z) \subseteq \pi' \cup \sigma(z)$, and therefore $\{\zeta\} \cup \pi \subseteq \rho(E(\pi')z)$. Since $\rho(E(\pi')z)$ is open, it follows that ζ belongs to a component of $\rho(E(\pi')z)$ which contains arbitrarily distant points of the complex plane and thus points of $\rho(T)$. 2.4 now implies $(E(\pi')z)(\zeta) \in [E(\pi')z]$. The assumptions of the invariance of $\mathfrak{Y}$ show that $[E(\pi')z] \subseteq \mathfrak{Y}$. Therefore $(E(\pi')z)(\zeta) \in \mathfrak{Y}$. But by 2.3, we have $(E(\pi')z)(\zeta) = E(\pi')z(\zeta)$; therefore

$$E(\pi')z(\zeta) \in \mathfrak{Y} .$$

Similarly one shows $E(\pi)z(\zeta) \in \mathfrak{Y}$. Therefore $z(\zeta) = E(\pi')z(\zeta) + E(\pi)z(\zeta) \in \mathfrak{Y}$. On the other hand, $E(\sigma)z = z$ implies by 2.3 $E(\sigma)z(\zeta) = z(\zeta)$. Therefore $z(\zeta) \in E(\sigma)\mathfrak{Y}$ as required.

It follows from the proof above that, under the conditions of the theorem, $z \in Y$ implies $z(\zeta) \in \mathfrak{Y}$, $\zeta \in \rho(z)$. The following example shows that without invariance of $\mathfrak{Y}$ under E, this need not hold even if T is a normal operator in Hilbert space. This, in turn, amplifies Example 2.2 by showing that even if T is spectral, and not merely prespectral, $\mathfrak{Y}$ is not necessarily invariant under E.

2.6. EXAMPLE. Let $\mathfrak{X}$ be the Hilbert space $L^2(\Omega)$, where Ω is the disc $\{\omega \mid |\omega| \leq 1\}$ in the complex plane. Let T be the operator of multiplication by ω. Then T is a bounded normal operator and spectral. We define $x \in \mathfrak{X}$ by

$$x[\omega] = \begin{cases} 1 & \text{if } \ \frac{1}{2} \leq |\omega| \leq 1 \\ 0 & \text{if } \ \quad |\omega| < \frac{1}{2} \,. \end{cases}$$

The maximal analytic extension of $T(\zeta)x$, $x(\zeta)$, exists for ζ not in the ring $\sigma(x) = \{\zeta \mid \frac{1}{2} \leq |\zeta| \leq 1\}$, and then

$$x(\zeta)[\omega] = \begin{cases} \dfrac{1}{\zeta - \omega} & \text{if } \ \frac{1}{2} \leq |\omega| \leq 1 \\ 0 & \text{if } \ \quad |\omega| < \frac{1}{2} \,. \end{cases}$$

We consider the subspace $\mathfrak{Y} = [x]$, which is invariant under $T(\zeta)$, $\zeta \in \rho(T)$ by 1.5.2, and contains x by 1.5.1. $[x]$ is the closure of the finite linear combinations of the functions $T(\zeta)x = x(\zeta)$ for $\zeta \in \rho(T) = \Omega'$.

Now, suppose that for a fixed ζ, $|\zeta| < \frac{1}{2}$, $x(\zeta)$ were approximable by these linear combinations in the Hilbertian norm. Since all these functions are holomorphic in $\sigma(x)$, $x(\zeta)$ would be uniformly approximable by the linear combinations on a closed ring τ concentric with and inner to $\sigma(x)$ [13, p. 96]. But this is impossible, since the approximants are rational functions with poles in the unbounded component of τ', while the only analytic continuation of $x(\zeta) \mid \tau$ to the other component is $1/(\zeta - \omega)$, which is not regular at ζ (v. [13, p. 25, Th. 16]).

It may also be directly shown that there exist x and σ such that $E(\sigma)x \notin [x]$.

We now give an example to show that the assumption that $T \mid \mathfrak{Y}$ is spectral cannot be dropped in Theorem 2.1 even if $\mathfrak{Y} = \mathfrak{X}$; i.e., a pre-spectral operator may have more than one resolution of the identity.

2.7. Example. We specialize Example 2.2, retaining its notation. We take for Ω the set of positive integers. Thus $\mathfrak{X} = M(\Omega)$ is the space usually denoted by (m). For μ we chose a function belonging to $\mathfrak{X}$ which satisfies

$$\mu(1) = 1\,; \tag{2.7.1}$$

$$\mu(j) \neq 1\,, \qquad j > 1\,; \tag{2.7.2}$$

$$\lim_j \mu(j) = 1\,. \tag{2.7.3}$$

As is well known [2, p. 34], there exists a real bounded linear functional $\lim_R$, defined on the space $(m)_R$ of all real bounded sequences, which has the following properties:

$$\text{If } x, y \in (m)_R \text{ and } y(j) = x(j+1)\,, \qquad j = 1, 2, \cdots, \tag{2.7.4$_R$}$$

then $\lim_R y = \lim_R x$;

$$(2.7.5)_R \qquad \underline{\lim}\, x(j) \leq \lim_R x \leq \overline{\lim}\, x(j)\ .$$

We define a functional lim on $\mathfrak{X}$ by $\lim x = \lim_R x' + i \lim_R x''$, where $x = x' + ix''$, $x', x'' \in (m)_R$. Evidently, lim is a bounded linear functional which enjoys the property (2.7.4) analogous to $(2.7.4)_R$. Further we have (T defined as in 2.2)

$$(2.7.6) \qquad \lim Tx = \lim x\ .$$

To see this, we write $(Tx)(j) = (\mu(j) - 1)x(j) + x(j)$. By the linearity of lim, it suffices to show that $\alpha(j) \to 0(\alpha(j) = \mu(j) - 1)$ implies $\lim \alpha x = 0$. This follows from $(2.7.5)_R$ on separating α and x into their real and imaginary parts. We define an operator $A \in B(\mathfrak{X})$ by $Ax = \lim x \cdot x_0$, where $x_0(j) = \delta_{1j}$ (Kronecker's symbol). Using (2.7,1), (2.7.6) we get $TA = AT$. On the other hand, A does not commute with E (defined in 2.2). Taking $\sigma = \{1\}$ we have, using (2.7.2), (2.7.4), $AE(\sigma)x = 0$ while $E(\sigma)Ax = \lim x \cdot x_0$. Hence the function F, defined by

$$F(\sigma) = E(\sigma) + AE(\sigma) - E(\sigma)A\ , \qquad \sigma \in \mathfrak{B}\ ,$$

differs from E. We show that F is a resolution of the identity for T. A straightforward calculation, based on the fact E is a spectral measure, shows that F is a spectral measure (In verifying that $F(\sigma)F(\delta) = F(\sigma \cap \delta)$, one uses the fact that $AE(\tau)A = 0$, $\tau \in \mathfrak{B}$). F is σ-additive with respect to the total linear manifold generated by the functionals x_j^* $(x_j^* x = x(j))$, $j \geq 2$ and $x^* = x_1^* - \lim$; since $x_j^* F(\sigma)x = x_j^* E(\sigma)x$ for $j \geq 2$, while $x^* F(\sigma)x = \chi_\sigma(1)(x(1) - \lim x)$. Since T commutes with E and A, T commutes with F. Finally, to see that $\sigma(T \mid F(\sigma)\mathfrak{X}) \subseteq \bar{\sigma}$, we assert that the restriction of $(T \mid E(\bar{\sigma})\mathfrak{X})(\zeta)$ to $F(\sigma)\mathfrak{X}$, $\zeta \in \bar{\sigma}'$, is an inverse of $(\zeta - T) \mid F(\sigma)\mathfrak{X}$. As shown in the proof of [6, Th. 5], the prespectrality of T implies $E(\bar{\sigma})AE(\bar{\sigma}) = AE(\bar{\sigma})$. Hence $E(\bar{\sigma})AE(\sigma) = AE(\sigma)$, whence it follows that $E(\bar{\sigma})F(\sigma) = F(\sigma)$. Therefore $F(\sigma)\mathfrak{X} \subseteq E(\bar{\sigma})\mathfrak{X}$, and the mentioned restriction is well defined. Let $x \in F(\sigma)\mathfrak{X}$. Then $\sigma(x) \subseteq \bar{\sigma}$, by 1.2, since $x \in E(\bar{\sigma})\mathfrak{X}$. Further, 1.1 and 2.3 imply

$$(T \mid E(\bar{\sigma})\mathfrak{X})(\zeta)x = x(\zeta) = (F(\sigma)x)(\zeta) = F(\sigma)x(\zeta)\ .$$

Thus the range of the restriction is included in $F(\sigma)\mathfrak{X}$. The truth of our assertion is now evident.

3. Conditions for operators of scalar type. If $T \in B(\mathfrak{X})$, the *full algebra* generated by T, denoted by $\mathfrak{A}(T)$, is the smallest subalgebra of $B(\mathfrak{X})$ which is closed in the norm topology of $B(\mathfrak{X})$, which is inverse-closed and which contains T and I [6, Def. 5]. Let σ be a compact subset of the complex plane. We denote by $R(\sigma)$ the set of rational functions regular on σ. $CR(\sigma)$ denotes the closure of $R(\sigma)$ in $C(\sigma)$. Following [1, p. 397], a compact nowhere dense set σ in the complex plane is called an *R-set* if and only if $CR(\sigma) = C(\sigma)$. For properties of R-sets used in the sequel see [1, p. 398] and the references there given.

3.1. THEOREM. *Let $S \in B(\mathfrak{X})$, then the following equivalent conditions are necessary in order that S be of scalar type*:

3.1.1. *There exists a constant $H < \infty$ such that for every $f \in R(\sigma(S))$*

$$\|f(S)\| \leq H \max_{\zeta \in \sigma(S)} |f(\zeta)| = H \|f(S)\|_{sp} = H \lim_n \|f(S)^n\|^{1/n} .$$

3.1.2. *There exists a constant $K < \infty$ such that for every $f \in R(\sigma(S))$*

$$\|f(S)\|^2 \leq K \|f(S)^2\| .$$

If $\mathfrak{X}$ is reflexive and $\sigma(S)$ is an R-set, each of the mentioned conditions is sufficient. Each of the following conditions implies 3.1.1:

3.1.3. *For every $x \in \mathfrak{X}$ there exists a constant $H(x)$ (independent of f) such that for every $f \in R(\sigma(S))$*

$$\|f(S)x\| \leq H \max_{\zeta \in \sigma(S)} |f(\zeta)| \cdot \|x\| = H(x) \|f(S)\|_{sp} \|x\| .$$

3.1.4. *The same; with $h(x)$, $f \in R(\sigma(S \mid [x]))$ and*

$$\|f(S \mid [x]) \leq h(x) \max_{\zeta \in \sigma(S \mid [x])} |f(\zeta)| = h(x) \|f(S \mid [x])\|_{sp}$$

3.1.5. *The same; with $k(x)$, $f \in R(\sigma(S \mid [x]))$ and*

$$\|f(S \mid [x])\|^2 \leq k(x) \|f(S \mid [x])^2\| .$$

3.1.3 *is implied by* 3.1.1. 3.1.4 *and* 3.1.5 *are necessary if S is of scalar type and satisfies the following condition*:

3.1.6. *If E is the resolution of the identity of S, $x \in \mathfrak{X}$ and $\sigma \in \mathfrak{B}$, then*

$$E(\sigma)x \in [x] .$$

Proof. For the equivalence of 3.1.1 and 3.1.2 see [9, p. 78] and for the necessity see the beginning of the proof of [6, Th. 13]. If one of them holds, then $\mathfrak{A}(S)$ is equivalent to $CR(\sigma(S))$, hence if $\sigma(S)$ is an R-set, to $C(\sigma(S))$. Therefore if $\mathfrak{X}$ is reflexive, S is of scalar type by [6, Th. 18 (IV)]. Since, from 1.5.2, $\sigma(S \mid [x]) \subseteq \sigma(S)$ and $\|f(S)x\| \leq \|f(S \mid [x])\| \cdot \|x\|$, the equivalent conditions 3.1.4, 3.1.5 imply 3.1.3. The proof that 3.1.3 implies 3.1.1 is much like the proof of the uniform boundness theorem. 3.1.3 and Baire's category theorem imply that at least one of the sets

$$G_j = \{x \in \mathfrak{X} \mid \|f(S)x\| \leq j \|f(S)\|_{sp} \|x\| \text{ , } f \in R(\sigma(S))\} \quad j = 1, 2, \cdots,$$

let it be the nth, contains a sphere $\{x \in \mathfrak{X} \mid \|x - x_0\| < r\}$, $r > 0$. 3.1.1 then easily follows with $H = n(2\|x_0\| + r)/r$. If S is of scalar type and satisfies 3.1.6, then every $[x]$ is invariant under E (because if $y \in [x]$, then $E(\sigma)x \in [y] \subseteq [x]$ by 1.5.3) and $S(\zeta)$ (by 1.5.2). Therefore, by 2.5, $S \mid [x]$ is of scalar type, and the necessity of 3.1.4, 3.1.5, which are 3.1.1, 3.1.2 for $S \mid [x]$, follows.

REMARKS. In case the conclusion of 1.2 holds, it may be convenient to replace $\sigma(S \mid [x])$ by $\sigma(x)$ in 3.1.4, 3.1.5. One always has $\sigma(x) \subseteq \sigma(S \mid [x])$. By slight modifications in the proof of [5, Lemma 1.10], one shows that, provided S is spectral, $\sigma(x) = \sigma(S \mid [x])$ (for every x) if and only if for every x and $\zeta \in \rho(x)$, $x(\zeta) \in [x]$. As remarked after 2.5, this is the case if 3.1.6 holds.

Taking S as in 2.2, 3.1.1 is obviously fulfilled. By an appropriate choice of Ω and μ, we may achieve that S is not spectral although $\sigma(S) =$ range of μ is an R-set. Thus these conditions fail to assure scalarity if $\mathfrak{X}$ is not reflexive.

We conclude the present section with some characterizations of finite dimensional cyclic subspaces.

3.2. THEOREM. *If S is of scalar type, satisfies 3.1.6 and $x \in \mathfrak{X}$, then the following conditions are equivalent*:

3.2.1. *$[x]$ is of finite dimension.*

3.2.2. *$\mathfrak{A}(S)x$ is of the second category in $[x]$ (or $x = 0$).*

3.2.3. *For each $y \in [x]$ there exists a $U(y) \in B(\mathfrak{X})$, commuting with S, such that $U(y)x = y$.*

3.2.4. *For each $y \in [x]$ there exists a $V(y) \in B([x])$, commuting with $S \mid [x]$, such that $V(y)x = y$.*

3.2.5. *$\sigma(x)$ is finite (equivalent to 3.2.1 by mere scalarity).*

Proof. Evidently we may assume $x \neq 0$. 3.2.1 $\Rightarrow$ 3.2.2 and 3.2.3: Since $\{f(S) \mid f \in R(\sigma(S))\}$ is dense in $\mathfrak{A}(S)$, $\mathfrak{A}(S)x$ is a dense linear submanifold of $[x]$. By 3.2.1, $\mathfrak{A}(S)x$ is of finite dimension; hence closed. Therefore $\mathfrak{A}(S)x = [x]$, whence 3.2.2 and 3.2.3 follow.

3.2.2 or 3.2.3 $\Rightarrow$ 3.2.4: Under either hypothesis the set

$$Z = \{z = U(z)x \mid U(z) \in B([x]), U(z)S = SU(z)\}$$

is of the second category in $[x]$. Suppose $f_n \in R(\sigma(S))$, $\|f_n(S)x\| = 1$

and $z \in Z$. Then $\{f_n(S)z\}$ is bounded since

$$\|f_n(S)z\| = \|f_n(S)U(z)x\| = \|U(z)f_n(S)x\| \\ \leq \|U(z)\| \|f_n(S)x\| = \|U(z)\| .$$

Therefore, by the uniform boundness theorem, $\{\|f_n(S) \mid [x]\|\}$ is bounded. Hence, if $f_n \in R(\sigma(S))$, $\|f_n(S)x\| = 1$ and $y \in [x]$, then $\{\|f_n(S)y\|\}$ is bounded. This shows that there exists a constant $c(y) < \infty$ such that $\|f(S)y\| \leq c(y)\|f(S)x\|$, $f \in R(\sigma(S))$. We define the transformation $V(y)$ on $\{f(S)x \mid f \in R(\sigma(S))\}$ by

$$V(y)f(S)x = f(S)y .$$

$V(y)$ is bounded by $c(y)$ on a dense linear submanifold of $[x]$. Therefore it is uniquely defined, and can be extended by continuity to a bounded operator on $[x]$. Evidently, this operator satisfies our requirements.

3.2.4 $\Rightarrow$ 3.2.5: We first show that for each $y \in [x]$ there exists a constant $c(y)$ such that

$$\|E(\sigma)y\| \leq c(y)\|E(\sigma)x\| , \qquad \sigma \in \mathfrak{B} .$$

As in the proof of 3.1, $S \mid [x]$ is of scalar type with the resolution of the identity $E \mid [x]$. By the commutativity theorem, mentioned in § 1, $E \mid [x]$ commutes with $V(y)$. Therefore for every Borel set σ

$$\|E(\sigma)y\| = \|E(\sigma)V(y)x\| = \|(E(\sigma) \mid [x])V(y)x\| \\ = \|V(y)(E(\sigma) \mid [x])x\| \leq \|V(y)\| \|E(\sigma)x\| .$$

This proves our statement. Hence, if we define

$$G_j = \{y \in [x] \mid \|E(\sigma)y\| \leq j\|E(\sigma)x\| , \sigma \in \mathfrak{B}\} , \qquad j = 1, 2, \cdots,$$

we have $\bigcup_j G_j = [x]$. Since the G_j's are closed, it follows by the usual category argument that there exists a constant $c < \infty$ such that

$$\|E(\sigma) \mid [x]\| \leq c\|E(\sigma)x\| , \qquad \sigma \in \mathfrak{B}.$$

Since the norm of a non null projection is at least 1, it follows that

$E(\sigma_n)x \to 0$, $\sigma_n \in \mathfrak{B} \Rightarrow$ There exists an n_0 such that $E(\sigma_n) \mid [x] = 0$ for $n \geq n_0$.

Now, suppose $\sigma(x)$ were infinite. Then we could represent it in the form $\sigma(x) = \bigcup_{n=0}^{\infty}\sigma_n$, where the σ_n are pairwise disjoint, $\sigma_0 \in \mathfrak{B}$ and σ_n, $n \leq 1$ are non void sets open relative to $\sigma(x)$ (we omit the easy proof). From the σ-additivity of E in the strong operator topology it follows that $E(\sigma_n)x \to 0$. Hence, by what was proved above, there exists an $m \geq 1$ such that $E(\sigma_m)x = 0$. $\sigma_m = \sigma(x) \cap \tau$, where τ is open in the complex plane. We have

$$E(\tau)x = E(\tau)E(\sigma(x))x = E(\tau \cap \sigma(x))x = E(\sigma_m)x = 0 .$$

Therefore $E(\tau')x = x$. Since τ' is closed, 1.2 implies $\sigma(x) \subseteq \tau'$. Thus we get $\sigma_m = \sigma(x) \cap \tau = \phi$, contradicting the choice of σ_m.

3.2.5 $\Rightarrow$ 3.2.1: Since we assumed $x \neq 0$, we have $\sigma(x) \neq \phi$. Let $\sigma(x) = \{\zeta_1, \cdots, \zeta_r\}$. If $y \in [x]$ there exist $f_n \in R(\sigma(S))$ such that $f_n(S)x \to y$. By 1.4, $f_n(S) = \int f_n(\zeta)E(d\zeta)$. Using Riemann's sums approximating the integral, we get

$$f_n(S)E(\sigma(x)) = \sum_{j=1}^{r} f_n(\zeta_j)E(\{\zeta_j\}) .$$

But $f_n(S)E(\sigma(x))x = f_n(S)x$; therefore

$$(*) \qquad \sum_{j=1}^{r} f_n(\zeta_j)E(\{\zeta_j\})x \to y .$$

Now

$$(**) \qquad E(\{\zeta_j\})x,\ j = 1, \cdots, r \text{ are linearly independent:}$$

If $\sum_{j=1}^{r} \alpha_j E(\{\zeta_j\})x = 0$, then operating with $E(\{\zeta_k\})$, we get $\alpha_k E(\{\zeta_k\})x = 0$. But $E(\{\zeta_k\})x \neq 0$ for otherwise

$$x = E(\sigma(x))x = E(\sigma(x) - \{\zeta_k\})x + E(\{\zeta_k\})x = E(\sigma(x) - \{\zeta_k\}x$$

would imply by 1.2 the contradiction $\sigma(x) \subseteq \sigma(x) - \{\zeta_k\}$. Therefore $\alpha_k = 0$. From (**) and (*) it follows by a well known argument that the sequences $\{f_n(\zeta_j)\}_{n=1}^{\infty}$ are bounded; hence compact. Therefore there exists a subsequence $\{n_k\}$ of the indices such that $f_{n_k}(\zeta_j) \to \alpha_j$, $j = 1, \cdots, r$. So

$$y = \sum_{j=1}^{r} \alpha_j E(\{\zeta_j\})x .$$

The vectors $E(\{\zeta_j\}x$, $j = 1, \cdots, r$, are independent of y, and thus span $[x]$.

4. Applications to unitary operators. To render the results of § 3 conveniently applicable, one should know beforehand of an operator that if it is spectral, it is of scaler type and satisfies Condition 3.1.6. We shall show that this is the case for a class of operators which includes the unitary operators in reflexive spaces. We lean heavily on [5]; and although some familiarity with this paper is assumed in the present section, it will be convenient to cite the pertinent definitions.

4.1. DEFINITION. Let the spectrum $\sigma(T)$ of an operator $T \in B(\mathfrak{X})$ lie in a closed rectifiable Jordan curve Γ_0. Suppose that Γ_0 is embedable in a family Γ_δ, $-\delta_0 \leq \delta \leq \delta_0$ $(0 < \delta_0 \leq \frac{1}{2})$, of closed rectifiable Jordan curves which satisfies the following conditions: Γ_{δ_1} is interior to Γ_{δ_2} for $-\delta_0 \leq \delta_1 < \delta_2 \leq \delta_0$. The curve Γ_δ is defined by a function $\zeta(\lambda, \delta)$, $-1 \leq \lambda \leq 1$, with $\zeta(-1, \delta) = \zeta(1, \delta)$. As λ increases from -1 to 1, the point $\zeta(\lambda, \delta)$ traces Γ_δ in a counterclockwise direction. For different values of λ, the arcs $\zeta(\lambda, \delta)$, $-\delta_0 \leq \delta \leq \delta_0$ do not intersect. They are rectifiable, and $|\delta|$ is the length of the subarc with endpoints $\zeta(\lambda, 0)$ and $\zeta(\lambda, \delta)$. Under these assumptions a nonnegative integer-valued function $\nu(\lambda)$ satisfying the condition

$$\| \delta^{\nu(\lambda)} T(\zeta(\lambda, \delta)) \| \leq 1 , \quad 0 < |\delta| < \delta_0, \quad -1 \leq \lambda \leq 1 ,$$

is called an *index function* for T.

4.2. THEOREM. *If U is a unitary spectral operator, it is of scalar type.*

Proof. This is essentially proved in [5]: It is easy to show that the spectrum of U lies in the unit circle and that if we embed the unit circle in the family of circles Γ_δ, $-\frac{1}{2} \leq \delta \leq \frac{1}{2}$, defined by $\zeta(\lambda, \delta) = (1 + \delta)e^{\pi i \lambda}$, $-\lambda \leq \lambda \leq 1$, then $\nu(\lambda) \equiv 1$ is an index function for U. Since $\zeta(\lambda, \delta)$ has continuous second partial derivatives, and the assumptions of [5, Lemma 3.16] hold, it follows from [5, Lemma 3.18] that $\int_{\sigma(U)} (U - \zeta)E(d\zeta) = 0$ or $U = \int \zeta E(d\zeta)$.

4.3. LEMMA. *Let $S \in B(\mathfrak{X})$ be spectral with index function $\nu(\lambda) \equiv 1$ with respect to $\zeta(\lambda, \delta)$ which has continuous second partial derivatives. Let $\mathfrak{X}$ be reflexive. Then $E(\{\zeta\})x \in [x]$, $x \in \mathfrak{X}$, $\zeta \in \Gamma_0$.*

Proof. Let $\zeta_0 \in \Gamma_0$. Then ζ_0 is of the form $\zeta_0 = \zeta(\lambda_0, 0)$. It is shown in the proof of [5, Th. 3.12 (III)] that there is a $y \in \mathfrak{X}$ and a sequence $\delta_n \to 0$ such that for $\zeta_n = \zeta(\lambda_0, \delta_n)$ we have

$$(\zeta_n - \zeta_0)S(\zeta_n)x(\to)y . \tag{4.3.1}$$

Further, (4.3.2) $(\zeta_0 - S)y = 0$,

$$x - y \in \overline{(\zeta_0 - S)\mathfrak{X}} . \tag{4.3.3}$$

$y \in [x]$ since, by (4.3.1), it is a weak limit of vectors in $[x]$, hence a strong limit of their linear combinations [2, p. 134. Th. 2]. (4.3.2) implies, by [6, Lemma 1], $E(\{\zeta_0\})y = y$. By (4.3.3), there exist z_n such that $(\zeta_0 - S)z_n \to y - x$, and by [5, Lemma 3.17] $E(\{\zeta_0\})(\zeta_0 - S) = 0$; therefore $E(\{\zeta_0\})(y - x) = 0$. It follows that $E(\{\zeta_0\})x = y \in [x]$.

4.4. LEMMA. *Under the hypotheses of* 4.3, *if $\zeta, \xi \in \Gamma_0$, $\zeta \neq \xi$ and*

$x \in \mathfrak{X}$, *then there exist* $z_k \in [x]$ *such that* $E(\{\zeta'\})E(\{\xi'\})x = \lim_k (S-\zeta)^2 \cdot (S-\xi)^2 z_k$.

Proof. Let $\zeta_0, \xi_0 \in \Gamma_0$, $\zeta_0 \neq \xi_0$. Then, by 1.5.1, 1.5.3 and 4.3, $u = E(\{\zeta_0\}')E(\{\xi_0\}')x \in [x]$. Therefore by 1.5.3 it is sufficient to establish the representation for u with $z_k \in [u]$. The argument follows closely part of the proof of [5, Lemmas 2.6, 2.10]. As in the proof of 4.3, there exist $\zeta_n \to \zeta_0$ such that

$$(\zeta_n - \zeta_0)S(\zeta_n)u(\to)E(\{\zeta_0\})u = 0 .$$

Thus

$$(\zeta_0 - S)S(\zeta_n)u = (\zeta_0 - \zeta_n)S(\zeta_n)u + u(\to)u .$$

Since $S(\zeta_n)u \in [u]$ and since weak convergence to u implies strong convergence of linear combinations, it follows that there exist $u_k \in [u]$ such that

$$(\zeta_0 - S)u_k \to u . \tag{4.4.1}$$

Operating on u_k with the identity

$$(\zeta_0 - S)^2 S(\zeta_n) = (\zeta_0 - \zeta_n)^2 S(\zeta_n) + (\zeta_0 - \zeta_n) + (\zeta_0 - S)$$

and letting n tend to infinity, we get

$$(\zeta_0 - S)u_k = \lim_n (\zeta_0 - S)^2 S(\zeta_n)u_k . \tag{4.4.2}$$

But $S(\zeta_n)u_k \in [u_k] \subseteq [u]$, hence (4.4.1), (4.4.2) show that there are $v_k \in [u]$ such that

$$(\zeta_0 - S)^2 v_k \to u . \tag{4.4.3}$$

Operating on (4.4.3) with $E(\{\xi_0\}')$, we get

$$(\zeta_0 - S)^2 E(\{\xi_0\}')v_k \to u . \tag{4.4.4}$$

But by 4.3 $E(\{\xi_0\}')v_k \in [v_k] \subseteq [u]$; therefore, by what has been proved thus far, $E(\{\xi_0\}')v_k$ is of the form

$$E(\{\xi_0\}')v_k = \lim (\xi_0 - S)^2 v_{kn} , \quad v_{kn} \in [u] . \tag{4.4.5}$$

From (4.4.4), (4.4.5) our lemma follows.

4.5. THEOREM. *If S is a spectral operator which satisfies the assumptions of* 4.3, *in particular if S is a spectral unitary operator in a reflexive space, then it satisfies Condition* 3.1.6.

Proof (After [5, Th. 2.11]). Since $E(\sigma) = E(\sigma \cap \Gamma_0)$, $\sigma \in \mathfrak{B}$, and since E is σ-additive in the strong operator topology, it suffices to show that $E(\sigma)x \in [x]$ for σ the closed proper subarcs of Γ_0. Let $\zeta = \zeta(\lambda, 0)$, $\xi = \zeta(\mu, 0)$, $\lambda \neq \mu$, be the ends of the arc

$$[\zeta, \xi] = \{\zeta(\alpha, 0) \mid \lambda \leq \alpha \leq \mu \text{ if } \lambda < \mu;\ \alpha \notin (\mu, \lambda) \text{ if } \mu < \lambda\} .$$

We show that $E([\zeta, \xi])x \in [x]$ (the case $\lambda = \mu$ cared for by 4.3). Since $I = E(\{\zeta\}) + E(\{\xi\}) + E(\{\zeta\}')E(\{\xi\}')$, we have

$$E([\zeta, \xi])x = E(\{\zeta\})x + E(\{\xi\})x + E([\zeta, \xi])E(\{\zeta\}')E(\{\xi\}')x .$$

By 4.3 we have to show that $E(\zeta, \xi])u \in [x]$, where $u = E(\{\zeta\}')E(\{\xi\}')x$. But by 4.4 there exists a sequence $z_k \in [x]$ such that

$$E([\zeta, \xi])u = \lim_k E([\zeta, \xi])(S - \zeta)^2(S - \xi)^2 z_k .$$

Thus we have only to show that $z \in [x]$ implies

$$E([\zeta, \xi])(S - \zeta)^2(S - \xi)^2 z \in [x] .$$

Let $\zeta_n = \zeta(\lambda_n, 0)$, $\xi_n = (\mu_n, 0)$, where the sequences $\lambda_n \to \lambda$, $\mu_n \to \mu$ are so chosen that if $\lambda < \mu$ then $\lambda_n < \lambda < \mu < \mu_n$, while if $\mu < \lambda$ then $\mu < \mu_n < \lambda_n < \lambda$. It is shown during the proof of [5, Th. 2.4] that, since S has 1 as an index function, $(S - \zeta)^2(S - \xi)^2$ is of the form

$$(S - \zeta)^2(S - \xi)^2 = \lim_n (I(\lambda, \mu) + I(\mu_n, \lambda_n)) , \tag{4.5.1}$$

where $I(\alpha, \beta)$, $-1 \leq \alpha, \beta \leq 1$, $\alpha \neq \beta$, are certain operators, the manner of definition of which is explained in [5, Lemma 2.4], which enjoy the properties:

(4.5.2) $I(\alpha, \beta)[x] \subseteq [x]$ ($I(\alpha, \beta)$ being a line integral of $S(\zeta)$).

(4.5.3) $\sigma(I(\alpha, \beta)y) \subseteq [\zeta(\alpha, 0), \zeta(\beta, 0)]$, $y \in \mathfrak{X}$ [5, Lemma 2.4].

Let $z \in [x]$. Then, by (4.5.1),

$$\begin{aligned} E([\zeta, \xi])(S - \zeta)^2(S - \xi)^2 z \\ = \lim_n (E([\zeta, \xi]I(\lambda, \mu)z + E([\zeta, \xi])I(\mu_n, \lambda_n)z) . \end{aligned} \tag{4.5.4}$$

But by (4.5.3) $\sigma(I(\lambda, \mu)z) \subseteq [\zeta, \xi]$, $\sigma(I(\mu_n, \lambda_n)z) \subseteq [\xi_n, \zeta_n]$ and hence by 1.2 $E([\zeta, \xi])I(\lambda, \mu)z = I(\lambda, \mu)z$ and

$$\begin{aligned} E([\zeta, \xi])I(\mu_n, \lambda_n)z &= E([\zeta, \xi])E([\xi_n, \zeta_n])I(\mu_n, \lambda_n)z \\ &= E(\phi)I(\mu_n, \lambda_n)z = 0 . \end{aligned}$$

Thus (4.5.4) takes the form

$$E([\zeta, \xi[)(S - \zeta)^2(S - \xi)^2 z = I(\lambda, \mu)z ,$$

and we may conclude $E([\zeta, \xi])(S - \zeta)^2(S - \xi)^2 z \in [x]$ from (4.5.2).

Generalizing the Hilbert space terminology, a two sided sequence of vectors $\{x_n\}_{n=-\infty}^{\infty}$ is called stationary if and only if the norm of any finite linear combination $\sum_{j=1}^{k}\alpha_j x_{j+h}$ is independent of h. If U is a unitary operator in $\mathfrak{X}$ and $x \in \mathfrak{X}$, then the sequence $\{U^n x\}_{n=-\infty}^{\infty}$ is stationary. Conversely, if $\{x_n\}_{n=-\infty}^{\infty}$ is stationary and $\mathfrak{Y}$ is the subspace spanned by this sequence, there exists a unique operator $U \in B(\mathfrak{Y})$ which satisfies $Ux_n = x_{n+1}$, an integer. U is unitary in $\mathfrak{Y}$ and is termed the shift operator of $\{x_n\}$. We call a stationary sequence spectral in case its shift operator is spectral.

The final statement of the following theorem replaces the problem of characterization of reflexive spaces every unitary operator of which is spectral by that of characterizing spectral stationary sequences. This "local" form of the problem seems more appropriate since the spectrality of every unitary operator in a space $\mathfrak{X}$ may depend not on "regular" properties of $\mathfrak{X}$ but on an irregularity which renders the class of unitary operators very sparse.

4.6. THEOREM. *Let U be a unitary operator in $\mathfrak{X}$. Then conditions* 3.1.1, 3.1.2 *and* 3.1.3 *are necessary in order that U be spectral. If $\mathfrak{X}$ is reflexive, then each of the conditions* 3.1.1 *to* 3.1.5 *is necessary and sufficient; and it is sufficient to let f in these conditions range over polynomials. For a reflexive $\mathfrak{X}$, U is spectral if and only if every stationary sequence it generates is spectral.*

Proof. The first statement follows from 4.2. and 3.1. It follows from 4.5 and from the fact that $\sigma(U)$, being a subset of the unit circle, s an R-set that if $\mathfrak{X}$ is reflexive, all the parts of Theorem 3.1 are applicable. Let $g \in R(\sigma(U))$. Using Cauchy's integral formula, it may be proved that there exists an admissible domain τ (in the sense of [4, Def. 2.2]) which contains $\sigma(U)$, such that g is uniformly approximable on τ by functions of the form

$$h(\zeta) = \sum_{j=1}^{k} \frac{1}{\zeta - \lambda_j}\,, \qquad \lambda_j \in \rho(U)$$

(τ may depend on g, but not on the approximants. Cf. [1, p. 398]). Since $\sigma(U)$ is contained in the unit circle, we may assume, diminishing τ if necessary, that the complement of $\bar{\tau}$ is either connected or consists of two components at most, one of which contains the point $\zeta = 0$. In either case it follows from [13, p. 47, Th. 15] that the functions h, and hence g, are uniformly approximable on τ by polynomials f in ζ and ζ^{-1}. Thus these polynomials form a dense subalgebra of $R(\sigma(U))$, and by the continuity of the functional calculus, the corresponding $f(U)$'s are dense in $\{g(U) \mid g \in R(\sigma(U))\}$ in the uniform operator topology. From the proof of 3.1 it is seen that we may replace $R(\sigma(U)$ and $R(\sigma(U \mid [x]))$

by any subalgebra of $R(\sigma(U))$ with these properties. Since U is unitary, the conditions of 3.1 remain invariant if the involved functions are multiplied by ζ^k, k an integer. Therefore polynomials in ζ will do. Finally it follows from what has been shown above that the subspace spanned by a stationary sequence $\{U^n x\}_{n=-\infty}^{\infty}$ is $[x]$. Thus the final statement follows from the fact that 3.1.4 is the same as 3.1.1 for the shift operator.

5. **Examples of non spectral unitary operators.** Let Ω be a compact Hausdorff space. The unitary operators in $C(\Omega)$ are the operators of the form $(Ux)(\omega) = \mu(\omega)x(h(\omega))$, $\omega \in \Omega$, where h is a homeomorphism of Ω on itself, $\mu \in C(\Omega)$ and $|\mu(\omega)| \equiv 1$. This is proved in [12, pp. 469–472] for the real case, but the proof can be modified to apply to the complex case too by the use of an argument of Arens in a similar situation (v. [9, p. 88]). The following theorem treats only the case that h is non-periodic; for the case that h is the identity mapping Cf. Example 2.2 above.

5.1. THEOREM. *Let Ω be a compact Hausdorff space, and let U of the form $(Ux)(\omega) = \mu(\omega)x(h(\omega))$ (h, μ as above) be a unitary operator in $C(\Omega)$. If h is non-periodic, then U is not spectral.*

Proof. By 4.2, 3.1 and the fact that $\sigma(U)$ is contained in the unit circle (actually, coincides with it), it is sufficient to show that there exists no finite constants H such that

(5.1.1) $\quad ||f(U))|| \leq H \max_{|\zeta|=1} |f(\zeta)|$, f a polynomial in ζ.

Let us calculate $||f(U)||$. If $f(\zeta) = \sum_{k=0}^{n} \alpha_k \zeta^k$, then

$$(f(U)x)(\omega) = \sum_{k=0}^{n} \alpha_k \mu(\omega)^k x(h^{0k}(\omega)) \,, \tag{5.1.2}$$

where h^{0k} denotes the kth iterate by substitution of $h(h^{00}(\omega) \equiv \omega)$. By hypothesis there exists an $\omega_0 \in \Omega$ such that the points $h^{0k}(\omega_0)$, $k = 0, 1, \cdots, n$ are distinct. Since Ω is Hausdorff, there exist pairwise disjoint open sets π_k, $k = 0, 1, \cdots, n$ such that $h^{0k}(\omega_0) \in \pi_k$. Since a compact Hausdorff space is normal, it follows by Urysohn's lemma that there exist functions $y_k \in C(\Omega)$ such that $y_k(h^{0k}(\omega_0)) = 1$, $y_k(\omega) = 0$ for $\omega \in \pi_k'$ and $0 \leq y_k(\omega) \leq 1$ on Ω. We define $x_0 \in C(\Omega)$ by

$$x_0(\omega) = \sum_{k=0}^{n} \overline{\operatorname{sgn}\,(\alpha_k \mu(\omega_0)^k)} y_k(\omega) \,.$$

Substitution in (5.1.2) gives

$$(f(U)x_0)(\omega_0) = \sum_{k=0}^{n} |\alpha_k \mu(\omega_0)^k| = \sum_{k=0}^{n} |\alpha_k| \,.$$

Since $\|x_0\| = 1$, $\|f(U)\| \geq \sum_{k=0}^{n} |\alpha_k|$ (actually $\|f(U)\| = \sum_{k=0}^{n} |\alpha_k|$). The necessary condition (5.1.1) now takes the form: there exists an $H < \infty$ such that for every polynomial $f(\zeta) = \sum_{k=0}^{n} \alpha_k \zeta^k$,

$$\sum_{k=0}^{n} |\alpha_k| \leq H \max_{|\zeta|=1} |f(\zeta)| .$$

To contradict this statement we use the following example of Hardy [8, § 14]. The series

$$\sum_{k=2}^{\infty} \frac{(-1)^k \binom{-i}{k}}{\log k} \zeta^k$$

converges uniformly for $|\zeta| = 1$, while the sum of the absolute values of its coefficients diverges. Therefore the polynomials which form its partial sums furnish us with the required counter example.

5.2. THEOREM. *In each of the sequence spaces l_p, $1 \leq p \leq \infty$, $p \neq 2$, there exists a non spectral unitary operator.*

Proof. If U is a unitary spectral operator in $\mathfrak{X}$, then necessarily [5, Assumption 1.14]:

(5.2.1) $M(U) = \sup \{\|x\| \mid x, y \in \mathfrak{X}, \|x + y\| = 1, \sigma(x) \cap \sigma(y) = \phi\} < \infty$.

This follows from the boundness of E by 1.2. Even if U is not spectral, the conclusion of 1.1 holds because $\sigma(U)$ is nowhere dense; and thus $\sigma(x)$ and $M(U)$ are definable. We show that in each of the considered spaces there exists a unitary operator U with $M(U) = \infty$.

Let p, $1 \leq p \leq \infty$, $p \neq 2$, be given. We denote by $\mathfrak{X}_j$ a space of the type $l_{p,n}$ or l_p (the last possibility is needed only for the remarks made after the theorem). If $\{\mathfrak{X}_j\}_{j=1}^{\infty}$ is a sequence of such spaces, we denote by $\sum_{j=1}^{\infty} \oplus \mathfrak{X}_j$ the Banach space of all sequences $\{x_j\}$ with $x_j \in \mathfrak{X}_j$ and

$$\|\{x_j\}\| = \left(\sum_{j=1}^{\infty} \|x_j\|^p\right)^{1/p} < \infty \quad (\text{if } p = \infty,\ \|\{x_j\}\| = \sup \|x_j\| < \infty).$$

If for each j, $T_j \in B(\mathfrak{X}_j)$, we denote by $\sum_{j=1}^{\infty} \oplus T_j$ the transformation T defined on (part of) $\sum_{j=1}^{\infty} \oplus \mathfrak{X}_j$ by $T\{x_j\} = \{T_j x_j\}$

5.3. LEMMA. *If $\mathfrak{X} = \sum_{j=1}^{\infty} \oplus \mathfrak{X}_j$ and for each j U_j is a unitary operator in $\mathfrak{X}_j$, then $U = \sum_{j=1}^{\infty} \oplus U_j$ is a unitary operator in $\mathfrak{X}$ and $M(U) \geq \sup_j M(U_j)$.*

Proof. That U is unitary is obvious. If $x_j \in \mathfrak{X}_j$ for a definite j, we denote by x_j^* the vector $\{y_k\} \in \mathfrak{X}$ defined by $y_j = x_j$, $y_k = 0$ for $k \neq j$.

Since the operation * is linear and norm preserving, $(\zeta - U_j)x_j(\zeta) = x_j$ for $\zeta \in \rho U_j(x_j)$ implies $(\zeta - U)x_j(\zeta)^* = x_j^*$ where $x_j(\zeta)^*$ is analytic on $\rho_{U_j}(x_j)$. Therefore $\sigma_U(x_j^*) \subseteq \sigma_{U_j}(x_j)$. It is obvious how to complete the proof.

Since l_p is linearly isometric to $\sum_{j=1}^{\infty} \oplus l_{p,n_j}$ where n_j are arbitrary natural numbers, 5.3 shows that we have only to find indices n_j and unitary operators U_j in l_{p,n_j} such that $\sup_j M(U_j) = \infty$. Let φ_j, $j = 1, \cdots, n$, be the natural basis of $l_{p,n}$. Henceforth U_n will denote the unitary operator in $l_{p,n}$ determined by the requirements $U_n\varphi_j = \varphi_{j+1(\text{mod}\, n)}$. The following lemmas will show that $\sup_n M(U_n) = \infty$, which will finish the proof.

We now use tensorial products as in [10]. If $x = (x_1, \cdots, x_n) \in l_{p,n}$, $y = (y_1, \cdots, y_m) \in l_{p,m}$, we define $x \otimes y$ to be the vector $(x_1y_1, x_1y_2, \cdots, x_1y_m, x_2y_1, x_2y_2, \cdots, x_2y_m, \cdots, x_ny_1, x_ny_2, \cdots, x_ny_m)$ of $l_{p,nm}$. This is a Kronecker product [7, p. 208], and the norm is a cross norm with respect to it, that is $\| x \otimes y \| = \| x \| \, \| y \|$. The tensorial product of linear operators, T in $l_{p,n}$ and S in $l_{p,m}$, is uniquely defined by the requirements $(T \otimes S)(x \otimes y) = Tx \otimes Sy$.

5.4. LEMMA. *If T, S are linear operators in $l_{p,n}$, $l_{p,m}$ respectively, then* $\sigma_{T\otimes S}(x \otimes y) = \{\eta\theta \mid \eta \in \sigma_T(x),\ \theta \in \sigma_S(y)\}$.

Proof. If T is an operator in a finite dimensional space and f is the minimum polynomial of x with respect to T, then $\sigma_T(x)$ is the set of zeros of f (cf. [5, p. 589]). We may assume that neither $\sigma_T(x)$ nor $\sigma_S(y)$ is empty since this case is trivial. In case $\sigma_T(x) = \{\eta\}$, $\sigma_S(y) = \{\theta\}$ the minimum polynomials are of the respective forms $(\zeta - \eta)^t$, $(\zeta - \theta)^s$ $(t, s \geq 1)$. By induction on t and s and use of the identity

$$(T \otimes S - \eta\theta)(x \otimes y) = (T - \eta) \otimes Sy + \eta x \otimes (S - \theta)y \ ,$$

one shows that the minimum polynomial of $x \otimes y$ with respect to $T \otimes S$, is of the form $(\zeta - \eta\theta)^r$, $r \geq 1$, and therefore $\sigma_{T\otimes S}(x \otimes y) = \{\eta\theta\}$ (actually we need only the case $t = s = 1$). In the general case $\sigma_T(x) = \{\eta_1, \cdots, \eta_a\}$, $\sigma_S(y) = \{\theta_1, \cdots, \theta_b\}$ we have by the finite dimensional case of the spectral theorem ([4, § 1] or [7, p. 132]) the resolutions $x = \sum_{i=1}^{a} x_i$, $y = \sum_{j=1}^{b} y_j$, where $\sigma_T(x_i) = \{\eta_i\}$, $\sigma_S(y_j) = \{\theta_j\}$. Let $\{\eta\theta \mid \eta \in \sigma_T(x), \theta \in \sigma_S(y)\} = \{\kappa_1, \cdots, \kappa_c\}$ and let z_k be the sum of the vectors $x_i \otimes y_j$ such that $\eta_i\theta_j = \kappa_k$. Since the x_i's are linearly independent and the y_j's are different from zero (by our assumption $x \neq 0$, $y \neq 0$), it follows that $z_k \neq 0$. Therefore, by the case of one point spectra, $\sigma_{T\otimes S}(z_k) = \{\kappa_k\}$. Since $x \otimes y = \sum_{k=1}^{c} z_k$ and since the minimum polynomial of a sum of vectors with minimum polynomials relatively prime in pairs is their product [7, p. 68], the statement of the lemma follows.

5.5. LEMMA. *If* $(m, n) = 1$, *then* $M(U_{nm}) \geq M(U_n)M(U_m)$.

Proof. $U_n \otimes U_m$ is determined by requirements of the form $(U_n \otimes U_m)\varphi_j = \varphi_{j\pi}$, where φ_j, $1 \leq j \leq nm$, is the natural basis of $l_{p,nm}$ and π is a permutation of the indices. Since $(m, n) = 1$, π is cyclic, and it is easily verified that there exists a unitary operator V in $l_{p,nm}$ such that $U_{nm} = V(U_n \otimes U_m)V^{-1}$, which implies that $M(U_{nm}) = M(U_n \otimes U_m)$. Since $l_{p,n}$ is of finite dimension, there exist vectors $x^{(1)}$, $y^{(1)}$ satisfying: $\sigma_{U_n}(x^{(1)}) \cap \sigma_{U_n}(y^{(1)}) = \phi$, $\| x^{(1)} + y^{(1)} \| = 1$ and $\| x^{(1)} \| = M(U_n)$. Let $x^{(2)}$, $y^{(2)}$ play a similar role with respect to U_m. Consider the vectors $x = x^{(1)} \otimes x^{(2)}$, $y = x^{(1)} \otimes y^{(2)} + y^{(1)} \otimes x^{(2)} + y^{(1)} \otimes y^{(2)}$. Since $\sigma(U_n)$ is the set of roots of unity of order n, $\sigma_{U_n}(x^{(1)})$ and $\sigma_{U_n}(y^{(1)})$ are sets of roots of unity of order n. Similarly for $\sigma_{U_m}(x^{(2)})$ and $\sigma_{U_m}(y^{(2)})$. Since $(m, n) = 1$, the representation of a root of unity order mn as a product of a root of unity of order n by one of order m is unique. Therefore it follows from 5.4 that $\sigma_{U_n \otimes U_m}(x) \cap \sigma_{U_n \otimes U_m}(y) = \phi$. By the cross property of the norm $\| x + y \| = \| (x^{(1)} + y^{(1)}) \otimes (x^{(2)} + y^{(2)}) \| = 1$ and $\| x \| = \| x^{(1)} \otimes x^{(2)} \| = M(U_n)M(U_m)$. Thus $M(U_{nm}) = M(U_n \otimes U_m) \geq M(U_n)M(U_m)$.

5.6. LEMMA. *For every given* p, $1 \leq p \leq \infty$, $p \neq 2$, *there exist an* $\eta > 1$ *and positive integers* k *and* m_0 *such that* $M(U_{km+1}) > \eta$ *for* $m \geq m_0$.

Proof. By calculating the eigenvectors of U_n, one shows that the vectors $x = (x_1, \cdots, x_n)$ with $\sigma_{U_n}(x)$ disjoint from $\sigma_{U_n}(y)$, where $y = (1, 1, \cdots, 1)$, are those which satisfy $\sum x_j = 0$. Thus

$$M(U_n) \geq \sup \left\{ \frac{\| x \|}{\| x + \alpha y \|} \middle| \sum x_j = 0,\ \alpha \text{ arbitrary} \right\} .$$

For $2 < p < \infty$ we chose $x = (1, \cdots, 1, -m/(n-m), \cdots, -m/(n-m)$, where 1 is repeated m times, and

$$\alpha = \frac{\left(\dfrac{m}{n-m}\right)^{(p-2)/(p-1)} - 1}{1 + \left(\dfrac{n-m}{m}\right)^{1/(p-1)}} .$$

Then if $n = km + 1$, $k \geq 2$ and

$$m \to \infty, \frac{\| x \|^p}{\| x + \alpha y \|^p}$$

tends to

$$\frac{(1 + t^{1/(p-1)})^p(1 + t^{1-p})}{(t^{1/(p-1)} + t^{-(p-2)/(p-1)})^p + t\left(\dfrac{t+1}{t}\right)^p}$$

where $t = k - 1$. Although the last expression tend to 1 as $t \to \infty$, it is not difficult to verify that it is greater than 1 for all sufficiently large values of t; hence a suitable integer $k = t + 1$ can be found. The case $1 < p < 2$ follows by duality: If $1/p + 1/q = 1$ then $M_q(U_n)$, where the subscript indicates that U_n is to be considered as an operator in $l_{q,n}$, is the maximum of the norms of the values of the resolution of the identity E of U_n. The resolution of the identity of $U_n^* = U_n^{-1}$ is E. Therefore $M_p(U_n^{-1}) = M_q(U_n)$. But U_n is unitarily equivalent in $l_{p,n}$ to U_n^{-1}. Therefore $M_p(U_n) = M_q(U_n)$; and since $2 < q < \infty$ the lemma is true in this case too. If $p = 1$, we may take $x = (1, \cdots, 1, -n + 1)$, $\alpha = -1$; while if $p = \infty$, we take the same x and $\alpha = n/2$.

Finally to see that 5.5 and 5.6 imply $\sup_n M(U_n) = \infty$, we have only to use the fact that each sequence $a_m = km + 1$, $m = 1, 2, \cdots$, contains an infinite subsequence of pairwise prime integers. As pointed out by Dr. Dov Jarden such a subsequence is obtained by defining inductively $m_1 = 1$, $m_{j+1} = a_{m_1}a_{m_2}\cdots a_{m_j}$.

REMARKS. For $p = 1, \infty$ the proof of 5.2 yields unitary operators which are not even prespectral. It applies also to subspaces which contain all finite sequences. It also follows from 5.2 that if Ω is a measure space which is not a finite union of atoms, then there exist non spectral unitary operators in the space $L_p(\Omega)$, $1 \leq p < \infty$, $p \neq 2$. An operator U in l_p, $1 \leq p < \infty$, $p \neq 2$, is unitary only if determined by $U\varphi_j = \lambda_j\varphi_{j\pi}$, $j = 1, 2, \cdots$, where $\{\varphi_j\}$ is the natural basis, π a permutation and $|\lambda_j| = 1$ ([2, p. 178]. The proof goes easily over to the complex case). We decompose π into disjoint cycles (including the possibility of infinite "cycles") and consider the unitary operators induced by U in the subspaces spanned by the φ_j's with j belonging to a definite cycle. One shows that $M(U) = \sup M(V)$, where V runs over the induced operators. Moreover, if we change the λ_j's into 1 and the cycle of V into a standard one, we obtain an operator W with $M(W) = M(V)$. Hence Condition (5.2.1) depends only on the length of the cycles determined by π. From Theorem 5.7 it will follow that if in particular at least one of these cycles is infinite, (5.2.1) does not hold. On the other hand, it follows from [5, Th. 3.11, Th. 3.12 (III)] that this condition is sufficient for spectrality of U if $1 < p < \infty$.

5.7. THEOREM. *Let $\bar{l}_p$, $1 \leq p \leq \infty$, $p \neq 2$, be the space of two-sided sequences $\{\alpha_j\}_{j=-\infty}^{\infty}$ with the obvious norm. Let φ_j, $-\infty < j < \infty$, be the natural basis of $\bar{l}_p$ and U the unitary operator defined by $U\varphi_j = \varphi_{j+1}$. Then U is not spectral.*

Proof. To facilitate the writing we assume $p < \infty$. From the proof of 4.6 through [6, Th. 18 (IV)] (cf. 3.1), it follows that if $H(U_n)$ is the infimum of possible constants in Condition 3.1.1 for polynomials, then $M(U_n) \leq H(U_n)$. Let K be a positive number. Then by the proof of 5.2 there exists an n such that $M(U_n) > 2K$, and therefore there exists a polynomial g such that $\| g(U_n \| > 2K \max_{\zeta^n=1} | g(\zeta) |$. If f is a polynomial, $f(U_n)$ depends only on the values f assumes at the nth roots of unity, and in a continuous manner. Therefore, by the approximation theorem of Weierstrass, there exists a polynomial $f(\zeta) = \sum_{k=0}^{s} \beta_k \zeta^k$ such that $\| f(U_n) \| > 2K \max_{\zeta^n=1} | f(\zeta) |$ and $2 \max_{\zeta^n=1} | f(\zeta) | > \max_{|\zeta|=1} | f(\zeta) |$; hence $\| f(U_n) \| > K \max_{|\zeta|=1} | f(\zeta) |$. Identifying $l_{p,n}$ with the subspace of $\bar{l}_p$ spanned by $\varphi_1, \cdots, \varphi_n$, we see that there exists an $x = \sum_{j=1}^{n} \alpha_j \varphi_j$ such that

$$\frac{\| f(U_n)x \|}{\| x \|} > K \max_{|x|=1} | f(\zeta) | . \tag{5.7.1}$$

It will simplify the notation if we assume, as we may, that the formal degree s of f is of the form $s = rn$, $r > 1$. Let t be a positive integer and consider the vector $x' = \sum_{m=1}^{r+t} \sum_{j=1}^{n} \alpha_j \varphi_{(m-1)n+j}$. Then

$$\begin{aligned} f(U)x' &= \sum_{k=0}^{s} \sum_{m=1}^{r+t} \sum_{j=1}^{n} \beta_k \alpha_j \varphi_{(m+1)n+j+k} \\ &= \sum_{u=1}^{2r+t} \sum_{v=1}^{n} (\sum \beta_k \alpha_j) \varphi_{(u-1)n+v} \end{aligned} \tag{5.7.2}$$

where the inner summation in the r.h.s. extends over the pairs j, k satisfying $(m-1)n + j + k = (u-1)n + v$, where $1 \leq j \leq n$, $0 \leq k \leq s = rn$ and $1 \leq m \leq r + t$. On the other hand

$$f(U_n)x = \sum_{k=0}^{s} \sum_{j=1}^{n} \beta_k \alpha_j \varphi_{j+k (\mathrm{mod}\, n)} = \sum_{v=1}^{n} (\sum \beta_k \alpha_j) \varphi_v , \tag{5.7.3}$$

where here the inner summation is over the pairs j, k satisfying $j + k = v (\mathrm{mod}\, n)$ with the same inequalities. For $r + 1 \leq u \leq r + t$, the coefficient of $\varphi_{(u-1)n+v}$ in (5.7.2) equals the coefficient of φ_v in (5.7.3). Therefore

$$\| f(U)x' \| \geq t^{1/p} \| f(U_n)x \| , \tag{5.7.4}$$

and on the other hand

$$\| x' \| = (r + t)^{1/p} \| x \| . \tag{5.7.5}$$

(5.7.1), (5.7.4) and (5.7.5) imply

$$\| f(U) \| > \left(\frac{t}{r+t} \right)^{1/p} K \max_{|\zeta|=1} | f(\zeta) | .$$

Letting t tend to infinity, we get $\| f(U) \| \geq K \max_{|\zeta|=1} | f(\zeta) |$. Since K is arbitrary and $\sigma(U)$ is contained in the unit circle, this shows that U does not satisfy Condition 3.1.1; hence it is not spectral.

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