AN INVERSION THEOREM FOR LAPLACE-STIELTJES TRANSFORMS

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E. Phragmén [2; p. 360] showed that under certain assumptions of boundedness for $F(x)$,

$$\lim_{s\to+\infty}\int_0^t F(\tau)[1-\exp(-e^{(t-\tau)s})]\,d\tau=\int_0^t F(\tau)d\tau\ .$$

If we write $1-\exp(-e^{s(t-\tau)})=\sum_1^\infty(-1)^{n+1}e^{nx(t-\tau)}/n!$ in the above formula, and interchange sum and integral, we formally obtain

$$\lim_{s\to\infty}\sum_1^\infty\frac{(-1)^{n+1}}{n!}e^{nst}\int_0^t e^{-ns\tau}F(\tau)d\tau=\int_0^t F(\tau)d\tau\ .$$

G. Doetsch [1; pp. 286–288] showed that for reals s, if $f(s)=\int_0^\infty e^{-s\tau}F(\tau)d\tau$ converges absolutely in some half-plane, then

$$\int_0^t F(\tau)d\tau=\lim_{s\to+\infty}\sum_1^\infty\frac{(-1)^{n+1}}{n!}f(ns)e^{nst}\ \text{for}\ t>0\ .$$

This paper will generalize this result to Laplace-Stieltjes transforms

$$f(s)=\int_0^\infty e^{-st}\,d\alpha(t) \tag{I}$$

and will eliminate the assumption of absolute convergence. Unless specifically written otherwise, all integrals will be evaluated from 0 to $+\infty$ and all summations from 1 to ∞. We shall need the following two propositions [3; pp 39,41]:

LEMMA 1. *If the integral*

$$f(s_0)=\int e^{-s_0t}\,d\alpha(t)$$

converges with $Rs_0>0$, *then*

$$f(s_0)=s_0\int e^{-s_0t}\,\alpha(t)dt-\alpha(0)$$

and $\int e^{-s_0t}\,\alpha(t)dt$ *converges absolutely if* s_0 *is replaced by any number with larger real part.*

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Lemma I remains valid for $Rs_0 < 0$ if we insist that $\alpha(\infty) = 0$. In this paper we shall make the following

Assumption. In (I), s is real and positive, and $\alpha(t)$ is of bounded variation in $(0, R)$, for every $R > 0$.

LEMMA 2. *If the integral*

$$\int_0^\infty e^{-s\tau}\, d\alpha(\tau)$$

converges for $s = s_0$ and if the real part γ of s_0 is positive, then $\alpha(\tau) = 0(e^{\gamma\tau})$ as $\tau \to \infty$.

We shall now prove some useful lemmas.

LEMMA 3. *If* (I) *converges in some half plane Γ, then*

$$\text{(a)} \qquad \lim_{s\to\infty}\left|\int_\sigma^\infty [1 - \exp(-e^{-s\tau})]d\alpha(\tau)\right| = 0 \text{ for fixed } \sigma > 0 \,,$$

$$\text{(b)} \qquad \lim_{\sigma\to\infty}\left|\int_\sigma^\infty [1 - \exp(-e^{-s\tau})]d\alpha(\tau)\right| = 0 \text{ for fixed } s > 0 \,.$$

Proof. Since $1 - \exp\,(-e^{-s\tau}) = O(e^{-s\tau})$ for $s,\tau \geq 0$, a standard argument involving integration by parts shows that

$$\int_\sigma^8 [1 - \exp(-e^{-s\tau})]d\alpha(\tau) = O\{e^{-s\sigma}\,[\,|\alpha(\sigma)| + s]\}$$

for $s \in \Gamma$ and $\sigma \geq 0$. The desired result now follows from Lemmas 1 and 2.

LEMMA 4. *If* (I) *converges in some half-plane Γ, then for $s \in \Gamma'$ where Γ' is a half-plane properly contained in Γ,*

$$\sum \frac{(-1)^{n+1}}{n!}\int e^{ns(t+\tau)}\, d\alpha(\tau) = \int d\alpha(\tau) \sum \frac{(-1)^{n+1}}{n!}\, e^{ns}(t-\tau) \,.$$

Proof. Upon integration by parts, application of Lemma 2, and some algebra, the desired equality takes the form

$$\sum_1^\infty \frac{(-1)^{n+1}}{(n-1)!}\int e^{ns(t-\tau)}\, \alpha(\tau)d\tau = \int \sum_1^\infty \frac{(-1)^{n+1}}{(n-1)!}\, e^{ns(t-\tau)}\, \alpha(\tau)d\tau \,.$$

To verify this latter equality, it suffices to show that

$$\sum_0^\infty \frac{1}{(n-1)!}\int e^{ns(t-\tau)}\, |\alpha(\tau)|d\tau < \infty \,,$$

but this follows from Lemma 1.

LEMMA 5. *If* (I) *converges in some half-plane Γ, then*

$$\alpha(t) - \alpha(0) = \lim_{s \to +\infty} \sum \frac{(-1)^{n+1}}{n!} f(ns)e^{nst}$$

for all non-negative t which are points of continuity of $\alpha(t)$.

Proof. We have

$$\sum \frac{(-1)^{n+1}}{n!} f(ns)e^{nst} = \int d\alpha(\tau) \sum \frac{(-1)^{n+1}}{n!} e^{ns(t-\tau)}$$

$$= \int [1 - \exp(-e^{s(t-\tau)})] d\alpha(\tau) ,$$

the interchange in the order of summation and integration being justified by Lemma 4. For $t = 0$ ($t > 0$) and a point of continuity of $\alpha(t)$, write the integral on the right as

$$\int_0^\delta + \int_\delta^\infty \left(\text{or, for } t > 0, \int_0^{t-\delta} + \int_{t-\delta}^{t-\delta} + \int_{t+\delta}^\infty \right) ,$$

with $0 < \delta < t$ chosen that the total variation of α on $[0, \delta]$ (respectively. $[t - \delta, t + \delta]$) is less than ε, and apply Lemma 3 to $\int_\delta^\infty \left(\text{resp.}, \int_{t+\delta}^\infty\right)$. We see that $\int_0^\delta \left(\text{resp.}, \int_{t-\delta}^{t+\delta}\right)$ is less than ε for all $s \geq 0$. (For $t > 0$, $\int_0^{t-\delta} = \alpha(t - \delta) - \alpha(0) - \int_0^{t-\delta} \exp[-e^{s(t-\tau)}] d\alpha(\tau)$, and this clearly tends to $\alpha(t - \delta) - \alpha(0)$ as $s \to \infty$. Thus the integral $\int_0^\infty$ is $\alpha(t) - \alpha(0) + o(1)$ as $s \to \infty$).

We can now prove our main result.

THEOREM *If* $\alpha(t) = [\alpha(t^+) + \alpha(t^-)]/2$ *for* $t > 0$ *and* (I) *converges for some* $s > 0$, *then*

$$\lim_{s \to \infty} \sum \frac{(-1)^{n+1}}{n!} f(ns)e^{nst} = \begin{cases} [\alpha(0^+) - \alpha(0)]\,(1 - e^{-1}), & t = 0 \\ \alpha(t) - \alpha(0) - [\alpha(t^+) - \alpha(t^-)]\,(e^{-1} - 1/2) , & t > 0 \end{cases}$$

Proof. Define

$$\beta(\tau) = \begin{cases} \alpha(\tau) - [\alpha(0^+) - \alpha(0)] \text{ sign } \tau, & \tau > 0 \\ \alpha(0) & , \tau = 0 \end{cases}$$

for $t = 0$, and

$$\beta(\tau) = \begin{cases} \alpha(\tau) - [\alpha(t^+) - \alpha(t^-)] \text{ sign } (\tau - t), & \tau > 0 \\ \alpha(t) & , \tau = 0 \end{cases}$$

for $t > 0$, β is then continuous at t, and

$$F(s) = \int_0^\infty e^{-s\tau}\, d\beta(\tau) = \begin{cases} f(s) - \alpha(0^+) + \alpha(0) & , t = 0 \\ f(s) - [\alpha(t^+) - \alpha(t^-)]e^{-st}, & t > 0 \,. \end{cases}$$

Now apply Lemma 5 with β and F substituted for α and f, respectively.

BIBLIOGRAPHY

1. G. Doetsch, *Handbuch der Laplace Trasformation* vol. **1**, Birkhäuser Basel, 1950.

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3. D. V. Widder, *The Laplace Transform*, Princeton University Press, 1946.

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