LOOPS WITH THE WEAK INVERSE PROPERTY

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Let the left and right inverses of an element x of a loop G be denoted by x^λ and x^ρ respectively, then G is said to have the inverse property if the two identities $x^\lambda(xy) = y$ and $(yx)x^\rho = y$ are satisfied by all elements x, y of G. Perhaps the two most basic properties of inverse property loops are that (i) the left, middle and right nuclei coincide, and that (ii) if every loop isotopic to G has the inverse property, then G is a Moufang loop[1]. More recently, R. Artzy has defined cross inverse property loops (G has the cross inverse property if any two elements x and y of G satisfy either of the two equivalent identities $x^\lambda(yx)=y$ and $(xy)x^\rho = y$), and has shown that the same two properties hold for these loops[2]. In the present paper, we shall consider (i) and (ii) for a class of loops which includes both of the classes already mentioned. In § 1 we introduce the weak inverse property and prove (i) for loops with this property. In § 2 and § 3 we discuss loops all of whose isotopes have the weak inverse property, and show that those loops are not necessarily Moufang loops but come very close (see Theorems 2 and 3). An interesting by-product of this investigation is the construction in § 3 of a class of loops, each of which is isomorphic to all its isotopes. The only previously known examples of such loops have been Moufang loops[3].

In dealing with isotopy and cross inverse property loops, Artzy does not discuss the question of whether a cross inverse property loop can arise as an isotope of an inverse property loop. In § 4 we answer this question in the negative.

1. Let G be a loop with identity element 1, then G will be said to satisfy the *weak inverse property*[4] if whenever three elements x, y, z of G satisfy the relation $xy \cdot z = 1$, they also satisfy the relation $x \cdot yz = 1$. Using the right inverse operator ρ, we may transform this definition into more usable form by observing that the relation $xy \cdot z = 1$ is equivalent to $z = (xy)^\rho$, and by substituting this into $x \cdot yz = 1$ to yield

$$(1) \qquad y \cdot (xy)^\rho = x^\rho .$$

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1. These concepts will all be defined in the body of the paper. A proof of these properties and those used in § 4 for inverse property loops and Moufang loops will be found in [3]. Note that Bruck uses the term "associator" instead of "nucleus" in his earlier papers.

2. See [1].

3. See [5].

4. Loops with this property have previously been considered in connection with nets. For a brief discussion of this and references, see [2].

Iterating this relation, we readily obtain

$$(2) \qquad (xy)^{\rho}\cdot x^{\rho^2} = y^{\rho} ,$$

and iterating again gives $x^{\rho^2}\cdot y^{\rho^2} = (xy)^{\rho^2}$, showing that ρ^2 is an automorphism of G. Since λ is the inverse of the operator ρ, λ^2 is also an automorphism, and applying it to (2) gives

$$(3) \qquad (xy)^{\lambda}\cdot x = y^{\lambda} ,$$

which is the dual of (1). From (3) it is easy to see that $x\cdot yz = 1$ implies $xy\cdot z = 1$, so that we could have equivalently defined G to have the weak inverse property if $xy\cdot z = 1$ whenever $x\cdot yz = 1$. It might also be remarked that if ρ is an anti-automorphism or automophism in a weak inverse property loop G, then equations (1) and (3) tell us that G has the inverse property or the cross inverse property respectively. Conversely, either of the latter two properties imply the weak inverse property.

Letting $R(y)$ and $L(y)$ denote right and left multiplication by the element y, we may rewrite (1) in the form $R(y)\rho L(y) = \rho$, which yields the two useful relations

$$(4) \qquad R^{-1}(y) = \rho L(y)\lambda , \text{ and } L^{-1}(y) = \lambda R(y)\rho .$$

To develop the properties of weak inverse property loops further, we shall need to introduce the concepts of isotopism and autotopism. Let G_0 be a loop consisting of the elements of G under a new binary operation "$\circ$" (the old operation shall be denoted by "$\cdot$"), and let U, V, W be three permutations on the elements of G satisfying the relation $xU\cdot yV = (x\circ y)W$ for all x, y of G. Then we shall say that G_0 is *isotopic* to G (or, equivalently, that it is an *isotope* of G) by means of the *isotopism* (U, V, W). In case "$\circ$" is just the original binary operation "$\cdot$", we shall call (U, V, W) an *autotopism*. Observe that if T is an automorphism of G, then it gives rise to the autotopism (T, T, T), and conversely. It is well known[5] that the set of isotopisms of G form a group under the operation $(U_1, V_1, W_1)(U_2, V_2, W_2) = (U_1U_2, V_1V_2, W_1W_2)$, and that the autotopisms form a subgroup.

LEMMA 1. *If (U, V, W) is an autotopism of a weak inverse property loop, then so are $(V, \lambda W\rho, \lambda U\rho)$ and $(\rho W\lambda, U, \rho V\lambda)$.*

Using (1) on the relation $xU\cdot yV = (xy)W$, we obtain $yV\cdot[(xy)W]^{\rho} = [xU]^{\rho}$. And making the substitution $x = (yz)^{\lambda}$ in this equation yields $yV\cdot[(z^{\lambda})W]_{\rho} = [(yz)^{\lambda}U]^{\rho}$, which tells us that $(V, \lambda W\rho, \lambda U\rho)$ is an autotopism. The other autotopism of the lemma arises in the same way using (3).

Next, we define the *left nucleus* of G to be the set of all elements

5. See [**3**], for example.

a of G satisfying the relation $ax \cdot y = a \cdot xy$ for every pair of elements x, y of G. We may equivalently characterize the left nucleus as the set of all a such that $(L(a), 1, L(a))$ is an autotopism of G. Similarly the dual concept of *right nucleus* may be characterized as the set of all a such that $(1, R(a), R(a))$ is an autotopism. If we now assume that a is in the right nucleus, then Lemma 1 tells us that $(R(a), \lambda R(a)\rho, 1)$ and $(\rho R(a)\lambda, 1, \rho R(a)\lambda)$ are autotopisms. From the latter it is clear that $(\lambda^2, \lambda^2, \lambda^2)(\rho R(a)\lambda, 1, \rho R(a)\lambda)^{-1}(\rho^2, \rho^2, \rho^2) = (L(a), 1, L(a))$ is an autotopism, so that a is in the left nucleus of G. On the other hand, from the former we get the equation $xa \cdot [(z^\lambda)a]^\rho = xz$, or $xa \cdot zL_a^{-1} = xz$ for all x, z of G. Setting $z = ay$ gives $xa \cdot y = x \cdot ay$ for all x, y of G, which is the definition of the element a being in the *middle nucleus*. Since all these steps are reversible, we have proved:

THEOREM 1. *The left, middle, and right nuclei of a weak inverse property loop coincide.*

2. We now turn to the question of when an isotope of G also has the weak inverse property. First of all, if G_0 is an isotope of an arbitrary loop G, then it is well known[5] that, up to isomorphism, "$\circ$" is given in terms of "$\cdot$" by the relation $x \circ y = xR^{-1}(g) \cdot yL^{-}({}^1f)$, for some fixed pair of elements f and g of G. If ρ_0 is the right inverse operator of G_0, then the weak inverse property in G_0 is equivalent to the identity $y \circ (x \circ y)^{\rho_0} = x^{\rho_0}$, or $yR^{-1}(g) \cdot [xR^{-1}(g) \cdot yL^{-1}(f)]^{\rho_0} L^{-1}(f) = x^{\rho_0}$. Setting $x = ug$ and $y = fv$, this becomes $(fv)R^{-1}(g) \cdot (uv)^{\rho_0} L^{-1}(f) = (ug)^{\rho_0}$, and using the weak inverse property in G yields

$$(ug)^{\rho_0 \lambda} \cdot (fv)R^{-1}(g) = [(uv)^{\rho_0} L^{-1}(f)]^\lambda \ . \tag{5}$$

Since fg is the unit of G_0 (as may be verified from its definition), the mapping ρ_0 is defined by the relation $fg = x \circ x^{\rho_0} = xR^{-1}(g) \cdot (x^{\rho_0})L^{-1}(f)$. Using (3), we may put this in the form $[(x^{\rho_0})L^{-1}(f)]^\lambda = (fg)^\lambda \cdot xR^{-1}(g)$, which leads to the formula $\rho_0 = R^{-1}(g)L((fg)^\lambda)\rho L(f)$. But (5) says precisely that $(R(g)\rho_0\lambda, L(f)R^{-1}(g), \rho_0 L^{-1}(f)\lambda)$ is an autotopism, which may now be rewritten as

$$(L([fg]^\lambda)R^{-1}(f), L(f)R^{-1}(g), R^{-1}(g)L([fg]^\lambda)) \tag{6}$$

after substituting for ρ_0 and using the first relation of (4).

Next, if (U, V, W) is an autotopism of G, and if f and g are the images of the identity under U and V respectively, then we may obtain the relations $UR(g) = W$, $VL(f) = W$ and $fg = 1W$ as special cases of the relation $xU \cdot yV = (xy)W$. Our autotopism may then be written in the form $(WR^{-1}(g), WL^{-1}(f), W)$. But then the isotope G_0 given by $x \circ y = xR^{-1}(g) \cdot yL^{-1}(f) = [xW^{-1} \cdot yW^{-1}]W$ is isomorphic to G, and hence has the weak inverse property. Conversely, if G_0 is isomorphic to G by the mapping W^{-1}, then $(WR^{-1}(g), WL^{-1}(f), W)$ will be an autotopism. We have shown:

LEMMA 2: *If f and g are two elements of a weak inverse property loop G, then the isotope G_0 given by $x \circ y = xR^{-1}(g) \cdot yL^{-1}(f)$ has the weak inverse property if and only if the expression in* (6) *is an autotopism. Furthermore, G_0 is isomorphic to G if and only if f and g are the images of the identity under the first two permutations of some autotopism.*

Consider now the special case of Lemma 2 with $g = 1$. The autotopism (6) is then $(L(f^\lambda)R^{-1}(f), L(f), L(f^\lambda))$, which may be transformed into $(L(f), \lambda L(f^\lambda)\rho, \lambda L(f^\lambda)R^{-1}(f)\rho) = (L(f), R^{-1}(f^\rho)R^{-1}(f^\rho)L(f))$ using Lemma 1. Applying this autotopism to the pair x, 1 gives the relation $L(f)R(f) = R^{-1}(f^\rho)L(f)$, which allows us to write our autotopism in the form

$$(7) \qquad (L(f), R^{-1}(f^\rho), L(f)R(f)) .$$

We shall also need to use this in the following equivalent form:

$$(8) \qquad L(fx) = R(f^\rho)L(x)L(f)R(f) = R(f^\rho)L(x)R^{-1}(f^\rho)L(f) .$$

From (7) and (8), it is clear that f is a weakened type of Moufang element of G[6]. Similar to the case of inverse and cross inverse property loops, one may say something about the structure of the set of elements which give isotopes with the weak inverse property, or which are the images of the identity under a permutation from some autotopism. However, since neither of these sets need form a subloop, this structure does not seem sufficiently interesting to be discussed except in the case where all isotopes have the weak inverse property, to which we turn next.

If all isotopes of G have the weak inverse property, then we may use (6), (7) and (8) for any elements f, g, x of G. In particular, if we take the inverse of (6) and set $f = 1$, we get $(L^{-1}(g^\lambda), R(g), L(g^\lambda)R(g))$. Applying this to $1, x$ gives $R(g)L(g) = L(g^\lambda)\, R(g)$, allowing us to write $(L^{-1}(g^\lambda), R(g), R(g)L(g))$, which is the dual of (7). Replacing f by g in (7) and multiplying by the inverse of its dual, we get $(L(g)L(g^\lambda), R^{-1}(g^\rho)R^{-1}(g), L(g)R(g)L^{-1}(g)R^{-1}(g))$. But each of these three permutations preserves the identity element of G, and hence, by an easy argument, they are all equal. Defining θ_g by

$$(9) \qquad \theta_g = L(g)L(g^\lambda) = R^{-1}(g^\rho)R^{-1}(g) = L(g)R(g)L^{-1}(g)R^{-1}(g) ,$$

we have shown that θ_g is an automorphism. From (9) we get the relation $R^{-1}(g^\rho) = \theta_g R(g)$, which allows us to put (7) in the form $(L(g), \theta_g R(g), L(g)R(g))$, which says that

6. In case the isotope G_0 defined by this element f is isomorphic to G, then f is a "companion" in the terminology of [**4**].

$$gx\cdot(z\theta_g\cdot g) = (g\cdot xz)\cdot g\ , \quad \text{for all} \quad g, x, z \quad \text{of} \quad G\ . \tag{10}$$

But if θ_g were the identity automorphism for all g, then (10) would be just one of the Moufang identities. For example, if G has the inverse property, then $L(g^\lambda) = L^{-1}(g)$, and θ_g is the identity. Similarly, if G has the cross inverse property, then $L(g^\lambda)=R^{-1}(g)$, and (9) yields $L(g)R^{-1}(g)= L(g)R(g)L^{-1}(g)R^{-1}(g)$, or $L(g) = R(g)$. Hence G is commutative and θ_g is again the identity. We have proved:

THEOREM 2. *If G is an inverse property, cross inverse property or commutative loop such that every isotope of G has the weak inverse property, then G is a Moufang loop.*

Now let α be the autotopism (6), and let β and γ be the special cases of this with $g = 1$ and $f = 1$ respectively, then $\beta\gamma\alpha^{-1}=(L(f^\lambda)R^{-1}(f), L(f), L(f^\lambda))\cdot(L(g^\lambda), R^{-1}(g), R^{-1}(g)L(g^\lambda))\cdot(R(f)L^{-1}([fg]^\lambda), R(g)L^{-1}(f), L^{-1}([fg]^\lambda)R(g)) = (L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f)L^{-1}([fg]^\lambda), 1, L(f^\lambda)R^{-\lambda}(g)L(g^\lambda) L^{-1}([fg]^\lambda)R(g))$. Applying this autotopism to the pair x, g gives $L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f) = R(g)L(f^\lambda)R^{-1}(g)L(g^\lambda)$. But from (8) this is just $L(g^\lambda f^\lambda)$, so that the first permutation of $\beta\gamma\alpha^{-1}$ is $L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda)$. Denoting this permutation by U, we thus have an autotopism of the form $(U, 1, W)$, or $xU\cdot y = (xy)W$ for all x, y of G. But setting $y=1$ in this equation gives $U = W$, and setting $x = 1$ shows that $U = L(u)$ where u is the image of the identity under U. Hence $u=(g^\lambda f^\lambda)L^{-1}([fg]^\lambda)$ is in the nucleus of G. Furthermore, in case $u = 1$ for all f and g of G, then we may conclude that $[fg]^\lambda = g^\lambda f^\lambda$ for all f, g of G, so that G has the inverse property. Since the nucleus of G is normal by a theorem of Bruck[7], we have proved:

THEOREM 3: *Let G be a loop all of whose isotopes have the weak inverse property, and let N be its nucleus. Then N is normal, and G/N is a Moufang loop.*

We turn next to a closer examination of the automorphism θ_g. First of all, if x is an arbitrary element of G and if b is an element of the nucleus, then $x\theta_b = b^\lambda\cdot bx = b^\lambda b\cdot x = x$, $b\theta_x=x^\lambda\cdot xb = x^\lambda x\cdot b = b$, and $(bx)^\lambda= [(bx)^\lambda\cdot b]\cdot b^{-1} = x^\lambda b^{-1}$. Also, $\theta_{bx} = \theta_x$, since $y\theta_{bx} = (bx)^\lambda\cdot(bx\cdot y) = x^\lambda b^{-1}\cdot (b\cdot xy) = x^\lambda\cdot xy = y\theta_x$ for any element y of G. Again, setting $x = g^\rho$ in (10) gives the relation $\theta_g = L(g^\rho)L(g)$, or $\theta_g = \theta_h$ for $h = g^\rho$ using (9). Using the iterates of this relation, we compute $x^{\lambda^i}\theta_x=x^{\lambda^i}L(x^{\lambda^{i+1}})L(x^{\lambda^{i+2}}) = x^{\lambda^{i+2}}$. As a special case of this we have $x\theta_x = x^\lambda\cdot xx = x^{\lambda^2}$, which may

7. See Theorem 4.1 of [**4**]. Although the statement of Bruck's theorem assumes that G is a Moufang loop, it is clear from the proof (and his Lemma 2.1) that he only uses the fact that every permutation of the form $L(x)$ or $R(x)$ occurs as the first permutation in some autotopism of G. This is true in our situation from (7), and from the special case $g = f_\rho$ of (6).

be transformed using the weak inverse property into $x^{\lambda^3}\cdot x^\lambda = (xx)^\lambda$, or $x^\lambda\cdot x^\rho = (xx)^\rho$. Similarly, $x\theta_x^{-1} = xx\cdot x^\rho = x^{\rho^2}$ leads to $x^\lambda\cdot x^\rho = (xx)^\lambda$. Hence $(xx)^\lambda = (xx)^\rho$, and so squares have unique inverses in G.

We now define a by the equation $xa = x\theta_x$, and observe that a will be in the nucleus. Since $1 = x^{\lambda^2}\cdot x^\lambda = xa\cdot x^\lambda = x\cdot ax^\lambda$, or $ax^\lambda = x^\rho$, we have $x^\lambda x^\lambda = (x^\lambda x^\lambda)^{\rho^2} = x^\rho x^\rho = x^\rho\cdot ax^\lambda = x^\rho a\cdot x^\lambda$, and hence $x^\rho a = x^\lambda$. But then $x^\rho a = x^\rho\theta_x$, and we can conclude from the properties developed in the last paragraph that $x^{\lambda^i}\cdot a = x^{\lambda^{i+2}}$ and $a\cdot x^{\lambda^i} = x^{\lambda^{i-2}}$. If K is the subloop of G generated by x, and A the cyclic subgroup generated by a, then A is contained in the nucleus of K and $xA = Ax$ from the relations just derived. But then A is normal, K/A is cyclic, and every element of K can be expressed in the form x^ia^j for some pair of integers i, j (note that x^i may be defined to be any element of K that maps into the ith power of the image of x in K/A). It is possible to determine how the elements of K multiply, and hence the structure of K, by an inductive argument. However, this can be avoided by exhibiting a loop, and proving that it is the free loop on one generator with the property that every isotope has the weak inverse property. These one-generator loops are of interest to us, on the one hand as proof that the class of loops we are studying is strictly larger than the class of Moufang loops, and on the other hand as examples of loops which are isomorphic to all their isotopes.

3. Let H be the set of all ordered pairs of integers $[i, k]$ under the binary operation defined by the following four equations

$$\begin{aligned} [2i, k][2j, m] &= [2i + 2j, k + m] \\ [2i + 1, k][2j, m] &= [2i + 2j + 1, k + m + j] \\ [2i, k][2j + 1, m] &= [2i + 2j + 1, m - k] \\ [2i + 1, k][2j + 1, m] &= [2i + 2j + 2, m - k - j]\ , \end{aligned} \tag{11}$$

where i, j, k, m are arbitrary integers. It is easy to verify that each product is uniquely defined by these equations and that H is a loop. As suggested by (11) it will be convenient hereafter to call an element of H odd or even if its first component is odd or even respectively. By checking each of the eight possible cases, the following result may easily be verified:

LEMMA 3. *A triple of elements u, v, w of H associate if and only if at least one of them is even. If all three are odd, then $u\cdot vw = (uv\cdot w)a$, where $a = [0, 1]$.*

COROLLARY 1. *The nucleus of H is the set of all even elements.*

Since three elements whose product is the identity cannot all be odd, we also have:

COROLLARY 2. *H has the weak inverse property.*

The fact that all isotopes of H also have the weak inverse property will follow from the following stronger result:

THEOREM 4. *Every isotope of H is isomorphic to it.*

Let H_0 be the isotope of H defined by $y\circ z = yR^{-1}(g)\cdot zL^{-1}(f)$, and let u and v be defined as follows: $u = [0, 0]$ if f is even, $u = [-1, 0]$ if f is odd, $v = [0, 0]$ if g is even, and $v = [1, 0]$ if g is odd. Then defining s and t by the relations $f = us$ and $sg = vt$, we observe that s and t are even, and hence in the nucleus. Thus,

$$L^{-1}(f) = L^{-1}(us) = [L(s)L(u)]^{-1} = L^{-1}(u)L^{-1}(s) ,$$

and

$$\begin{aligned} R^{-1}(g)R^{-1}(s) &= [R(s)R(g)]^{-1} = R^{-1}(sg) = R^{-1}(vt) \\ &= [R(v)R(t)]^{-1} = R^{-1}(t)R^{-1}(v) . \end{aligned}$$

Using these relations we have

$$\begin{aligned} y\circ z &= yR^{-1}(g)\cdot zL^{-1}(u)L^{-1}(s) = yR^{-1}(g)\cdot s^{-1}\cdot zL^{-1}(u) \\ &= yR^{-1}(g)R^{-1}(s)\cdot zL^{-1}(u) = yR^{-1}(t)R^{-1}(v)\cdot zL^{-1}(u) \\ &= yR^{-1}(t)R^{-1}(v)\cdot zR^{-1}(t)L^{-1}(u)R(t) \\ &= [yR^{-1}(t)R^{-1}(v)\cdot zR^{-1}(t)L^{-1}(u)]R(t) . \end{aligned}$$

But then, defining the isotope $y\otimes z=[yR(t)\circ zR(t)]R^{-1}(t)=yR^{-1}(v)\cdot zR^{-1}(u)$, we see that, up to isomorphism, we need only consider the four cases where $f = u$ and $g = v$.

Now, if $y\circ z = yR^{-1}(v)\cdot z$ where $v = [1, 0]$, define $G_\times$ by $y\times z = [yR(v)\cdot zR(v)]R^{-1}(v) = [y\cdot zR(v)]R^{-1}(v)$, and up to isomorphism we may consider the isotope $G_\times$ instead of G_0. From Lemma 3, we observe that $y\times z = yz$ if either y or z is even, and $y\times z = yz\cdot a^{-1}$ if both are odd. Similarly, if $y\circ z=y\cdot zL^{-1}(u)$ where $u = [-1, 0]$, we define $G_\otimes$ by $y\otimes z= [yL(u)\circ zL(u)]L^{-1}(u) = [yL(u)\cdot z]L^{-1}(u)$, and computing $y\otimes z$ from Lemma 3, we find that $y\otimes z = y\times z$. Finally, if $y\circ z =yR^{-1}(v)\cdot zL^{-1}(u)$ where $u = [-1, 0]$ and $v = [1, 0]$, we would like to show that $y\circ z = y\times z$. But this is equivalent to $yR^{-1}(v)\cdot zL^{-1}(u) = [y\cdot zR(v)]R^{-1}(v)$, or $pq\cdot v = pv\cdot(uq\cdot v)$, where we have right-multiplied by v and set $y=pv$ and $z=uq$. Using Lemma 3, this identity may be easily checked for all four cases of p and q odd and even.

It now only remains to show that the isotope $G_\times$ is isomorphic to G. Letting T be the permutation sending $[i, k]$ into $[-i, -k]$, we shall verify that $y \times z = (yT \cdot zT)T^{-1}$. If either y or z is even, it is easy to check visually from (11) that $yT \cdot zT = (yz)T$. And if both are odd, then we have

$$\begin{aligned}([2i+1, k]T \cdot [2j+1, m]T)T^{-1} &= ([-2i-1, -k][-2j-1, -m])T \\ &= [-2i-2j-2, -m+k+j+1]T \\ &= [2i+2j+2, m-k-j-1] = [2i+1, k][2j+1, m] \cdot a^{-1},\end{aligned}$$

to complete the proof.

Now let K be the free loop on one generator with the property that every isotope has the weak inverse property. Then we may induce a homomorphism φ of K onto H by sending the generator x of K onto the element $[1, 0]$, which generates H. Under this homomorphism, the element a, defined at the end of § 3, can be seen to go onto $[0, 1]$ (by mapping the relation $x^{\rho}a = x^{\lambda}$, for example). If A is the cyclic subgroup of K generated by a, then no element of A is in the kernel of φ since $[0, 1]$ has infinite order in H. But $K/A \to H/\varphi(A)$ also has no kernel, since both are infinite cyclic, and hence, φ is an isomorphism.

THEOREM 5. *The loop H defined by the relations* (11) *is the free loop on one generator with the property that every isotope has the weak inverse property.*

It might be pointed out that every homomorph of H also has the property that it is isomorphic to all its isotopes. By imposing the relations $x^{4m} = a^n = 1$ for integers $m \geq 1$ and $n \geq 2$, we get a loop of order $4mn$ with this property, which is not a group.

4. In this section we shall prove that a cross inverse property loop can only be isotopic to an inverse property loop if it is commutative (and hence already satisfies the inverse property itself). In addition to clarifying the relation between two well known classes of weak inverse property loops, this result is of interest to us here because the method of proof is identical with those used in the rest of the paper.

Let G be an inverse property loop and let G_0 be the isotope given by $a \circ b = ag^{-1} \cdot f^{-1}b$. If G_0 has the cross inverse property, then $(a \circ b) \circ a^{\rho_0} = b$, or $(ag^{-1} \cdot f^{-1}b)g^{-1} \cdot f^{-1}a^{\rho_0} = b$, where ρ_0 is the right inverse operator in G_0. Setting $a = x^{-1}g$ and $b = y^{-1}$ gives

$$(x^{-1} \cdot f^{-1}y^{-1})g^{-1} \cdot f^{-1}(x^{-1}g)^{\rho_0} = y^{-1},$$

or

$$(x^{-1} \cdot f^{-1}y^{-1})g^{-1} = y^{-1} \cdot [f^{-1}(x^{-1}g)^{\rho_0}]^{-1},$$

and taking the inverse of both sides yields $g(yf \cdot x) = [f^{-1}(x^{-1}g)^{\rho_0}]y$. Using the special case obtained by setting $y = 1$, we may rewrite this equation as $g(yf\cdot x)=(g\cdot fx)y$. Finally, replacing y by yf^{-1} gives $(g\cdot fx)\cdot yf^{-1}=g(yx)$. We are motivated by this relation to define an *anti-autotopism* $[U, V, W]$ to be an ordered triple of permutations on G satisfying $xU\cdot yV=(yx)W$ for all pairs of elements x and y of G. We may then express our last relation by saying that

$$(12) \qquad [L(f)L(g), R^{-1}(f), L(g)]$$

is an anti-autotopism. By adapting Lemma 1 to the case of anti-autotopisms and of loops with the inverse property, it is easy to verify that if $[U, V, W]$ is an anti-autotopism, then so is $[W, \rho V\rho, U]$. Hence (12) may also be put in the form

$$(13) \qquad [L(g), L(f), L(f)L(g)] .$$

But if $[U_1, V_1, W_1]$ and $[U_2, V_2, W_2]$ are anti-autotopisms, then $(xy)W_1W_2=(yU_1\cdot xV_1)W_2=xV_1U_2\cdot yU_1V_2$, so that (V_1U_2, U_1V_2, W_1W_2) is an autotopism. Hence the anti-autotopism $[U, V, W]$ has an inverse anti-autotopism given by $[V^{-1}, U^{-1}, W^{-1}]$. Using this information, we may combine (12) and (13) to get the autotopism

$$\begin{aligned}[L(g), L(f), L(f)L(g)]&[L(f)L(g), R^{-1}(f), L(g)]^{-1}\\ &=[L(g), L(f), L(f)L(g)][R(f), L^{-1}(g)L^{-1}(f), L^{-1}(g)]\\ &=(L(f)R(f), L^{-1}(f), L(f)).\end{aligned}$$

Then the analogue of Lemma 1 for inverse property loops allows us to conclude that $(L(f), R(f), L(f)R(f))$ is also an autotopism, which shows that f is in the Moufang nucleus of G. Since the inverse property and cross inverse property are both symmetric, we may conclude by duality that g is also in the Moufang nucleus. But then the isotope G_0 given by $x\circ y=xg^{-1}\cdot f^{-1}y$ will also have the inverse property, as was to be proved.

It might be remarked that a non-commutative cross inverse property loop may not be obtained from an inverse property loop even by allowing anti-isotopisms. This is because every anti-isotopism is the product of an ordinary isotopism and the canonical anti-isotopism given by $x\circ y=y\cdot x$, which clearly preserves both the inverse property and the cross inverse property.

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