THE FRATTINI SUBGROUP OF A p-GROUP

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The Frattini subgroup $\Phi(G)$ of a group G is defined as the intersection of all maximal subgroups of G. It is well known that some groups cannot be the Frattini subgroup of any group. Gaschütz [3, Satz 11] has given a necessary condition for a group H to be the Frattini subgroup of a group G in terms of the automorphism group of H. We shall show that two theorems of Burnside [2] limiting the groups which can be the derived group of a p-group have analogues that limit the groups which can be Frattini subgroups of p-groups.

We first state the two theorems of Burnside.

THEOREM A. *A non-abelian group whose center is cyclic cannot be the derived group of a p-group.*

THEOREM B. *A non-abelian group, the index of whose derived group is p^2, cannot be the derived group of a p-group.*

We shall prove the following analogues of the theorems of Burnside.

THEOREM 1. *If H is a non-abelian group whose center is cyclic, then H cannot be the Frattini subgroup $\Phi(G)$ of any p-group G.*

THEOREM 2. *A non-abelian group H, the index of whose derived group is p^2, cannot be the Frattini subgroup $\Phi(G)$ of any p-group G.*

We shall require four lemmas, the first two of which are due to Blackburn and Gaschütz, respectively.

LEMMA 1. [1, Lemma 1] *If N is a normal subgroup of the p-group G such that the order of N is p^2, then the centralizer of N in G has index at most p in G.*

LEMMA 2. [3, Satz 2] *If $H = \Phi(G)$ for a p-group G and N is a subgroup of H that is normal in G, then $\Phi(G/N) = \Phi(G)/N$.*

LEMMA 3. *If $N = \{a\} \times \{b\}$ is a subgroup of order p^3 normal in the p-group G such that N is contained in $\Phi(G)$, and if $\{a\}$ is a group of order p^2 in the center of $\Phi(G)$, then N is in the center of $\Phi(G)$.*

Proof. N normal in G implies that N contains a group C of order p which is in the center of G. If C is not contained in $\{a\}$ the proof

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is trivial, hence we may assume $C = \{a^p\}$. Since an element of order p in a p-group cannot be conjugate to a power of itself the possible conjugates of b under G are

$$b, ba^p, \cdots, ba^{(p-1)p} .$$

The index of the centralizer of b in G is equal to the number of conjugates of b under G, hence is at most p. Thus b is in the center of $\Phi(G)$, and the lemma follows.

LEMMA 4. *If H is a non-abelian group of order p^3 then there is no p-group G such that $\Phi(G) = H$.*

Proof. If $H = \Phi(G)$ for a p-group G, then H is normal in G and must contain a group N of index p which is also normal in G. Then N is a group of order p^2, hence (Lemma 1) the centralizer C of N in G has index at most p in G. Therefore C contains H, and N is in the center of H. Since the center of H has order p this is a contradiction, and the lemma follows.

We can now prove Theorems 1 and 2.

Proof of Theorem 1. We proceed by induction on the order of H. The theorem is true if H has order p^3 (Lemma 4). Suppose H is group of minimal order for which the theorem is false, and let C of a subgroup of H of order p which is contained in the center of G. Then (Lemma 2)

$$\Phi(G/C) = \Phi(G)/C = H/C .$$

Thus the induction hypothesis implies that H/C cannot be a non-abelian group with cyclic center. We consider two cases: H/C is abelian; or, the center of H/C is non-cyclic.

Case 1. Suppose H/C is abelian. Since H is not abelian, and C has order p, we conclude that C is the derived group of H. Thus H/C, which coincides with its center, is not cyclic, and we are in Case 2.

Case 2. Suppose that the center Z of H/C is non-cyclic. The elements of order p in Z form a characteristic subgroup P of Z. Since Z is not cyclic, P is also not cyclic and hence has order at least p^2. Thus we can find subgroups $\bar{M}$ and $\bar{N}$ of P which are normal in G/C and have orders p and p^2, respectively, where $\bar{M}$ is contained in $\bar{N}$. Let M and N be the subgroups of G which map on $\bar{M}$ and $\bar{N}$. Then M and N are subgroups of H which contain C and are normal in G; M and N have orders p^2 and p^3, respectively, and M is contained in N.

We see from Lemma 1 that the centralizer of M in G has index at most p in G, hence M is in the center of H, which is cyclic. Also, N is abelian since N is contained in H and the index of M in N is p. Now $\bar{N}$ is contained in P, hence is not cyclic. Therefore N is a non-cyclic group which (Lemma 3) is in the center of H. Since the center of H is cyclic this is a contradiction, and the proof is complete.

Proof of Theorem 2. We denote the derived group of a group K by K'. Suppose G is a p-group such that $\Phi(G) = H$ where $H' \neq \{1\}$ and $(H: H') = p^2$. Let N be a normal subgroup of G which has index p in H'. Then G/N is a p-group such that (Lemma 2)

$$\Phi(G/N) = \Phi(G)/N = H/N .$$

But $(H/N)' = H'/N \neq \{1\}$, and the order of H/N is

$$(H: N) = (H: H')(H': N) = p^3 .$$

Thus H/N is a non-abelian group of order p^3 which is the Frattini subgroup of the p-group G/N. This is impossible (Lemma 4) and the theorem follows.

REMARK 1. The only properties of the Frattini subgroup used in the proof of Theorems 1 and 2 are the following: $\Phi(G)$ is a characteristic subgroup of G which is contained in every subgroup of index p in G; and, $\Phi(G/N) = \Phi(G)/N$ whenever N is normal in G and contained in $\Phi(G)$. Thus if we have a rule ψ which assigns a unique subgroup $\psi(G)$ to every p-group G, then Theorems 1 and 2 will hold after replacing "the Frattini subgroup $\Phi(G)$" by "the subgroup $\psi(G)$" if $\psi(G)$ satisfies the following conditions.

(1) $\psi(G)$ is a characteristic subgroup of G.

(2) $\psi(G)$ is contained in $\Phi(G)$.

(3) $\psi(G/N) = \psi(G)/N$ if N is normal in G and N is contained in $\psi(G)$.

In particular, if $\psi(G) = G'$, the derived group of G, we have the theorems of Burnside. The proofs are unchanged.

REMARK 2. Blackburn [1] has used Theorem A to characterize the groups having two generators which are the derived group of a p-group. Using Theorem 1 it is easy to see that Blackburn's proof establishes the following

THEOREM 3. *If $H = \Phi(G)$ for a p-group G and if H has at most two generators, then H contains a cyclic normal subgroup N such that H/N is cyclic.*

References

1. N. Blackburn, *On prime-power groups*, Proc. Camb. Phil. Soc. **53** (1957), 19-27.
2. W. Burnside, *On some properties of groups whose orders are powers of primes* Proc. Lond. Math. Soc. (2) **11** (1912), 225-45.
3. W. Gaschütz, *Über die Φ-Untergruppe endlicher Gruppen.* Math. Z. **58** (1953) 160-170.

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