CHARACTERIZATIONS OF TREE-LIKE CONTINUA

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1. Introduction. It has been conjectured by J. R. Isbell that every one dimensional continuum with trivial Čech homology (arbitrary coefficient group) is tree-like. In this note we give an example showing the conjecture is false. Moreover, the example has the Čech homology groups, the Čech cohomology groups, and the Čech fundamental group (see [3]) of a point. Also, the example cannot be mapped essentially onto a circle, but can be mapped essentially onto a "figure 8". We precede the example with two characterizations of tree-like continua.

2. Preliminaries. Throughout this note by a continuum we will mean a compact connected metric space and unless otherwise specified by a complex we will mean a finite complex. Also, by a linear graph we will mean a one dimensional connected complex.

For this section let X be any one dimensional continuum, K be any linear graph, and $\mathscr{C}(X)$ be the collection of all essential finite open covers of order two of X. For $U \in \mathscr{U} \in \mathscr{C}(X)$ let $\mathscr{N}(\mathscr{U})$ denote the nerve (see page 68 of [5]) of $\mathscr{U}$ and $\sigma(U)$ denote that vertex in $\mathscr{N}(\mathscr{U})$ corresponding to U. Note that for any $\mathscr{U} \in \mathscr{C}(X)$, $\mathscr{N}(\mathscr{U})$ is a linear graph. Where $\mathscr{U} \in \mathscr{C}(X)$ and $x \in X$ let $\Delta(\mathscr{U}, x)$ be the simplex in $\mathscr{N}(\mathscr{U})$ which has as vertices the collection of all $\sigma(U)$ such that $x \in U \in \mathscr{U}$. Where $\mathscr{U} \in \mathscr{C}(X)$, a continuous function f from X to $\mathscr{N}(\mathscr{U})$ is said to be a $\mathscr{U}$-canonical mapping provided that $f(x) \in \Delta(\mathscr{U}, x)$ for all $x \in X$. Where f is a continuous function from X to K, let $\mathscr{S}(f)$ be the collection of all non-empty inverse images under f of open stars of vertices in K. Note that $\mathscr{S}(f) \in \mathscr{C}(X)$. Where f is a continuous function from X to K let f' be that simplicial mapping from $\mathscr{N}(\mathscr{S}(f))$ to K which satisfies the condition.

$$f'(\sigma(f^{-1}\,[\text{open star of } v])) = v$$

for all vertices v in K such that

$$f^{-1}\,[\text{open star of } v]$$

is non-empty. Where $\mathscr{V}$ and $\mathscr{U}$ are two elements of $\mathscr{C}(X)$ such that $\mathscr{V}$ refines $\mathscr{U}$, a simplicial mapping p from $\mathscr{N}(\mathscr{V})$ to $\mathscr{N}(\mathscr{U})$ is said to be a projection if

$$p(\sigma(V)) = \sigma(U)$$

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implies that $V \subset U$ for all $V \in \mathscr{V}$ and $U \in \mathscr{U}$. If $\mathscr{V}$ refines $\mathscr{U}$ then there is always a projection from $\mathscr{N}(\mathscr{V})$ to $\mathscr{N}(\mathscr{U})$ (see page 135 of [5]).

P1. For any continuous function f from X to K there exists another, say g, which is homotopic to f and is such that $g[X]$ is a subcomplex of K.

This is proved by deforming the map f so as to uncover the interior of any simplex whose interior is not completely covered by f and keeping f fixed on the rest of the complex.

P2. For any $\mathscr{U} \in \mathscr{C}(X)$ there exists a $\mathscr{U}$-canonical mapping g from X to $\mathscr{N}(\mathscr{U})$ such that $g[X]$ is a subcomplex of $\mathscr{N}(\mathscr{U})$.

The existence of a $\mathscr{U}$-canonical mapping f from X to $\mathscr{N}(\mathscr{U})$ is established on page 286 of [2]. We use the method described for proving P1 to deform f to a mapping g such that $g[X]$ is a subcomplex of $\mathscr{N}(\mathscr{U})$. Under this natural deformation, g is $\mathscr{U}$-canonical.

P3. If $\mathscr{U} \in \mathscr{C}(X)$ and c is a $\mathscr{U}$-canonical mapping from X to $\mathscr{N}(\mathscr{U})$ then $\mathscr{L}(c)$ refines $\mathscr{U}$.

The proof is immediate from the fact that

$$c^{-1}\,[\text{open star of } \sigma(U)] \subset U$$

for any $U \in \mathscr{U}$.

P4. If f is a continuous function from X onto K then f' is a simplicial isomorphism from $\mathscr{N}(\mathscr{L}(f))$ onto K.

This is a special case of proposition D on page 69 of [5].

P5. If f is a continuous function from X onto K, $\mathscr{U} \in \mathscr{C}(X)$, $\mathscr{U}$ refines $\mathscr{L}(f)$, p is a projection from $\mathscr{N}(\mathscr{U})$ to $\mathscr{N}(\mathscr{L}(f))$, and c is a canonical mapping from X to $\mathscr{N}(\mathscr{U})$ then $f'\,pc$ is homotopic to f.

For any $x \in X$, let $S(x)$ be the smallest simplex in K which contains $f(x)$. It follows immediately from the definitions that $f'\,pc(x)$ and $f(x)$ are both in $S(x)$ for any $x \in X$. Therefore $f'\,pc$ is homotopic to f.

3. Two characterizations of tree-like continua. A one dimensional continuum X is said to be tree-like provided that every open cover of X can be refined by a finite open cover having nerve a tree, that is, having nerve a simply connected linear graph. A continuous mapping f from X to K is said to be inessential if it is homotopic to a constant map — otherwise it is said to be essential.

We shall prove the following two theorems simultaneously:

THEOREM 1. *A given one dimensional continuum X is tree-like if* *and only if every continuous mapping of X into any linear graph is inessential.*

THEOREM 2. *A given one dimensional continuum X is tree-like if and only if for every $\mathscr{U} \in \mathscr{C}(X)$ there exists an element $\mathscr{V}$ of $\mathscr{C}(X)$ which refines $\mathscr{U}$ and is such that any projection from $\mathscr{N}(\mathscr{V})$ to $\mathscr{N}(\mathscr{U})$ is inessential.*

Proof.

Part A. Suppose that X is tree-like and $\mathscr{U}$ is any element of $\mathscr{C}(X)$. Since X is tree-like we may take $\mathscr{V} \in \mathscr{C}(X)$ such that $\mathscr{V}$ refines $\mathscr{U}$ and $\mathscr{N}(\mathscr{V})$ is a tree. Since $\mathscr{N}(\mathscr{V})$ is a tree this nerve is contractible and any mapping (in particular any projection) from $\mathscr{N}(\mathscr{V})$ to $\mathscr{N}(\mathscr{U})$ is inessential.

Part B. Suppose that for any $\mathscr{U} \in \mathscr{C}(X)$ there exists $\mathscr{V} \in \mathscr{C}(X)$ such that $\mathscr{V}$ refines $\mathscr{U}$ and any projection from $\mathscr{N}(\mathscr{V})$ to $\mathscr{N}(\mathscr{U})$ is inessential. Let f be any continuous mapping of X into any linear graph K. In view of P1 we may assume that f is onto. Now we have that $\mathscr{L}(f) \in \mathscr{C}(X)$. Take $\mathscr{U} \in \mathscr{C}(X)$ such that $\mathscr{U}$ refines $\mathscr{L}(f)$ and any projection from $\mathscr{N}(\mathscr{U})$ to $\mathscr{N}(\mathscr{L}(f))$ is inessential. Let p be such a projection and let c be a canonical mapping from X to $\mathscr{N}(\mathscr{U})$.

By P5, the composite mapping $f' pc$ is homotopic to f. Since p is inessential so are $f' pc$ and f.

Part C. Suppose that every continuous mapping of X into any linear graph is inessential. Let $\mathscr{O}$ be any open cover of X. Since X is a one dimensional continuum we may take $\mathscr{U} \in \mathscr{C}(X)$ such that $\mathscr{U}$ refines $\mathscr{O}$.

Let c be a canonical mapping of X into $\mathscr{N}(\mathscr{U})$ such that $c[X]$ is a subcomplex of $\mathscr{N}(\mathscr{U})$. Let K be the universal covering space of $\mathscr{N}(\mathscr{U})$ with projection π. The space K is a complex (in general infinite) and π is simplicial. Since by our hypothesis c is inessential there exists a continuous mapping c^* from X to K such that $\pi c^* = c$. Let $T = c^*[X]$. Then T is a tree. By P4 the nerve $\mathscr{N}(\mathscr{L}(c^*))$ is isomorphic to T and hence is a tree. Also $\mathscr{L}(c^*)$ refines $\mathscr{L}(c)$ which refines $\mathscr{U}$ which refines $\mathscr{O}$ and therefore $\mathscr{L}(c^*)$ refines $\mathscr{O}$.

4. A group theoretic Lemma. The group theoretic situation discussed in this section is fundamental to the construction of the example in the following section.

Let G be a free non-Abelian group on two generators a and b. Let h be that endomorphism of G characterized by

$$h(a) = aba^{-1}b^{-1}$$

and

$$h(b) = a^2b^2a^{-2}b^{-2} .$$

Let Q be the set of all ordered pairs (α, n) such that n is a integer and α is either a or b. For any $(\alpha, n) \in Q$ let $e(\alpha, n) = \alpha^n$. Let S be the collection of all finite sequences $\{(\alpha_i, n_i)\}_{i=1}^{r}$ in Q such that

$$n_i \neq 0 \qquad \text{for } i = 1, \cdots, r$$

and

$$a_i \neq a_{i+1} \qquad \text{for } i = 1, \cdots, r-1$$

For each element g of G other than the identity there exists a unique $\{(\alpha_i, n_i)\}_{i=1}^{r}$ in S such that

$$g = \prod_{i=1}^{r} e(\alpha_i, n_i) .$$

In this case $\prod_{i=1}^{r} e(\alpha_i, n_i)$ is called the *preferred* representation of g and r is called the *length* of g.

LEMMA. *If g is an element of positive length r in G then the length of $h(g)$ is greater than or equal to $3r$.*

In order to prove the Lemma we will prove by induction the following somewhat stronger proposition:

(*) If g is any element of G with preferred representation $\prod_{i=1}^{r} e(\alpha_i, n_i)$ and $h(g)$ has preferred representation $\prod_{i=1}^{s} e(\beta_i, m_i)$ then

(i) $s \geqq 3r$,
(ii) $\alpha_r = a$ and $n_r > 0$ imply $\beta_s = b$ and $m_s = -1$,
(iii) $\alpha_r = a$ and $n_r < 0$ imply $\beta_s = a$ and $m_s = -1$,
(iv) $\alpha_r = b$ and $n_r > 0$ imply $\beta_s = b$ and $m_s = -2$,
and
(v) $\alpha_r = b$ and $n_r < 0$ imply $\beta_s = a$ and $m_s = -2$.

Observing that

$$h(a^{-1}) = bab^{-1}a^{-1}$$

and

$$h(b^{-1}) = b^2a^2b^{-2}a^{-2}$$

proposition (*) is obviously true for $r = 1$.

Suppose that proposition (*) is true for the positive integer r. Let g be any element of G with length $r + 1$.

Say

$$g = \prod_{i=1}^{r} e(\alpha_i, n_i)\, e(\alpha, n)$$

and this is the preferred representation of g. Let $f = \prod_{i=1}^{r} e(\alpha_i, n_i)$. This is the preferred representation of f. Let

$$h(f) = \prod_{i=1}^{s} e(\beta_i, m_i)$$

be the preferred representation of $h(f)$. Note that

$$h(g) = h[f \cdot e(\alpha, n)] = h(f) \cdot h(\alpha, n)$$

$$= \prod_{i=1}^{s} e(\beta_i, m_i)\ h(\alpha, n)\ .$$

In order to conclude (i) — (v) of (*) we break the situation down into the following cases:

Case I $\quad \alpha_r = a,\ n_r > 0,\ \alpha = b,$ and $n > 0$.
Case II. $\quad \alpha_r = a,\ n_r > 0,\ \alpha = b,$ and $n < 0$.
Case III. $\quad \alpha_r = a,\ n_r < 0,\ \alpha = b,$ and $n > 0$.
Case IV. $\quad \alpha_r = a,\ n_r < 0,\ \alpha = b,$ and $n < 0$.
Case V. $\quad \alpha_r = b,\ n_r > 0,\ \alpha = a,$ and $n > 0$.
Case VI. $\quad \alpha_r = b,\ n_r \quad 0,\ \alpha = a,$ and $n < 0$.
Case VII. $\quad \alpha_r = b,\ n_r < 0,\ \alpha = a,$ and $n > 0$.
Case VIII. $\quad \alpha_r = b,\ n_r < 0,\ \alpha = a,$ and $n < 0$.

For convenience let k be the absolute value of n.
Case I. Define $\{\gamma_j, q_j\}_{j=1}^{4k}$ by

$$\begin{aligned} \gamma_{4i+1} &= a,\ q_{4i+1} = 2 \\ \gamma_{4i+2} &= b,\ q_{4i+2} = 2 \\ \gamma_{4i+3} &= a,\ q_{4i+3} = -2 \\ \gamma_{4i+4} &= b,\ q_{4i+4} = -2 \end{aligned}$$

for $i = 0, 1, 2, \cdots, k - 1$. Then

$$\prod_{i=1}^{s} e(\beta_i, m_i) \prod_{j=1}^{4k} e(\gamma_j, q_j)$$

is the preferred representation of $h(g)$ and $h(g)$ has length $s + 4k \geqq 3r + 4k \geqq 3r + 3 \geqq 3(r + 1)$.
Also

$$\gamma_{4k} = b \text{ and } q_{4k} = -2\ .$$

Case II. Define $\{\gamma_j, q_j\}_{j=2}^{4k}$ by

$$\begin{aligned} \gamma_{4i+1} &= b,\ \ q_{4i+1} = 2 \\ \gamma_{4i+2} &= a,\ \ q_{4i+2} = 2 \\ \gamma_{4i+3} &= b,\ \ q_{4i+3} = -2 \\ \gamma_{4i+4} &= a,\ \ q_{4i+4} = -2 \end{aligned}$$

for $i = 0, 1, 2, \cdots, k-1$ and $4i + 1 \neq 1$.

Then

$$\prod_{i=1}^{s-1} e(B_i, m_i)\ e(b, 1) \prod_{j=2}^{4k} e(\gamma_j, q_j)$$

is the prefered representation of $h(g)$ and $h(g)$ has lengh

$$s + 4k - 1 \geqq 3r + 4k - 1 \geqq 3r + 3 \geqq 3(r + 1)\ .$$

Also

$$\gamma_{4k} = a \text{ and } q_{4k} = -2\ .$$

These two cases are representative of all of them. In every case the length of $h(g)$ is either $s + 4k$ or $s + 4k - 1$ and hence is greater than or equal to $3(r + 1)$. Cancellation can occur in at most one place and that is where the terminal factor of $h(f)$ lies next to the initial factor of $h(\alpha, n)$. Conditions (ii) – (v) of (*) follow immediately.

5. **An example.** Let C be the collection of all complex numbers having modulus 1. Let

$$B = [C \times \{1\}] \cup [\{1\}] \times C]$$

and let $b_0 = (1, 1)$. In geometrical terms B is the union of two tangent circles and b_0 is the point of tangency. Define the function f from B to itself by the following formulas:

$$f(u, 1) = \left\{\begin{array}{ll} (u^4, 1) & \text{for } 0 \leqq \arg(u) \leqq \pi/2 \\ (1, u^4) & \text{for } \pi/2 \leqq \arg(u) \leqq \pi \\ (u^{-4}, 1) & \text{for } \pi \leqq \arg(u) \leqq 3\pi/2 \\ (1, u^{-4}) & \text{for } 3\pi/2 \leqq \arg(u) \leqq 2\pi \end{array}\right\}$$

and

$$f(1, v) = \left\{\begin{array}{ll} (v^8, 1) & \text{for } 0 \leqq \arg(v) \leqq \pi/2 \\ (1, v^8) & \text{for } \pi/2 \leqq \arg(v) \leqq \pi \\ (v^{-8}, 1) & \text{for } \pi \leqq \arg(v) \leqq 3\pi/2 \\ (1, v^{-8}) & \text{for } 3\pi/2 \leqq \arg(v) \leqq 2\pi \end{array}\right\}$$

for all $u, v \in C$ where u^{-1} and v^{-1} are the complex conjugates of u and v respectively. Note that f is continuous, onto, and takes b_0 into b_0.

We will represent the fundamental group $\pi(\mathrm{B}, b_0)$ of B with base point b_0 as the homotopy classes of continuous mappings σ from C to B such that $\sigma(1) = b_0$. Let a be that element of $\pi(B, b_0)$ represented by the continuous mapping α from C to B defined by

$$\alpha(u) = (u, 1)$$

for all $u \in C$. Let b be that element of $\pi(B, b_0)$ represented by the continuous mapping β from C to B defined by

$$\beta(u) = (1, u)$$

for all $u \in C$. It is well known that $\pi(B, b_0)$ is a free non-Abelian group on the two generators a and b. It follows immediately from the definition of the group operation on $\pi(B, b_0)$ that the natural induced endomorphism f^* of $\pi(B, b_0)$ satisfies the conditions

$$f^*(a) = aba^{-1}b^{-1}$$

and

$$f^*(b) = a^2b^2a^{-2}b^{-2} .$$

Therefore the group $\pi(B, b_0)$ and the endomorphism f^* of this group satisfy the hypothesis of the group theoretic lemma in the preceding section. It also follows that the induced endomorphisms $f_\#$ and $f^\#$ on the one dimensional homology group $H_1(B)$ and on the one dimensional cohomology group $H^1(B)$ respectively are the zero endomorphisms — no matter what coefficient group is used.

We define the space M to be the limit of the inverse system

$$B \xleftarrow{f} B \xleftarrow{f} B \xleftarrow{f} \cdots .$$

M may be described in a more elementary but more tedious way as the intersection of certain nest of closed tubular neighborhoods of "figure 8's" in three dimensional Euclidean space.

Q1. M is a continuum

This follows immediately from results in Chapter VIII of [2].

In establishing some of the other properties it will be convenient to give a more explicit definition of M. The set M is the collection of all sequences

$$x = \{x_i\}_{i=1}^{\infty}$$

of points in B such that

$$f(x_{i+1}) = x_i$$

for all i. For each i we define the projection π_i from M to B by the formula

$$\pi_i(x) = x_i$$

for all $x \in M$. The collection of all subsets of M of the form $\pi_i^{-1}[U]$ where i is any positive integer and U is any open subset of B from a basis for the topology of M. Moreover, the intersection of any two of these basic open sets is another. For all $i < j$ we let

$$\pi_i^j = f^{j-i}$$

be the mapping from B onto itself obtained by $j - i$ iterations of f. Let m_0 be that point in M defined by

$$\pi i(m_0) = b_0$$

for all i.

Q2. For each i, π_i is a continuous mapping of M onto B.
This follows from Corollary 3.9 on page 218 of [2].

Q3. For any open cover $\mathscr{U}$ of M there exists a positive integer j and a finite open cover $\mathscr{V}$ of B such that the collection of all $\pi_j^{-1}[V]$ for $V \in \mathscr{V}$ refines $\mathscr{U}$.

Proof. Let $\mathscr{U}$ be any open cover of M. Since M is compact $\mathscr{U}$ may be refined by a finite cover $\mathscr{B}$ of basic open sets. Let F be a finite set of pairs having first coordinate a positive integer and second coordinate an open subset of B such that $\mathscr{B}$ is the collection of all $\pi_i^{-1}[N]$ for $(i, N) \in F$. Let j be greater than any of the first coordinates of members of F. Let $\mathscr{V}$ be the collection of all $[\pi_i^j]^{-1}[N]$ for $(i, N) \in F$. Then for each $(i, N) \in F$ we have

$$[\pi_j^{-1}][\pi_i^j]^{-1}[N] = [\pi_i^j \pi_j]^{-1}[N] = \pi_i^{-1}[N] .$$

Therefore, $\mathscr{B}$ is the set of all $\pi_j^{-1}[V]$ for $V \in \mathscr{V}$ and $\mathscr{B}$ refines $\mathscr{U}$.

Q4. M is one dimensional.

Proof. Let $\mathscr{U}$ be any open cover of M. By Q3 we may take a positive integer j and a finite open cover $\mathscr{V}$ of B such that the collection of all $\pi_j^{-1}[V]$ for $V \in \mathscr{V}$ refines $\mathscr{U}$. Since B is a one dimensional continuum we may take $\mathscr{W}$ to be a finite open cover of B which refines $\mathscr{U}$ and is of order 2. Now the collection of all $\pi_j^{-1}[W]$ for $W \in \mathscr{W}$ is of order 2. The collection of all $\pi_j^{-1}[W]$ for $W \in \mathscr{W}$ is a finite open cover of M which refines $\mathscr{U}$ and is of order 2. Therefore, the continuum M has dimension less than or equal to one. We need only observe that it contains more than one point to see that it is one dimensional.

Since M is one dimensional all of the higher groups of M are trivial and we do not mention them further.

Q5. $H_1(M)$ and $H^1(M)$ are zero for an arbitrary coefficient group.

Proof. The Čech homology and Čech cohomology satisfy the continuity axiom (see Chapter X of [2]). Therefore, $H_1(M)$ is isomorphic to the limit of the inverse system

$$H_1(B) \xleftarrow{f^\#} H_1(B) \xleftarrow{f^\#} H_1(B) \xleftarrow{f^\#} \cdots\cdots$$

and $H^1(B)$ is isomorphic to the limit of the direct system

$$H^1(B) \xrightarrow{f^\#} H^1(B) \xrightarrow{f^\#} H^1(B) \xrightarrow{f^\#} \cdots$$

We have already observed that both f and f are the zero homomorphisms. Therefore both $H_1(M)$ and $H^1(M)$ are the zero groups.

Q6. M cannot be mapped essentially onto the circle.

Proof. Making use of the fact that M is a one dimensional compact space and $H^1(M)$ with integer coefficients is zero, we see that Q6 follows immediately from the corollary on page 150 in [5] to Hopf's extension theorem.

Q7. The Čech fundamental group $\pi(M, m_0)$ of M with base point m_0 is zero.

Proof. The Čech fundamental group also satisfies the continuity axiom (see [3]) and agrees with the usual fundamental group on complexes. Therefore $\pi(M, m_0)$ is isomorphic to the limit of the inverse system.

$$\pi(B, b_0) \xleftarrow{f^*} \pi(B,b_0) \xleftarrow{f^*} \pi(B, b_0) \xleftarrow{f^*} \cdots$$

of non-Abelian groups and homomorphisms. We now apply the group theoretic lemma of the previous section. Suppose there is an element g other than the identity in this inverse limit. Then $g = \{g_i\}_{i=1}^{\infty}$ where for each i

$$g_i \in \pi(B, b_0)$$

and $f^*(g_{i+1}) = g_i$. Moreover we may take n such that g_i is not equal to the identity for all $i \geqq n$. Therefore for any $i > n$, g_i has positive length,

$$(f^*)^{i-n}(g_i) = g_n,$$

and g_n has length greater than or equal to $3(i - n)$. This says that g_n has infinite length, a contradiction.

We have yet to establish that M is not tree-like. For this purpose we make use of Theorem 1 although we could just as well use Theorem 2.

We construct an essential mapping q of M onto B. For each i, let

f^i be the mapping from B to itself obtained by i iterations of f. Note that for all $i < j$

$$f^i \pi_i^j = f^j .$$

Define the mapping q from M to B by

$$q(x) = f(x_1) .$$

Actually $q = f\pi_1$, but for clarity we use this different notation. Note that for any i

$$q(x) = f^i(x_i) .$$

Q8. For every positive integer i the mapping f^i from B into itself is essential.

Proof. According to the algebraic lemma of the preceding section the endomorphism f^* of $\pi(B, b_0)$ is an automorphism. Therefore, the endomorphism $(f^i)^*$ which equals $(f^*)^i$ is also an automorphism. Now since the group $\pi(B, b_0)$ is not zero, the automorphism $(f^i)^*$ is not zero and the mapping f^i is essential.

Q9. The continuous mapping q from M onto the "figure 8" B is essential and therefore M is not tree-like.

Proof. By way of contradiction, suppose that q is inessential. Let $\tilde{B}$ be the universal covering space of B with projection p. Since the mapping q from M to B is inessential there exists a continuous mapping q^* from M to $\tilde{B}$ such that $pq^* = q$.

Let $\mathscr{E}$ be an open cover of $q^*[M]$ by open sets of $\tilde{B}$ such that for any $E \in \mathscr{E}$ the mapping p_E obtained by restricting p to E is a homeomorphism of E onto the open set $p[E]$ in B. Let $\mathscr{U}$ be the collection of all inverse images under q^* of elements of $\mathscr{E}$. According to Q3 we may take a positive integer i and an open cover $\mathscr{V}$ of B such that the collection of all $\pi_i^{-1}[V]$ for $V \in \mathscr{V}$ refines $\mathscr{U}$.

Define the function c from B to $\tilde{B}$ as follows:

$$c(x) = q^*(z)$$

for any $x \in B$ and $z \in \pi_i^{-1}[x]$.
Clearly

$$pc(x) = pq^*(z) = q(z) = f^i(z_i) = f^i(x)$$

for any $x \in B$ and $z \in \pi_i^{-1}[x]$. Therefore, if c is a continuous function it will cover f^i.

To show that c is single valued, let x be any point in B and let $z, z' \in \pi_i^{-1}[x]$.
Then

$$q(z) = q(z') = f^i(x) .$$

Therefore $pq^*(z) = pq^*(z')$. By the choice of i and $\mathscr{V}$ we know that there exists $E \in \mathscr{E}$ such that

$$q^*[\pi_i^{-1}[x]] \subset E .$$

Therefore $q^*(z), q^*(z') \in E$. We also know that p restricted to E is a homeomorphism. Therefore $q^*(z) = q^*(z')$ and we have that c as defined above is single valued.

It is immediate from the fact that $\pi_i[M] = B$ that the domain of c is B.

In order to show continuity note that for any $x \in V \in \mathscr{V}$,

$$c(x) = (p_E)^{-1} f^i(x)$$

where E is an element of $\mathscr{E}$ such that

$$q^*[\pi_i^{-1}[V]] \subset E .$$

Therefore c is continuous on each member of $\mathscr{V}$, an open cover of B, and therefore c is continuous on all of B.

Now we have lifted the map f^i from B to B to the map c from B to the universal covering space of B. Since B is a linear graph the sub-continuum $c[B]$ of $\tilde{B}$ is contractible and hence c is inessential. Since $pc = f^i$ the map f^i is inessential, a contradiction of Q8.

Further remarks. Theorems 1 and 2 give us two conditions each of which is equivalent to saying that a given one dimensional continuum X is tree-like. We list without proof some other likely characterizations:
(1) X has no non-trivial connected generalized covering space.
(2) X cannot be mapped essentially into the Universal one dimensional curve.
(3) X is an inverse limit of 2-cells.
Condition (3) leads us to stating another question. First let us say that a continuum X is disk-like if it is an inverse limit of 2-cells or equivalently if every open cover of X can be refined by one which has nerve a disk. The tree-like continua are precisely the one dimensional disk-like continua. Also, those disk-like continua that can be imbedded in the plane are precisely the continua which can be imbedded in the plane and do not separate the plane.

Question. Does every disk-like continuum have the fixed point property?

Obviously an affirmative answer to this problem would give affirmative answers to the fixed point problem for tree-like continua and for those sub-continua of the plane which do not separate it.

The continuum M described in this paper gives further insight into the difficulties of generalizing the definition of the fundamental group. We may conclude that any generalization of the fundamental group, which agrees with the usual fundamental group on complexes, and which also satisfies the continuity axiom cannot distinguish the tree-like continua from the other one-dimensional continua. This difficulty will be described more explicitly in another paper which will include the verification of condition (1) of this section.

The referee pointed out our lack of reference to the known results on the fixed point problem for continua which are inverse limits of n-cells with $n \neq 2$. We remark that snake-like continua (those which are inverse limits of 1-cells) have the fixed point property (see [4]) but there exist cube-like continua (those which are inverse limits of 3-cells) which do not have the fixed point property (see [1]).

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