SPECTRAL THEORY FOR LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Introduction. The study of boundary value problems for systems of first order differential equations was begun by Bliss in 1926 [1]. Such problems are of interest not only because they include boundary value problems for single equations of arbitrary order, but also because they arise in the calculus of variations and relativistic quantum mechanics. Until now, attention has been concentrated on boundary value problems on a finite interval [1, 2, 8], but an application to a particular boundary value problem on an infinite interval has also been considered [6]. It seems reasonable to expect that the theory of boundary value problems and eigenfunction expansions on an infinite interval for a single differential equation of arbitrary order can be extended to first order systems. In this paper, the extension will be carried out along lines similar to those used by the author in [3]. It will be shown that all the results obtained in [3] can be formulated so as to be valid for systems. Vector and matrix notation will be used extensively, and as a result, formulae will take a simpler and more natural form than in [3].

The elements of a matrix A will be denoted by A_{ij}, and the components of a row or column vector f will be denoted by f_i in the usual manner. The adjoint of a matrix A, written A^*, will be the matrix with $\bar{A}_{ji}$ in the ith row, jth column, the bar indicating the complex conjugate. The adjoint f^* of a row or column vector f will be the column or row vector respectively with components $\bar{f}_i$. It is easily seen that $(AB)^* = B^*A^*$, whether A and B are vectors or matrices such that AB is defined. If A is a matrix and α is a scalar, then $(\alpha A)^* = \bar{\alpha}A^*$. Also, if A is a Hermitian matrix ($A = A^*$), and f and g are column vectors, then it is easy to see that $(f^*Ag)^* = g^*Af = \overline{(f^*Ag)}$. The matrix dA/dt, or A', will be the matrix with elements A'_{ij}, and the vector df/dt, or f', will be the vector with components f'_i. Any analytic properties, such as continuity or differentiability, postulated for a vector or matrix will be understood to be assumed for each element separately.

1. The expansion theorem. Let A_0, A, B be $n \times n$ continuous complex-valued matrix functions of t defined on an interval $I = (a, b)$, not necessarily a bounded interval, with not all elements of B vanishing identically on I and with A_0 non-singular at every point of I. We are

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interested in boundary value problems for the linear system of differential equations

$$(1) \qquad A_0x' + Ax = \lambda Bx \, ,$$

where x is an n-dimensional column vector. The adjoint system is defined to be

$$(2) \qquad -(A_0^*y)' + A^*y = \lambda B^*y \, .$$

The system (1) is called symmetric if there exists a transformation $y = C(t)x$, with C a non-singular continuously differentiable matrix on I, which transforms (1) into (2) for all values of λ. It can easily be shown (cf. [8]) that (1) is symmetric if and only if

$$(3) \qquad (A_0^*C)' - A_0^*CA_0^{-1}A - A^*C = 0, \qquad B^*C + A_0^*CA_0^{-1}B = 0 \, .$$

If (3) is satisfied, it may easily be verified that

$$(4) \qquad -(A_0^*Cx)' + A^*Cx - \lambda B^*Cx = -A_0^*CA_0^{-1}(A_0x' + Ax - \lambda Bx) \, .$$

It may be shown by integration by parts that if f and g are two differentiable vector functions vanishing at the ends of the interval I, then

$$\int_I (Cg)^*(A_0f' + Af - \lambda Bf)dt = \int_I [-(A_0^*Cg)' + A^*Cg - \bar{\lambda}B^*Cg]^* f dt \, .$$

If the system (1) is symmetric, (4) yields

$$(5) \qquad \int_I (Cg)^*(A_0f' + Af - \lambda Bf)dt$$
$$= -\int_I (A_0g' + Ag - \bar{\lambda}Bg)^* A_0^{*-1}C^*A_0 f dt \, .$$

Let $C_0^1(I)$ denote the set of continuously differentiable vector functions which vanish identically outside some compact subinterval of I. A symmetric linear system (1) is called definite if

(i) the matrix $S = C^*B$ is Hermitian, so that $C^*B = B^*C$,

(ii) $\int_I f^*Sf dt \geqq 0$ for any $f \in C_0^1(1)$, and

(iii) $A_0u' + Au = 0, Bu = 0$ on any subinterval J of I implies that u vanishes identically on J.

In view of these conditions,

$$(6) \qquad [f, g] = \int_I g^*Sf dt$$

may be regarded as an inner product on $C_0^1(I)$. Let H be the Hilbert space completion of $C_0^1(I)$ in the inner product (6). Then H is the set of equivalence classes of vector functions f such that $\int_I f^*Sf\,dt < \infty$. The norm in H will be denoted by $||f||$.

Let D denote the set of functions f in $C_0^1(I)$ such that

$$A_0 f' + Af = Bp \tag{7}$$

for some p in H. Although p may not be uniquely determined as a function by (7), the function Bp is uniquely determined. If p_1 and p_2 are elements of H with $Bp_1 \equiv Bp_2$, then

$$||p_1 - p_2||^2 = \int_I (p_1 - p_2)^* C^* B(p_1 - p_2)dt = 0 ,$$

and p_1 and p_2 define the same element of H. Thus the equation (7) determines a unique element p of H. We define an operator L in H with domain D, by defining $Lf = p$ for $f \in D$, with p determined by (7).

LEMMA 1. *If the system* (1) *is symmetric and definite, then the operator L is symmetric on D.*

Proof. Let $f, g \in D$, with p as in (7) and

$$A_0 g' + Ag = Bq . \tag{8}$$

Then,

$$\begin{aligned}
[Lf, g] &= \int_I g^*Sp\,dt = \int_I g^*B^*Cp\,dt = -\int_I g^*A_0^*CA_0^{-1}Bp\,dt \\
&= -\int_I g^*A_0^*CA_0^{-1}(A_0f' + Af)dt \\
&= -[g^*A_0^*Cf]_a^b + \int_I (g^*A_0^*C)'f\,dt - \int_I g^*A_0^*CA_0^{-1}f\,dt \\
&\quad -[g^*A_0^*Cf]_a^b + \int_I g^{*\prime}A_0^*Cf\,dt + \int_I g^*[(A_0^*C)' - A_0^*CA_0^{-1}A]f\,dt ,
\end{aligned}$$

using (3), (7), and integration by parts. Also,

$$\begin{aligned}
[f, Lg] &= \int_I q^*Sf\,dt = \int_I q^*B^*Cf\,dt = \int_I (A_0g' + Ag)^*Cf\,dt \\
&= \int_I (g^{*\prime}A_0^*Cf + g^*A^*Cf)dt ,
\end{aligned}$$

using (8). Thus

$$\begin{aligned}
[Lf, g] - [f, Lg] &= -[g^*A_0^*Cf]_a^b \\
&\quad + \int_I g^*[(A_0^*C)' - A_0^*CA_0^{-1}A - A^*C]f\,dt .
\end{aligned}$$

The integral vanishes because of (3), and the first term on the right side vanishes because f and g vanish outside a compact subinterval of I. Therefore $[Lf, g] = [f, Lg]$, and L is symmetric on D.

Throughout this paper, we shall assume that (1) is symmetric and definite, and that the symmetric operator L has a self-adjoint extension T, considered as an operator in H. If A_0, A, and B have real coefficients, then L is a real operator and always has at least one self-adjoint extension ([9], p. 329).

LEMMA 2. *There exists a matrix $k(t, s, \lambda)$ with the following properties*:

(i) *k is continuous on $I \times I$ for fixed λ except on $t = s$, and analytc in λ for fixed t, s,*

(ii) *$k(s + 0, s, \lambda) - k(s - 0, s, \lambda)$ is the identity matrix E for $s \in I$ and any λ,*

(iii) *the columns of k satisfy* (1) *as functions of t for $t \neq s$,*

(iv) *if J is any compact subinterval of I and f is any function in $C_0^1(J)$, then*

$$f(t) = \int_J k(t, s, \lambda)[A_0(s)f'(s) + A(s)f(s) - \lambda B(s)f(s)]ds \ , \tag{9}$$

for $t \in J$.

Proof. Let $\Phi(t, \lambda)$ be a fundamental matrix solution of (1), that is, a matrix whose columns are linearly independent solutions of (1). This matrix is non-singular for all $t \in I$, and can be chosen so that all its elements are analytic in λ for each fixed t. For $t < s$, define $k(t, s, \lambda) = 0$, and for $t \geqq s$, define $k(t, s, \lambda) = \Phi(t, \lambda)\Phi^{-1}(s, \lambda)$. The properties (i)-(iii) are immediate consequences of this definition, and the property (iv) follows from the variation of constants formula ([5], p. 74).

The function $k(t, s, 0)$ will be denoted by $k(t, s)$. In this section, we will use only $k(t, s)$, but the more general $k(t, s, \lambda)$ will be required later. An expression such as $k(t,\ )$ will stand for $k(t, s)$, considered as a function of s for any fixed t. Let J be any compact subinterval of I and let θ_J be a real continuously differentiable scalar functions which is 1 on J and which vanishes identically outside some compact subinterval of I. Let $z(t, s) = C^{-1}(s)k^*(t, s)\theta_J(s)$, an $n \times n$ matrix. It is clear that the columns $z_i(t,\ )$ of $z(t,\ )$ are continuously differentiable vectors which vanish outside a compact subinterval of I, and that each $z_i(t,\ )$ is an element of H. If f belongs to D and vanishes identically outside J, then we can write

$$f(t) = \int_J \theta_J(s)k(t, s)B(s)p(s)ds = \int_J z^*(t, s)C^*(s)B(s)p(s)ds$$

$$= \int_I z^*(t, s)S(s)p(s)ds ,$$

using (7), (9), and $S = C^*B$. This means that each component f_i of f $(i = 1, \cdots, n)$ can be written

$$f_i(t) = \int_J z_i^*(t, s)S(s)p(s)ds = [p, z_i(t, \ )] = [Lf, z_i(t, \ )] . \tag{10}$$

We will make use of the theory of direct integrals and the spectral theorem as given in [7]. The notation will be similar, but not identical, to that used in [3]. The elements of the direct integral $L^2(\sigma, \nu)$ are $\nu(\lambda)$-dimensional vectors $F(\lambda)$, and the inner product

$$(F, G) = \int_R \sum_{k=1}^{\nu(\lambda)} F_k(\lambda)\overline{G}_k(\lambda)d\sigma(\lambda)$$

of two elements F, G of $L^2(\sigma, \nu)$ will be denoted by $\int_R G^*(\lambda)F(\lambda)d\sigma(\lambda)$, in analogy to our other notation. R will always mean the real line.

We can now state the result of this section.

THEOREM 1. *Let T be a self-adjoint extension with domain D_T of the operator L defined for a symmetric definite system* (1). *The spectral theorem furnishes a direct integral $L^2(\sigma, \nu)$ and a unitary transformation U from H to $L^2(\sigma, \nu)$ which diagonalizes T. This transformation is given by*

$$(Uf)(\lambda) = \int_I E^*(t, \lambda)S(t)f(t)dt , \tag{11}$$

and its inverse by

$$(U^{-1}F)(t) = \int_R E(t, \lambda)F(\lambda)d\sigma(\lambda) , \tag{12}$$

with the integrals converging to the functions in the norms of the Hilbert spaces $L^2(\sigma, \nu)$ and H respectively. Here, $E(t, \lambda)$ is a matrix function with n rows and $\nu(\lambda)$ columns, whose elements have locally square-integrable derivatives with respect to t. The columns of $E(t, \lambda)$ are improper eigenfunctions (not necessarily belonging to H) *of the differential equation* (1) *for almost all λ. If λ_0 is an eigenvalue of T, then the columns of $E(t, \lambda_0)$ are proper eigenfunctions.*

Proof. Let $L^2(\sigma, \nu)$ be a suitable direct integral and let U be the unitary mapping of H to $L^2(\sigma, \nu)$ which diagonalizes the self-adjoint extension T of L. The fact that U is unitary is expressed by the Parseval formula

$$[f, g] = (Uf, Ug) = \int_R (Ug)^*(\lambda)(Uf)(\lambda)d\sigma(\lambda) . \tag{13}$$

Let f belong to D_T, the domain of T, and let g be any function in H such that Sg vanishes identically outside some compact subinterval J of I. Let $F = Uf, G = Ug, Z_i = Uz_i, E^i(t, \lambda) = \lambda Z_i^*(t, \lambda)$, where z_i is as in (10). Then

$$\begin{aligned} f_i(t) &= [Tf, z_i(t, \ )] = (UTf, Z_i(t, \ )) = (\lambda Uf, Z_i(t, \ )) \\ &= (F, E^{i*}(t, \ )) = \int_R E^i(t, \lambda)F(\lambda)d\sigma(\lambda) , \end{aligned}$$

using (10), (13), and the spectral theorem. In addition,

$$\begin{aligned} [f, g] &= \int_J g^*Sf dt = \int_J \sum_{i=1}^n [g^*S]_i f_i dt \\ &= \int_J \sum_{i=1}^n [g^*(t)S(t)]_i \left\{\int_R E^i(t, \lambda)F(\lambda)d\sigma(\lambda)\right\} dt \\ &= \int_R \left\{\int_J \sum_{i=1}^n [g^*(t)S(t)]_i E^i(t, \lambda)dt\right\} F(\lambda)d\sigma(\lambda) , \end{aligned}$$

the interchange in the order of integration being justified by the absolute convergence of the integral. We define the $n \times \nu(\lambda)$ matrix $E(t, \lambda)$ with rows $E^i(t, \lambda)$. Then we can write

$$[f, g] = \int_R \left\{\int_J g^*(t)S(t)E(t, \lambda)dt\right\} F(\lambda)d\sigma(\lambda) .$$

On the other hand,

$$[f, g] = \int_R G^*(\lambda)F(\lambda)d\sigma(\lambda) ,$$

and thus

$$G^*(\lambda) = \int_J g^*(t)S(t)E(t, \lambda)dt ,$$

or,

$$G(\lambda) = \int_J E^*(t, \lambda)S(t)g(t)dt$$

for almost all λ.

For $g \in D, g$ vanishing identically outside J, we have seen that $(Ug)(\lambda) = \int_J E^*(t, \lambda)S(t)g(t)dt$. If $A_0g' + Ag = Bp$, then $Bp = BTg = 0$ outside J, $STg = 0$ outside J, and we can apply the above relation to Tg, obtaining

$$(UTg)(\lambda) = \int_J E^*(t, \lambda)S(t)Tg(t)dt .$$

Since

$$(UTg)(\lambda) = \lambda(Ug)(\lambda) = \int_J E^*(t, \lambda)S(t)g(t)dt ,$$

we obtain

$$\int_J E^*(t, \lambda)S(t)[Tg(t) - \lambda g(t)]dt = 0 , \tag{14}$$

when λ does not belong to a set N_g of measure zero, with N_g dependent on g. The same is true for a sequence g_j of functions when λ does not belong to the null set $N = \bigcup_{i=1}^{\infty} N_{g_j}$. We choose the sequence g_j dense in $D \cap C_0^1(J)$, and then (14) holds for all $g \in D \cap C_0^1(J)$ if $\lambda \notin N$. We let $E(t, \lambda) = 0$ for $\lambda \in N$, and then (14) holds for all λ. Since $S = C^*B$, (14) yields

$$\int_J E^*(t, \lambda)C^*(t)[B(t)Tg(t) - \lambda B(t)g(t)]dt = 0 ,$$

or

$$\int_J E^*(t, \lambda)C^*(t)[A_0(t)g'(t) + A(t)g(t) - \lambda B(t)g(t)]dt = 0 .$$

Thus the columns of $C(t)E(t, \lambda)$ are weak solutions of (1) on J. It follows from a well-known theorem on weak solutions of partial differential equations that the columns of $C(t)E(t, \lambda)$ have locally square-integrable derivatives with respect to t which are continuous after correction on a null set for each λ, and that each column is a solution of (1). This theorem is easily proved using the properties of $k(t, s, \lambda)$. Since $C(t)$ is non-singular, the columns of $E(t, \lambda)$ are improper eigenfunctions of (1).

The matrix E depends on the compact subinterval J. Let E' be another matrix with the same properties, corresponding to an interval $J' \supseteqq J$. Then

$$\int_J [E^*(t, \lambda) - E'^*(t, \lambda)]S(t)g(t)dt = 0$$

for almost all λ, independent of $g \in C_0^1(J)$. It follows that $S(t)E(t, \lambda) - S(t)E'(t, \lambda) = 0$ for λ outside some null set. For λ in this null set we redefine $E(t, \lambda) = E'(t, \lambda) = 0$. The columns of $E(t, \lambda) - E'(t, \lambda)$ satisfy $Bu = 0$. At the same time, since E and E' are eigenfunctions of (1), they satisfy $A_0u' + Au = \lambda Bu = 0$. By hypothesis (iii) in the definiteness of (1), $E(t, \lambda) = E'(t, \lambda)$ on J for all λ. By taking a sequence of compact subintervals J tending to I, we can extend E uniquely to a matrix function defined for $t \in I$ and all λ.

If λ_0 is an eigenvalue of T, then σ has a jump, which we may assume to be a jump of 1, at λ_0. We choose $F = 0$ except at λ_0, and $F_j(\lambda_0) = \delta_{jk}$ for any fixed index $k \leqq \nu(\lambda_0)$. Then $F \in L^2(\sigma, \nu)$ and

$$(U^{-1}F)(t) = \int_R E(t, \lambda)F(\lambda)d\sigma(\lambda) = E_k(t, \lambda_0) ,$$

the kth column of $E(t, \lambda_0)$, an element of H. Thus the columns of $E(t,\lambda_0)$ are proper eigenfunctions of T if λ_0 is an eigenvalue of T.

The inversion formulae (11), (12), obtained for functions f in D_T which vanish identically outside a compact subinterval J, can be extended to all functions in D_T by a standard density argument. They are valid with the integrals converging to the functions in the norms of the appropriate Hilbert spaces. These formulae give the expansion of an arbitrary function $f \in D_T$ in eigenfunctions of the system of differential equations (1). The proof of Theorem 1 is now complete.

To prepare for the next section, we write the expansion formulae in a different form. Let $\Phi(t, \lambda)$ be a fundamental matrix solution of (1), with each element analytic in λ for fixed t. The matrix $E(t, \lambda)$ can be expressed in terms of $\Phi(t, \lambda)$ by

$$E(t, \lambda) = \Phi(t, \lambda)R(\lambda) , \tag{15}$$

where $R(\lambda)$ is a matrix with n rows and $\nu(\lambda)$ columns whose elements are functions of λ only. With the use of (15), the Parseval equality (13) takes the form

$$\begin{aligned} \|f\|^2 &= \int_R F^*(\lambda)F(\lambda)d\sigma(\lambda) \\ &= \int_R \left[\int_I f^*(t)S(t)E(t, \lambda)dt\right]\left[\int_I E^*(s, \lambda)S(s)f(s)ds\right]d\sigma(\lambda) \\ &= \int_R \left[\int_I f^*(t)S(t)\Phi(t, \lambda)R(\lambda)dt\right]\left[\int_I R^*(\lambda)\Phi^*(s, \lambda)S(s)f(s)ds\right]d\sigma(\lambda) \\ &= \int_R (Vf)^*(\lambda)R(\lambda)R^*(\lambda)(Vf)(\lambda)d\sigma(\lambda) , \end{aligned}$$

where

$$(Vf)(\lambda) = \int_I \Phi^*(t, \lambda)S(t)f(t)dt . \tag{16}$$

The formula

$$d\rho(\lambda) = R(\lambda)R^*(\lambda)d\sigma(\lambda) \tag{17}$$

defines a Hermitian positive semi-definite $n \times n$ matrix, called a spectral matrix. Let H^* be the Hilbert space of all complex-valued n-dimensional vector functions $F(\lambda)$ such that

$$\int_R \sum_{k=1}^{n} \bar{F}_j(\lambda)F_k(\lambda)d\rho_{jk}(\lambda) = \int_R F^*(\lambda)d\rho(\lambda)F(\lambda) < \infty ,$$

with inner product

$$(F, G) = \int_R G^*(\lambda) d\rho(\lambda) F(\lambda) .$$

Then (16) defines a unitary mapping of H onto H^* which diagonalizes T. A straightforward computation gives

$$(V^{-1}F)(t) = \int_R \Phi(t, \lambda) d\rho(\lambda) F(\lambda) . \tag{18}$$

2. Green's function and the spectral matrix. Let T be a self-adjoint extension of L as in §1, and let $R_\lambda = (T-\lambda)^{-1}$, for $Im\lambda \neq 0$, be the resolvent of T, a bounded operator in H.

THEOREM 2. *There exists an $n \times n$ matrix $G(t, s, \lambda)$ defined for $t, s \in I$, $Im\lambda \neq 0$, such that*

$$S(t)R_\lambda f(t) = \int_J G(t, s, \lambda) S(s) f(s) ds , \tag{19}$$

where J is a compact subinterval of I, $t \in J$, and $f \in C_0^1(J)$. This matrix G, called the Green's matrix of the operator T, has the following properties:

(i) G is analytic in λ for fixed t, s and $Im\lambda \neq 0$, is continuous in (t, s) on $I \times I$ for fixed λ except on the diagonal $t = s$

(ii) $G(s+0, s, \lambda) - G(s-0, s, \lambda) = E$ for $s \in I$, $Im\lambda \neq 0$

(iii) $G(t, s, \lambda)S(s) = S(t)G^*(s, t, \bar{\lambda})$

(iv) considered as functions of t, the columns of G satisfy (1) if $t \neq s$

(v) G is uniquely determined by T

(vi) if $f \in C_0^1(I)$, then $S(t)f(t) = \int_I G(t, s, \lambda) S(s)(T-\lambda) f(s) ds$.

Proof. (cf. [7], p. 14). If $f \in C_0^1(J)$, $g \in C_0^1(I)$, then

$$\begin{aligned}[f, g] &= \int_I g^*(t) S(t) f(t) dt = [R_\lambda f, (T-\lambda) g] \\ &= \int_J [(T-\lambda) g(t)]^* S(t) R_\lambda f(t) dt ,\end{aligned} \tag{20}$$

by (6) and the definition of the resolvent. We make use of a matrix $k(t, s, \lambda)$ as in Lemma 2. Let s_0 be any point of J, V a neighbourhood of s_0 whose closure is contained in J, and θ_V a real scalar function in $C_0^1(J)$ which is equal to 1 on V. For $t \in I$, $s \in V$, define

$$p(t, s) = (T_t - \lambda)[k(t, s, \lambda)(1 - \theta_V(t))] , \tag{21}$$

the subscript t indicating that the operator is applied to $k(t, s, \lambda)(1-\theta_V(t))$ considered as a function of t for fixed s. The result of application of an operator to a matrix will be understood as the matrix whose columns are obtained by applying the operator to the columns of the original matrix. For fixed $s \in V$, $p(\ ,s)$ vanishes except on a "ring" contained in $J - V$, and the matrix function $p(t, s)$ is continuous on $I \times V$. Consider $v(t) = \int_J p^*(s, t)S(s)R_\lambda f(s)ds$. If $g \in C_0^1(V)$, then

$$\int_V g^*(t)v(t)dt = \int_V g^*(t)\left[\int_J p^*(s, t)S(s)R_\lambda f(s)ds dt\right.$$
$$= \int_J\left[\int_V g^*(t)p^*(s, t)dt\right]S(s)R_\lambda f(s)ds\ , \tag{22}$$

However, if $u(s, g) = \int_J k(s, t, \lambda)g(t)dt$, then

$$\begin{aligned}\int_V p(s, t)g(t)dt &= \int_V (T_s - \lambda)k(s, t, \lambda)g(t)dt \\ &\quad - \int_V (T_s - \lambda)k(s, t, \lambda)\theta_V(s)g(t)dt \\ &= g(s) - (T_s - \lambda)\theta_V(s)u(s, g)\ ,\end{aligned}$$

using the properties of k and (21). Substituting in (22),

$$\begin{aligned}\int_V g^*(t)v(t)dt &= \int_J g^*(s)S(s)R_\lambda f(s)ds - \int_J [(T_s - \lambda)\theta_V(s)u(s, g)]^*S(s)R_\lambda f(s)ds \\ &= \int_J g^*(s)S(s)R_\lambda f(s)ds - \int_J \theta_V(s)u^*(s, g)S(s)f(s)ds \\ &= \int_J g^*(s)S(s)R_\lambda f(s)ds - \int_J\int_J \theta_V(s)g^*(t)k^*(s, t, \lambda)S(s)f(s)ds\ ,\end{aligned}$$

using (20) and the definition of $u(s, g)$. Since this holds for all $g \in C_0^1(V)$, we obtain $S(t)R_\lambda f(t) = v(t) + \int_J \theta_V(s)k^*(s, t, \lambda)S(s)f(s)ds$ for almost all $t \in V$. If $k_1(s, t, \lambda) = R_\lambda^* p(s, t)$, then

$$\begin{aligned}v(t) &= \int_J p^*(s, t)S(s)R_\lambda f(s)ds = [R_\lambda f, p(\ , t)] = [f, R_\lambda^* p(\ , t)] \\ &= [f, k_1(\ , t, \lambda)] = \int_J k_1^*(s, t, \lambda)S(s)f(s)ds\ ,\end{aligned}$$

using the definition of the adjoint operator R_λ^*. It is clear that $k_1(s, t, \lambda)$ is continuous on $V \times I$ for fixed λ, $Im\lambda \neq 0$, since R_λ^* is the inverse of a differential operator. Now

$$S(t)R_\lambda f(t) = \int_J [k_1^*(s, t, \lambda) + \theta_V(s)k^*(s, t, \lambda)]S(s)f(s)ds\ .$$

and the definition

$$G(t, s, \lambda) = k_1^*(s, t, \lambda) + \theta_V(s)k^*(s, t, \lambda) \tag{23}$$

yields (19). As this can be done for any $s_0 \in J$, (19) holds for all $t, s \in J$. The analogue of this result in [3] is proved incorrectly, as has been pointed out to the author by Professor M. H. Stone. A correct proof can be given essentially following the argument used here. The matrix G depends on the interval J, but is uniquely determined by J. If J' is another compact subinterval of I which contains J, and G' is the corresponding matrix, it is easy to see that $G(t, s, \lambda) = G'(t, s, \lambda)$ for t, $s \in J$, $Im\lambda \neq 0$. Thus, by taking a sequence of compact subintervals J tending to I, we can extend G uniquely to a matrix function defined for $t, s \in I$.

The remainder of the proof consists of the verification of the properties of the Green's matrix. The property (vi) follows immediately from the definition of the resolvent and (19). Since $R_\lambda^* = R_{\bar\lambda}$, $[R_\lambda f, g] = [f, R_{\bar\lambda} g]$ for any $f, g \in C_0^1(I)$. Then

$$\int_I g^*(t)S(t)R_\lambda f(t)dt = \int_I [R_{\bar\lambda} g(s)]^* S(s)f(s)ds = \int_I [S(s)R_{\bar\lambda} g(s)]^* f(s)ds ,$$

and, using (19), this yields

$$\int_I \int_I g^*(t)[G(t, s, \lambda)S(s) - S(t)G^*(s, t, \bar\lambda)]f(s)dsdt = 0 .$$

Since this holds for all $f, g \in C_0^1(I)$, we obtain

$$G(t, s, \lambda)S(s) = S(t)G^*(s, t, \bar\lambda) , \tag{24}$$

which is property (iii), for almost all $s, t \in I$. As $k_1(s, t, \lambda)$ is continuous, (23) shows that $G(t, s, \lambda)$ has the same analytic behaviour as $k^*(s, t, \lambda)$, in particular the same discontinuity at $s = t$, and the properties (i) and (ii) follow from Lemma 2 of §1. In view of the continuity of the matrices involved, (24) must actually be true for all $s, t \in I$. To prove (iv), we begin with (vi), written as

$$\begin{aligned} S(t)f(t) &= \int_I G(t, s, \bar\lambda)C^*(s)\left[A_0(s)\frac{d}{ds} + A(s) - \bar\lambda B(s)\right]f(s)ds \\ &= \int_I [C(s)G^*(t, s, \bar\lambda)]^*\left[A_0(s)\frac{d}{ds} + A(s) - \bar\lambda B(s)\right]f(s)ds . \end{aligned}$$

using the definition $S = C^*B$. Application of (5) yields

$$\begin{aligned} S(t)f(t) = &- \int_I \left[\left(A_0(s)\frac{d}{ds} + A(s) - \lambda B(s)\right)\right. \\ &\left. \times\, G^*(t, s, \bar\lambda)\right]^* A_0^{*-1}(s)C^*(s)A_0(s)f(s)ds . \end{aligned} \tag{25}$$

Since (25) is true for all $f \in C_0^1(I)$, the columns of $G^*(t, s, \bar\lambda)$, considered as functions of s, satisfy (1) for $t \neq s$. This, together with (24), proves (iv).

If there were two Green's matrices for $Im\lambda \neq 0$, their difference would be continuous everywhere and would be an eigenfunction of the operator T. As the spectrum of the self-adjoint operator T is real, this is impossible, and the Green's matrix is therefore unique. This completes the proof of Theorem 2.

Now we express the Green's matrix in terms of the fundamental matrix solution $\Phi(t, \lambda)$ of (1) introduced at the end of §1. From the properties of the Green's matrix, it is easy to deduce that G may be written

$$G(t, s, \lambda) = S(t)\Phi(t, \lambda)P^{+}(\lambda)\Phi^{*}(s, \bar{\lambda}) \qquad (s \geqq t) \tag{26}$$
$$G(t, s, \lambda) = S(t)\Phi(t, \lambda)P^{-}(\lambda)\Phi^{*}(s, \bar{\lambda}) \qquad (s \leqq t) .$$

The matrices P^{+} and P^{-} are analytic in λ except possibly on the real axis, and $P^{-*} = P^{+}$. We define the matrix $P = \frac{1}{2}(P^{+} + P^{-})$, and then P is analytic for $Im\lambda \neq 0$ and Hermitian.

THEOREM 3 (Titchmarsh-Kodaira formula). *The Green's matrix G of T is related to the spectral matrix ρ associated with the fundamental matrix solution Φ of* (1) *by the formula*

$$P(\mu) = \int_{-\infty}^{\infty} d\rho(\lambda)/(\lambda - \mu) , \tag{27}$$

where P is as defined above, and (27) *is to be taken in the sense that $P(\mu) - \int_{-N}^{N} d\rho(\lambda)/(\lambda - \mu)$ is analytic across the real axis on the interval $(-N, N)$.*

Proof. Let $f \in D_T$, $F = Vf$. Then, by (18),

$$f(t) = \int_{R} \Phi(t, \lambda)d\rho(\lambda)F(\lambda) .$$

Let

$$u(t) = \int_{R} \Phi(t, \lambda)d\rho(\lambda)F(\lambda)/(\lambda - \mu) .$$

Then

$$A_0u' + Au - \mu Bu = \int_{R} \lambda B(t)\Phi(t, \lambda)d\rho(\lambda)F(\lambda)/(\lambda - \mu)$$
$$- \int_{R} \mu B(t)\Phi(t, \lambda)d\rho(\lambda)F(\lambda)/(\lambda - \mu) = B(t)f(t) ,$$

or $u = R_{\mu}f$. Thus

$$\mu(Vu)(\lambda) = \mu\int_{I} \Phi^{*}(t, \lambda)S(t)u(t)dt = \int_{I} \Phi^{*}(t, \lambda)C^{*}(t)\mu B(t)u(t)dt$$

$$= \int_I \Phi^*(t, \lambda)C^*(t)[A_0(t)u'(t) + A(t)u(t) - \mu B(t)f(t)]dt$$
$$= \int_I \Phi^*(t, \lambda)C^*(t)[A_0(t)u'(t) + A(t)u(t)]dt - (Vf)(\lambda)$$
$$= V(Tu)(\lambda) - (Vf)(\lambda) = \lambda(Vu)(\lambda) - (Vf)(\lambda) ,$$

using (16), $u = R_\mu f$, and the fact that V diagonalizes T. Thus $(\lambda - \mu)(Vu)(\lambda) = (Vf)(\lambda)$. Applying the Parseval equality to u and f, $[u, f] = \int_R (Vf)^*(\lambda)d\rho(\lambda)(Vu)(\lambda) = \int_R F^*(\lambda)d\rho(\lambda)F(\lambda)/(\lambda - \mu)$, which is $F^*(\mu)\left[\int_{-N}^{N} d\rho(\lambda)/(\lambda - \mu)\right]F(\mu)$ plus a matrix which is analytic unless μ is real and $|\mu| \geqq N$. On the other hand, $S(t)u(t) = \int_I G(t, s, \mu)S(s)f(s)ds$, and $[u, f] = \int_I\int_I f^*(t)G(t, s, \mu)S(s)f(s)dsdt$, which, using (26), is equal to $F^*(\mu)P(\mu)F(\mu)$ plus an analytic function. Letting f run through a dense subset of H, which means, that F runs through a dense subset of H^*, we conclude that $P(\mu) - \int_{-N}^{N} d\rho(\lambda)/(\lambda - \mu)$ is analytic unless μ is real and $|\mu| \geqq N$.

Another form of the Titchmarsh-Kodaira formula is

$$\rho(\lambda) = \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_\delta^{\lambda+\delta} [P(\mu + i\varepsilon) - P(\mu - i\varepsilon)]d\mu ,$$

with ρ normalized to be continuous from the right and $\rho(0) = 0$, and with the formula interpreted in the same way as (27). The proof is exactly the same as the corresponding proof in [3], a straightforward inversion.

3. **Boundary conditions.** Let D_0 be the set of functions f in H such that $A_0 f' + Af$ exists almost everywhere on I and such that (7) is satisfied for some p in H. Let T_0 be the operator in H with domain D_0 defined by $T_0 f = p$ for $f \in D_0$, p as in (7). We assume that T_0 has at least one self-adjoint restriction. Let R_λ be the resolvent of some self-adjoint restriction of T_0, so that

$$S(t)R_\lambda f(t) = \int_I G(t, s, \lambda)S(s)f(s)ds ,$$

for $f \in H$, $Im\lambda \neq 0$. Then R_λ is a bounded operator for $Im\lambda \neq 0$, mapping H into D_0, whose adjoint is $R_{\bar\lambda}$. Let $\varepsilon(\lambda)$ be the eigenspace of T_0 corresponding to the value λ, the set of all solutions in D_0 of the differential system (1).

LEMMA 3. *T_0 is a closed operator whose domain consists of all $f \in H$ of the form $f = R_\lambda h + w$, where $h \in H$, $w \in \varepsilon(\lambda)$, for any λ with $Im\lambda \neq 0$.*

Proof. Since R_λ maps H into D_0 and $\varepsilon(\lambda)$ is containined in D_0, it is clear that every f of this form belongs to D_0. Conversely, suppose $f \in D_0$ is given. Let $h = T_0 f - \lambda f$, $w = f - R_\lambda h$. Then

$$T_0 w = T_0 f - T_0 R_\lambda h = T_0 f - \lambda R_\lambda h - h = T_0 f - h - \lambda(h - w) = \lambda w ,$$

and thus $w \in \varepsilon(\lambda)$, while $f = R_\lambda h + w$. If f is written in this way, $T_0 f - \lambda f = h$. If f_k is a sequence in D_0 such that $f = \lim f_k$ and $f^* = \lim T_0 f_k$ exist, we can write $f_k = R_\lambda(T_0 f_k - \lambda f_k) + w_k$, and deduce that $w = \lim w_k$ exists and belongs to $\varepsilon(\lambda)$. Letting $k \to \infty$, we obtain $f = R_\lambda(f^* - \lambda f) + w$, which implies $f \in D_0$ and $T_0 f = f^*$. This proves that T_0 is closed.

Since T_0 is closed and its domain D_0 is dense in H, T_0 has a closed adjoint T_0^* whose domain D_0^* is dense in H. Also, $T_0 = T_0^{**} = (T_0^*)^*$. For any subspace M of H, we let $H - M$ denote the orthogonal complement of M in H.

LEMMA 4. *D_0^* consists of all $g \in D_0$ of the form $g = R_\lambda z$, where $z \in H - \varepsilon(\bar{\lambda})$. The operator T_0^* is a restriction of T_0 and is closed and symmetric.*

Proof. $g^* = T_0^* g$ means

$$[T_0 f, g] = [f, g^*] \tag{28}$$

for every $f \in D_0$. By Lemma 3, any $f \in D_0$ may be written $f = R_{\bar{\lambda}} h + w$, with $h \in H$, $w \in \varepsilon(\bar{\lambda})$, and then $T_0 f = \bar{\lambda} f + h$. Substitution in (28) gives

$$[R_{\bar{\lambda}} h + w, g^*] = [\bar{\lambda} f + h, g] = [\bar{\lambda} R_{\bar{\lambda}} h + \bar{\lambda} w + h, g] ,$$

or

$$[h, \lambda R_{\bar{\lambda}}^* g + g - R_{\bar{\lambda}}^* g^*] + [w, \lambda g - g^*] = 0$$

for all $h \in H$, $w \in \varepsilon(\bar{\lambda})$. Then $g^* - \lambda g = z$ is orthogonal to $\varepsilon(\bar{\lambda})$, or $z \in H - \varepsilon(\bar{\lambda})$, and $g = R_{\bar{\lambda}}^*(g^* - \lambda g) = R_\lambda z$. Since R_λ maps H into D_0, g belongs to D_0. Thus $D_0^* \subseteqq D_0$. As it is assumed that there exists a self-adjoint restriction T of T_0 with domain D_T, $D_0 \supseteqq D_T \supseteqq D_0^*$, and since T is symmetric, its restriction T_0^* is also symmetric.

As we have seen in Lemma 1,

$$[T_0 f, g] - [f, T_0 g] = g^*(a) A_0^*(a) C(a) f(a) - g^*(b) A_0^*(b) C(b) f(b)$$

for $f, g \in D_0$. Here, $g^*(t) A_0^*(t) C(t) f(t)$ is a bilinear form in f, g which is non-degenerate for all $t \in I$ and skew-Hermitian. We define

$$\langle f g \rangle = g^*(a) A_0^*(a) C(a) f(a) - g^*(b) A_0^*(b) C(b) f(b) .$$

A homogeneous boundary condition is a condition on $f \in D_0$ of the form $\langle f\alpha\rangle = 0$, where α is a fixed function in D_0 The conditions

$$\langle f\alpha_j\rangle = 0, \qquad (j = 1, \cdots, p) \tag{29}$$

are said to be linearly independent if the only set of complex numbers $\gamma_1, \cdots, \gamma_p$ for which $\sum_{j=1}^{p} \gamma_j \langle f\alpha_j\rangle = 0$ identically in $f \in D_0$ is $\gamma_1 = \cdots = \gamma_p = 0$. Since $[T_0 f, g] - [f, T_0^* g] = \langle fg\rangle$ for $f \in D_0, g \in D_0^*$, it is easily seen that these boundary conditions are linearly independent if and only if the functions $\alpha_1, \cdots, \alpha_p$ are linearly independent $(\mathrm{mod}\, D_0^*)$. A set of p linearly independent boundary conditions (29) is said to be self-adjoint if $\langle \alpha_j \alpha_k\rangle = 0$ for $j, k = 1, \cdots, p$. Two sets of boundary conditions are said to be equivalent if the sets of functions satisfying the two sets of conditions are identical.

The assumption that T_0^* has a self-adjoint extension is equivalent to the assumption that the linear spaces $\varepsilon(i)$ and $\varepsilon(-i)$ have the same dimension τ, the defect index of T_0^*. By exactly the same proof as that used in [3], originally used in [4], we can obtain the following relation between self-adjoint extensions of T_0^* and boundary conditions.

THEOREM 4. *If T is a self-adjoint extension of T_0^** (or, equivalently, restriction of T_0) *with domain D_T, then there exists a self-adjoint set of τ linearly independent boundary conditions such that D_T is the set of all $f \in D_0$ satisfying these conditions. Conversely, corresponding to a self-adjoint set of τ linearly independent boundary conditions, there exists a self-adjoint extension T of T_0^* whose domain D_T is the set of all $f \in D_0$ satisfying these boundary conditions.*

4. **Examples.** The results of this paper include as a special case the corresponding results for a single differential equation of arbitrary order as obtained in [3]. For simplicity, we consider only equations of even order with real coefficients. Let L and M be formally self-adjoint linear differential operators of orders $2r$ and $2s$ respectively $(r > s)$. Then L and M can be written

$$Lu = \sum_{i=0}^{r} [p_{r-i} u^{(i)}]^{(i)}, \qquad Mu = \sum_{i=0}^{s} [q_{s-i} u^{(i)}]^{(i)},$$

where p_{r-i}, q_{s-i} are real functions having continuous derivatives up to order i on I. We assume $p_0 \neq 0$ on I. It is not difficult to verify, as suggested in ([5], p. 206, problem 19), that the differential equation $Lu = \lambda Mu$ is equivalent to a system (1). If we let x be the vector with components $(x_1, \cdots, x_{2r})$, with

$$x_j = u^{(j-1)} [j = 1, \cdots, r], \; x_{r+j} = (-1)^j [p_{r-j} u^{(j)} + (p_{r-j-1} u^{(j+1)})'$$

$$+ \cdots + (p_0 u^{(r)})^{(r-j)}] + (-1)^{j+1}[q_{s-j}u^{(j)} + (q_{s-j-1}u^{(j+1)})'$$
$$+ \cdots + (q_0 u^{(s)})^{(s-j)}],\ [j = 1, \cdots, r]\ ,$$

understanding zero for any expression q_{-k}, $k > 0$, we obtain the system

$$
\begin{aligned}
&- x'_{r+1} + p_r x_1 = \lambda q_s x_1 \\
&- x'_{r+2} - p_{r-1}x_2 - x_{r+1} = - \lambda q_{s-1}x_2 \\
&\qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
&- x'_{r+s} - (-1)^s p_{r+1-s}x_s + x_{r+s-1} = (-1)^s \lambda q_0 x_s \\
&\qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
&- x'_{2r} - (-1)^r p_1 x_r + x_{2r-1} = 0 \\
&x'_1 - x_2 = 0 \\
&x'_2 - x_3 = 0 \\
&\quad \cdot \quad \cdot \quad \cdot \\
&x'_{r-1} - x_r = 0 \\
&x'_r - (-1)^r x_{2r}/p_0 = 0\ ,
\end{aligned}
\tag{30}
$$

which is of the form (1), where

$$A = \begin{pmatrix} 0_r & -E_r \\ E_r & 0_r \end{pmatrix}, \qquad A = \begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix},$$

$$P = \begin{pmatrix} p_r & & & & \\ & -p_{r-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & (-1)^{r-1}p_1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \cdots 0 \\ 1 \cdots 0 \\ \cdot \quad\ \ \cdot \\ 0 \cdot\cdot\ 10 \\ 0 \cdots 1 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 \cdots 0 \\ \cdot \cdots\cdot \\ 0 \cdots (-1)^r/p_0 \end{pmatrix}, \qquad B = \begin{pmatrix} q_s & & & & & \\ & -q_{s-1} & & & & \\ & & \cdot & & & \\ & & & (-1)^s q_0 & & \\ & & & & \cdot & \\ & & & & & 0 \end{pmatrix}$$

E_r denoting the r-dimensional unit matrix, 0_r the r-dimensional zero matrix, and all elements not shown being zero. It is an immediate consequence of (31) that the system (30) is its own adjoint. The set of functions D may be regarded as the set of scalar functions with $2r$ continuous derivatives on I which vanish identically outside some compact subinterval of I, the condition (7) being no restriction. The norm is given by

$$||f||^2 = \sum_{i=0}^{s} \int_I (-1)^i q_{s-i}(t) |f^{(i)}|^2 dt \, ,$$

and to make the problem definite in the sense of §1, we must assume $(-1)^i q_{s-i}(t) \geqq 0$ $(i = 0, 1, \cdots, s)$. With this restriction, we obtain the eigenfunction expansion theorem, the existence of the Green's function, the Titchmarsh-Kodaira formula, and the nature of the boundary conditions as in [3] from the results of this paper.

A problem which has arisen in relativistic quantum mechanics (cf. [6]) involves the pair of differential equations

$$(32) \qquad x_1' = q_1(t)x_2 + \lambda x_2, \qquad x_2' = -q_2(t)x_1 - \lambda x_1 \, ,$$

where q_1 and q_2 are real and continuous on $0 \leqq t < \infty$. This is of the form (1) with $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & -q_1 \\ q_2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The adjoint system is

$$(33) \qquad y_1' = q_2(t)y_2 + \lambda y_2, \qquad y_1' = -q_1(t)y_1 - \lambda y_1 \, ,$$

and (32) may be transformed into (33) by $y = Cx$ with $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $||f||^2 = \int_I (|f_1|^2 + |f_2|^2) dt$, where $f = (f_1, f_2)$. It can be determined that $\rho(\lambda) = \begin{pmatrix} \lambda/\pi & 0 \\ 0 & 0 \end{pmatrix}$. If $(u(x, \lambda), v(x, \lambda))$ is a solution of (32), the expansion formulae are

$$f_1(t) = \frac{1}{\pi} \int_R F_1(\lambda) u(t, \lambda) d\lambda, \; f_2(t) = \frac{1}{\pi} \int_R F_1(\lambda) v(t, \lambda) d\lambda,$$

$$F_1(\lambda) = \int_I [f_1(t) \bar{u}(t, \lambda) + f_2(t) \bar{v}(t, \lambda)] dt \, ,$$

with F_2 not appearing because ρ has rank 1. Possibly this approach can be used to prove the existence of eigenfunction expansions in more general applications, but its usefulness will be limited by the difficulty in computing the spectral matrix.

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