CONCERNING CERTAIN LOCALLY PERIPHERALLY SEPARABLE SPACES

L. B. TREYBIG

In 1954, F. Burton Jones raised the question [2] "Is every connected, locally peripherally separable [3], metric space separable?" In this paper it will be shown that there exists a connected, semi-locally-connected, space Σ satisfying R. L. Moore's axioms 0 and C[1], in which every region has a separable boundary, every pair of points is a subset of some separable continuum[2], and the set of all points at which Σ is not locally separable is separable. It will also be shown that every compactly connected, locally peripherally separable, metric space is completely separable.

PART 1

Let S' denote the set of all points of the Euclidean plane E. A square disk in E will be said to be horizontal if it has two horizonta sides. A point set in E will be called an *H-disk* only if that set is a horizontal square disk. By the *width* of a square disk will be meant the length of one of its sides.

Let K denote a definite H-disk of width d. Let $R_0(K)$ *denote the H-disk of* width $d/4$ whose center is on the vertical line that contains the center of K, and whose upper side lies at a distance of $d/16$ below the upper side of K. Let $R_{00}(K)$ and $R_{01}(K)$ denote the H-disks of width $d/8$ whose upper sides are at a distance of $d/32$ above the lower side of $R_0(K)$ and whose centers are on the vertical lines containing the left and right sides, respectively, of $R_0(K)$.

In general, for each positive integer n let $U_n(K)$ denote a collection of 2^n mutually exclusive congruent H-disks such that

(1) $R_{01}(K)$ and $R_{00}(K)$ are the elements of $U_1(K)$,

(2) if n is a positive integer and y is an element of $U_n(K)$, and x and z are H-disks of width $d/4(2)^{n+1}$ whose centers lie on the same vertical lines as the left and right sides of y, respectively, and whose upper sides lie at a distance of $d/32(2)^n$ above the lower side of y, then x and z are elements of $U_{n+1}(K)$.

If n is a positive integer and $R_{x_1x_2\cdots x_n}(K)$ is an elements of $U_n(K)$, then let the elements x and y of $U_{n+1}(K)$ whose centers lie on the same

Presented to the American Mathematical Society, June 15, 1957; received by the editors September 4, 1958. This paper is part of a dissertation submitted to the Graduate School of the University of Texas in partial fulfillment of the requirements for the Ph. D. degree.

[1] The proof that every space which satisfies axioms 0 and C is metric is due to R. L. Moore.

[2] A continuum is a connected, closed set.

vertical lines as the left and right sides of $R_{x_1x_2\cdots x_n}(K)$, respectively, be denoted by $R_{x_1x_2\cdots x_n0}(K)$ and $R_{x_1x_2\cdots x_n1}(K)$, respectively. Let $C(K)$ be a collection to which x belongs if and only if x is $R_0(K)$ or in one of the collections $U_1(K), U_2(K), \cdots$.

Let $L(K)$ denote the H-disk of width $d/8$ whose center is on the same vertical line as the center of K, and whose lower side is at a distance of $3d/16$ above the lower side of K. Let $P_l(K)$ and $P_r(K)$ denote the left and right-hand end points, respectively, of the lower side of $L(K)$. Let $M(K)$ denote the point set such that a point P belongs to it if and only if P is a point of the interval $P_l(K)P_r(K)$ such that there is no nonnegative integer p and positive integer q such that $PP_l(K)/P_l(K)P_r(K) = p/2^q$. Let $I(K)$ denote the collection to which x belongs if and only if x is a vertical interval containing a point of $M(K)$, and with both end points on the boundary of $L(K)$. Let an interval i of $I(K)$ be denoted by $i_x(I(K))$ if and only if it is true that if P is the lowest point of i, then $P_l(K)P/P_l(K)P_r(K) = x$.

Let R denote some definite H-disk. Let $R_0(R)$ be denoted by Q_0; let $R_{00}(R)$ and $R_{01}(R)$ be denoted by Q_{00} and Q_{01}, respectively. Let $R_{000}(R)$, $R_{001}(R)$, $R_{010}(R)$, and $R_{011}(R)$ be denoted by Q_{000}, Q_{001}, Q_{010}, and Q_{011}, respectively, and so forth. Let $C(R)$ be denoted by C_1 and let $I(R)$ be denoted by I_0.

Let C_2 denote the collection to which x belongs if and only if x is an element of $C(y)$, for some element y of C_1 distinct from Q_0. Let $R_0(Q_{00})$ be denoted by $Q_{00,0}$; let $R_{01}(Q_{00})$ be denote by $Q_{00,01}$. In general, let $R_x(Q_y)$ be denoted by $Q_{y,x}$. Also, if Q_x is in C_1 and $x \neq 0$, let $I(Q_x)$ be denoted by I_x.

In general, let C_{n+1} denote the collection to which x belongs if and only if x is an element of $C(y)$, for some elements y of C_n, which, in case x_n is 0, is distinct from $Q_{x_1,x_2,\cdots,x_n}$. Let the element $R_{x_{n+1}}[R_{x_n}[R_{x_{n-1}}[\cdots[R_{x_1}(R)]\cdots]]]$ of C_{n+1} be denoted by $Q_{x_1,x_2,\cdots,x_{n+1}}$. Also if w is the element $Q_{x_1,x_2,\cdots,x_n}$ of C_n and $x_n \neq 0$, then let $I(w)$ be denoted by $I_{x_1,x_2,\cdots,x_n}$. For each n let I_n be the collection to which x belongs if and only if there is an element $Q_{x_1,x_2,\cdots,x_n}$ of C_n such that $x_n \neq 0$ and x is in $I(Q_{x_1,x_2,\cdots,x_n})$.

Let W denote the point set to which a point P belongs if and only if P belongs to C_n^*[3] for each positive integer n. For each positive integer n let B_n denote the collection of all the boundaries of the elements of C_n. The boundary of $Q_{x_1,x_2,\cdots,x_n}$ will be denoted by $J_{x_1,x_2,\cdots,x_n}$.

Let S denote $[I_0^* + I_1^* + \cdots] + [B_1^* + B_2^* + \cdots] + W$.

Let C' be a collection to which w belongs if and only if w is R or $I(w)$ is a subset of S and there is a positive integer n such that w is in C_n.

[3] C_n^* Means the sum of all the point sets of the collection C_n.

For each positive integer n let H_n denote a collection to which x belongs if and only if x is the common part of S and the interior of some square of $[B_n + B_{n+1} + \cdots]$. For each element $Q_{x_1,x_2,\cdots,x_n}$ of C_n, let the set of all points of S in the interior of $J_{x_1,x_2,\cdots,x_n}$ be denoted by $r_{x_1,x_2,\cdots,x_n}$.

For each positive integer n let K_n denote a collection to which x belongs if and only if, either (1) x is a segment of an arc lying on some square J of $(B_1 + B_2 + \cdots)$, having length less than $1/4^n$ times the perimeter of J, and intersecting no square of the collection $(B_1 + B_2 + \cdots)$ except J, or (2) x is the sum of two straight line segments p and q intersecting at their midpoints and lying on different squares J_p and J_q of $(B_1 + B_2 + \cdots)$, such that p and q each have length less than $1/4^n$ times the perimeters of J_p and J_q, respectively, and such that neither p nor q intersects three squares of $(B_1 + B_2 + \cdots)$.

Suppose x is a positive number such that $i_x[I_{j_1,j_2,\cdots,j_n}]$ is an interval of $I_{j_1,j_2,\cdots,j_n}$. For each positive integer n there exists a unique pair (k_n, x_n) such that k_n is a non-negative integer, x_n is a positive number less than one, and $x = (k_n + x_n)/2^n$. By $i_n[i_x(I_{j_1,j_2,\cdots,j_n})]$ will be meant the vertical interval $i_{x_n}(I(y))$, where y is the H-disk of $U_n[Q_{j_1,j_2,\cdots,j_n}]$ with only k_n disks of $U_n(Q_{j_1,j_2,\cdots,j_n})$ to the left of it.

Suppose, for some y in C', P is the highest point of the interval $i_x(I(y))$. By $R_n(P)$ will be meant the sum of all the sects z such that either

(1) for some positive integer d greater than or eqal to n, z is the subset of $i_d[i_x(I(y))]$ with length $1/2^n$ times the length of $i_d[i_x(I(y))]$ that contains the lowest point of $i_d[i_x(I(y))]$, or

(2) z is the subset of $i_x[I(y)]$ with length $1/2^n$ times the length of $i_x(I(y))$ that contains the highest point of $i_x(I(y))$.

For each positive integer n let L_n denote a collection such that x belongs to it if and only if there exists a positive integer d greater than or equal to n, an element y of C', and an interval of the collection $I(y)$ such that if P denotes the highest point of that interval, then $x = R_d(P)$.

For each positive integer n let N_n denote a collection to which x belongs if and only if either

(1) for some element y of C' there exists an interval i of the collection $I(y)$ such that x is a segment of i of length less than $1/2^n$ times the length of i, or

(2) for some element y of C' there exists an element i of $I(y)$ such that x is a sect lying in i, containing the lowest point of i and of length less than $1/2^n$ times the length of i.

For each positive integer n let G_n denote a collection to which x belongs if and only if it lies in $H_n + K_n + L_n + N_n$. S is the set of all points of Σ. A subset r of S is a *region* in Σ if and only if r belongs to G_1^4.

R. L. Moore's axioms 0 and C are as follows:

Axiom 0. Every region is a point set.

Axiom C. There exists a sequence $G_1, G_2, \cdots$ such that

(1) for each positive integer n, G_n is a collection such that each element of G_n is of region and G_n covers S,

(2) for each n, G_{n+1} is a subcollection of G_n,

(3) if A is a point, B is a point and R is a region containing A, then there exists a positive integer n such that if x is a region of G_n containing A and y is a region of G_n intersecting x, then

(a) y is a subset of R and

(b) if B is not A, y does not contain B,

(4) if $M_1, M_2, \cdots$ is a sequence of closed point sets such that for each n there exists a region g_n of G_n such that M_n is a subset of $\bar{g}_n$ and for each n M_n contains M_{n+1}, then there is a point common to all the point sets of this sequence.

It is obvious that in the space Σ each region has a countable, and therefore separable, boundary, and that the sequence $G_1, G_2, \cdots$ defined for the space Σ satisfies conditions (1) and (2) of axiom C. It will be shown that it also satisfies conditions (3) and (4) of this axiom.

Suppose that P is a point of W, that $r = r_{x_1, x_2, \cdots, x_n}$ is a region of H_n containing P, and that Q is a point of r distinct from P. If q is a region containing a point of W, then q must belong to H_1. Since each element of C_{n+1} which contains P has a side of length less than or equal 1/4 times the length of a side of $Q_{x_1, x_2, \cdots, x_n}$, and each element of C_{n+2} which contains P has a side of length less than or equal $1/4^2$ times the length of side of $Q_{x_1, x_2, \cdots, x_n}$, and so forth; it is obvious that there is a $d > n$ such that if q is a region of H_d which contains P, then $\bar{q}$ does not intersect Q and is a subset of r. Suppose that x and y are two intersecting regions of G_{n+1} such that x contains P. x belongs to H_{n+1} and is therefore a subset of r. Every region of G_{n+1} which intersects x is a subset of r, so clearly, y is a subset of r.

Now suppose that P is a point of $J_{x_1, x_2, \cdots, x_n}$ of B_n and r is a region containing P, and Q is a point of r distinct from P. There exists a circle J in E with center at P such that every point of S in the interior of J belongs to r, but Q is not in the interior of J. There exists a positive integer d such that $1/4^d$ times the perimeter of any square of $(B_1 + B_2 + \cdots)$ to which P belongs is less than the radius of J, and such that no region of H_d contains P. If R^1 is a region of G_{d+1} containing P, then $\bar{R}^1$ does not contain Q and is a subset of r. If $n > d + 2$

[4] The collection G_1 of regions is a basis for the space Σ.

and x and y are two intersecting regions of G_n such that x contains P, then $x+y$ is a subset of r.

Now suppose that P is a point of $i_x(I(y))$, for y in C', and that r is a region containing P and that Q is a point of r distinct from P.

Case 1. Suppose P is not the highest point of $i_x(I(y))$. There exists a segment t containing P, or a sect in case P is the lowest point of $i_x(I(y))$, such that t is a subset of r and does not contain Q nor the highest point of $i_x(I(y))$. There exists a positive number ε such that every point of $i_x(I(y))$ which is at a distance from P of less than ε lies in t. There exists a positive integer d such that

(1) no region of L_d intersects t and no region of H_d intersects $i_x(I(y))$, and

(2) $1/2^d$ times the length of $i_x(I(y))$ is less than ε. Therefore, if k is a region of G_{d+1} containing P, then $\bar{k}$ is a subset of r and does not contain Q. Also, if x and y are two intersecting regions of G_{d+2} such that x contains P, then $x+y$ is a subset of r.

Case 2. Suppose P is the highest point of $i_x(I(y))$. Whether Q belongs to $i_x(I(y))$ or there is a positive integer p such that Q belongs to $i_p[i_x(I(y))]$ or r is in H_1 and Q does not belong to $i_x(I(y)) + i_1[i_x(I))] + i_2[i_x(I(y))] + \cdots$, there is a positive integer d such that

(1) $R_d(P)$ does not contain Q and is a subset of r, and

(2) no region of H_d contains P. If k is a region of G_{d+1} containing P, then $\bar{k}$ is a subset of r and does not contain Q. Also, if x and y are two intersecting regions of G_{d+3} such that x contains P, then $x+y$ is a subset of r.

Therefore $G_1, G_2, \cdots$ satisfies the third part of axiom C.

Suppose that $M_1, M_2, \cdots$ is a sequence of closed point sets such that

(1) for each n M_n contains M_{n+1}, and

(2) for each n there is a region g_n of G_n such that M_n is a subset of $\bar{g}_n$.

In case, for each n, g_n is in H_n, then by definition of W, there is a point common to $M_1, M_2, \cdots$ because some point of W can be easily shown to be a limit point or point of M_n for each n.

In case there is a positive integer j such that g_j belongs to K_j, then for $n > j$, g_n belongs to K_n. But $M_j, M_{j+1}, \cdots$ is a sequence of closed and compact point sets such that for $n \geqq j$ M_n contains M_{n+1}. So there is a point common to $M_j, M_{j+1}, \cdots$ and thus common to $M_1, M_2, \cdots$.

In case there is a positive integer j such that g_j belongs to N_j, then for $n > j$, g_n belongs to N_n. So, for the same reason as in the previous case, there is a point common to $M_1, M_2, \cdots$.

The only case not considered is the one where there is a positive integer j_1 such that, for $n \geqq j_1$, g_n belongs to L_n. In this case g_{j_1} must be $R_{x_1}(P)$ for some point P and positive integer x_1. There is a positive integer $j_2 > j_1$ such that $g_{j_2} = R_{x_2}(P)$, where $x_2 > x_1$. There is a positive integer $j_3 > j_2$ such that $g_{j_3} = R_{x_3}(P)$, for $x_3 > x_2$, and so forth. P is common to the sets $R_{x_1}(P), R_{x_2}(P), \cdots$. But if P does not belong to each of the sets $M_{j_1}, M_{j_2}, \cdots$ then there is a positive integer d such that $\bar{R}_{x_d}(P)$ contains no point of M_{x_j} for any j. But $R_{x_d}(P)$ contains $M_{j_{d+1}}$. So P is common to the sets $M_{j_1}, M_{j_2}, \cdots$ and thus common to $M_1, M_2, \cdots$.

Thus, Σ satisfies the fourth part of axiom C.

In order to show that Σ is connected, an indirect argument will be used. Suppose that S is the sum of two mutually separated sets H and K. Since $W + (B_1^* + B_2^* + \cdots)$ is connected, let H' be the one of the sets H and K that contains this set and let K' be the other. There exists an element y of C' such that for some $x\, i_x[I(y)]$ is a subset of K'. But there exists a positive integer d_1 such that for $n \geqq d_1$, $i_n[i_x(I(y))]$, belongs to K'. There exists a positive integer d_2 such that for $n \geqq d_2\, i_n[i_{d_1}(i_x(I(y)))]$ belongs to K'. So, obviously, there is a positive integer sequence, $d_1, d_2, \cdots$ such that if j is a positive integer and $n \geqq d_j$, then $i_n(i_{d_{j-1}}(i_{d_{j-2}}(\cdots i_{d_1}(i_x(I(y)))\cdots)))$ belongs to K'. But from this fact it is easily seen that some point of W is a limit point of K'. So Σ is connected.

It has been shown that in any space satisfying axioms 0 and C (1) if M is a separable point set, M is completely separable, and (2) if M is separable, any subset of M is separable.

In order to show that any two points of S lie in a separable continuum, suppose first that P and Q are two points of S. Obviously, $(B_1^* + B_2^* + \cdots)$ is separable and connected, and therefore $W + (B_1^* + B_2^* + \cdots)$ is a separable continuum. In case P and Q both lie in $W + (B_1^* + B_2^* + \cdots)$, this continuum has the desired properties. In case P does not belong to this set, P belongs to $i_x[I(y)]$ for some y in C'. Let M_P be the set to which point R belongs if and only if, either

(1) there is a finite positive integer sequence $x_1, x_2, \cdots, x_n$ such that R belongs to $i_{x_1}[i_{x_2}[\cdots i_{x_n}[i_x(I(y))]\cdots]]$, or

(2) there is a positive integer q such that R belongs to $i_q[i_x(I(y))]$, or

(3) R belongs to $i_x[I(y)]$. $M_P + (B_1^* + B_2^* + \cdots) + W$ is a separable continuum. If Q does not belongs to this set, let M_Q be a set related to Q like M_P was related to P. The continuum $M_P + M_Q + (B_1^* + B_2^* + \cdots) + W$ is separable.

The statement that Σ is locally separable at the point P means that there is a region R containing P such that R is separable. Alexandroff [1] has shown that if β is a connected, locally completely separable, space satisfying axioms 0 and C, then β is completely separable. It is interesting to note that Σ is locally separable, and therefore locally completely separable, at each point except those of a separable set, and yet, Σ is not separable.

Σ is obviously locally separable at all points not belonging to W. Since every region that contains a point of W contains uncountably many mutually exclusive domains, Σ is not locally separable at any point of W. Furthermore $(B_1^* + B_2^* + \cdots)$ is separable, and so $\overline{(B_1^* + B_2^* + \cdots)}$ is separable, and thus, since W is a subset of the latter, W is separable.

Σ is said to be *semi-locally-connected* [5] at point P if and only if it is true that if R is a region containing P, R contains a region R' containing P such that $S - R$ does not intersect infinitely many components of $S - R'$. Σ is said to be semi-locally-connected if and only if Σ is semi-locally-connected at each point.

The space Σ is obviously semi-locally-connected because S minus any region has only a finite number of components.

PART 2

Suppose that Σ is a space satisfying the conditions specified on the first page of this paper.

For each positive integer j let G_j denote the collection of all open sets which have diameter less that j^{-1}.

Let P denote some definite point, and suppose n is a positive integer such that no countable subcollection of G_n covers S. Let R_n be some region of G_n which contains P, let $H_1 = \{R_n\}$, and let K_1 be the boundary of R_n.

For each point Q of S let $\Delta(Q)$ be the least integer $j > n$ such that some region $R(Q)$ of G_n contains every region of G_j that intersects a region of G_j that contains Q.

It has been shown that in a space satisfying these axioms if L is a separable point set and G is a collection of open sets covering L, then some countable subcollection of G covers L. Therefore, there is a countable point set T_1 dense in K_1 such that the collection H_2 of all $R(Q)$'s, for Q's in T_1, covers K_1. Let K_2 be the sum of the boundaries of all the sets in $H_1 + H_2$. There is a countable point set T_2 dense in K_2 such that the collection H_3 of all $R(Q)$'s for Q's in T_2, covers K_2. Let K_3 be the sum of the boundaries of the sets in $H_1 + H_2 + H_3$, and so forth.

There is a point B not in the closure of $H = (H_1 + H_2 + \cdots)^*$. Let M be a compact continuum containing P and B.

Case 1. Suppose some point A of $M - M \cdot H$ is a limit point of $K = K_1 + K_2 + \cdots$.

Let R_1' be a region of G_n containing A, let Q_1 be a point of $T = T_1 + T_2 + \cdots$ in R_1', and let x_1 be the largest integer i such that R_1' belongs to G_i. Let R_2' be a region of C_{x1+1} containing A such that $\bar{R}_2'$ lies in $R_1' - Q_1$. Let Q_2 be a point of T in R_2' and let x_2 be the largest integer i such that R_2' is in G_i. Obtain R_3', Q_3, and x_3 similarly, and so forth. $n \leqq x_1 < x_2 < x_3 < \cdots$. For each i, $\Delta(Q_i) > x_i$. Otherwise, for some i, $R(Q_i)$ would contain R_i', and thus A. However, there is a positive integer $t > n$ such that if x, y, and z are regions of G_t such that $x \cdot y$ and $y \cdot z$ exist and x contains A, then R_n contains $x + y + z$. For some $s > t$, $\Delta(Q_s) > t$. But R_n contains every region of G_t that intersects a region of G_t that contains G_s. So $\Delta(Q_s) \leqq t$, which is a contradiction.

Case 2. Suppose no point of $M - M \cdot H$ is a limit point of K. For each point Q of $M - M \cdot H$ let g_Q be a region containing Q such that g_Q contains no point of $K + P$. Some finite subcollection C of the g_Q's covers this set of limit points. Let $D = H - H \cdot \bar{C}^*$. Let C_1 be the component of $M - M \cdot \bar{D}$ which contains B. Some point z of $M \cdot \bar{D}$ is a limit point of C_1. But z lies in a region r of H, and therefore C_1 would intersect the boundary of r, and thus contain a limit point of K. This yields a contradiction.

Since, for each n, some countable subcollection of G_n covers S, Σ is completely separable.

REFERENCES

1. Paul Alexandroff, *Uber die Metrization der im Kleinen Kompakten topologischen Raume*, Mathematische Annalen, **92** (1924) 294–301.

2. F. B. Jones, *On a property related to separability in metric spaces*, Journal of the Elisha Mitchell Scientific Society, **70** (1954), 30-33.

3. ———, *A theorem concerning locally peripherally separable spaces*, Bull. Amer. Math. Soc. **41** (1935), 437–439.

4. R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloq. Publ., vol 13 1932.

5. G. T. Whyburn, *Semi-locally-connected sets*, Amer. J. Math. **61** (1939), 733–749.

THE UNIVERSITY OF TEXAS