PROJECTIONS ONTO THE SUBSPACE OF COMPACT OPERATORS

E. O. THORP

Introduction. The purpose of this paper is to establish the following theorem.

THEOREM. *Suppose U and V are Banach spaces and that there are bounded projections P_1 from U onto X and P_2 from V onto Y. Then there are no bounded projections from the space of bounded operators on U into V onto the closed subspace of compact operators, in the following cases:*

1. *X is isomorphic* [1] *to ℓ^p, $1 \leq p < \infty$; Y is isomorphic to ℓ^q, $1 \leq p \leq q \leq \infty$ or c_0 or c.*
2. *X is isomorphic to c_0; Y is isomorphic to ℓ^∞, c_0 or c.*
3. *X is isomorphic to c; Y is isomorphic to ℓ^∞.*

NOTATION. If X and Y are Banach spaces, $[X, Y]$ is the set of bounded linear operators from X into Y. ℓ^∞ is the set of bounded sequences with the sup norm.

A space X is said to have a countable basis if there is a countable subset of elements of X, called a basis, such that each $x \in X$ is uniquely expressible as

$$x = \sum_{i=1}^{\infty} \xi_i \varphi_i$$

in the sense that

$$\lim_{n\to\infty} \| x - \sum_{i=1}^{n} \xi_i \varphi_i \| = 0 .$$

If X and Y are spaces with countable bases (φ_i) and (ψ_i) respectively and A is a bounded linear transformation from X into Y, then A can be represented by an infinite matrix (a_{ij}), with

$$A\varphi_j = \sum_{i=j}^{\infty} a_{ij} \psi_i$$

[2]. In what follows, the basis used for ℓ^p will be given by $\varphi_j = (0, 0, \cdots, 0, 1, 0, 0, \cdots)$ where there is a 1 in the jth place and 0 elsewhere. Similarly for ψ_i. The matrix representations of operators will all be with respect to these bases.

Received April 29, 1959. The author thanks Professor Angus Taylor for proposing this problem and thanks both him and Professor Richard Arens for helpful discussions.

Proof of the theorem. The details of the proof are given below only for $X = \ell^p, 1 \leq p < \infty$, and $Y = \ell^q, 1 \leq p \leq q < \infty$. The proof for the remaining pairs is similar and is indicated in a remark at the end.

DEFINITION. Let E be the function on $[\ell^p, \ell^q], 1 \leq p \leq q < \infty$, which sends an operator whose matrix is (a_{ij}) into the operator whose matrix is $(a_{ij}\delta_{ij})$, i.e. the non-diagonal matrix elements are replaced by zero and the diagonal elements are unaltered.

LEMMA 1. *E is a projection with $\|E\| = 1$, range the diagonal operators, and null-space the operators with $a_{ii} = 0$, all i.*

Proof. E is additive and homogeneous as easily follows from [2]. $E^2 = E$, and the characterization of the range and null-spaces are apparent.

From the chain

$$\begin{aligned}\infty > \|A\| &= \sup_{\|x\|_p \leq 1} \|Ax\|_q \geq \sup_j \|A\varphi_j\|_q \\ &= \sup_j \|\sum_i a_{ij}\psi_j\|_q \geq \sup_j \|a_{jj}\psi_j\|_q = \sup_j |a_{jj}| \\ &\geq \sup_{\Sigma|\xi_i|^p \leq 1} (\sum |a_{ii}\xi_i|^p)^{1/p} \geq \sup_{\|x\|_p \leq 1} (\sum |a_{ii}\xi_i|^q)^{1/q} = \|EA\|,\end{aligned}$$

where the last $\geq$ is by Jensen's inequality, we see that E sends bounded operators into bounded operators and, further, $\|E\| = 1$. Also

$$\|EA\| \leq \sup_j |a_{jj}| .$$

In fact,

$$\|EA\| = \sup_j |a_{jj}|$$

because

$$\|EA\| \geq \sup_j \|EA\varphi_j\| = \sup_j |a_{jj}| .$$

LEMMA 2. *The mapping γ from the set of diagonal operators onto ℓ^∞ defined by $\gamma(a_{ii}) = (a_{11}, a_{22}, \cdots)$ is an isometry which carries the compact diagonal operators onto c_0.*

Proof. That γ is an isometry from the diagonal operators onto ℓ^∞ follows from the previous observation that $\|EA\| = \sup_j |a_{jj}|$. Hence it suffices to show that the compact diagonal operators are exactly those with the additional condition $\lim_i |a_{ii}| = 0$. This condition is necessary;

otherwise for some $\varepsilon > 0$ there is an infinite index set I such that $|a_{ii}| \geq \varepsilon$ whenever $i \in I$. Then the bounded sequence $(\varphi_i)_{i \in I}$ would be carried into the sequence $(a_{ii}\psi_i)_{i \in I}$, which has no convergent subsequence, showing (a_{ii}) is not compact. The condition is sufficient because, if $\|x\|_p \leq 1$ then

$$\left(\sum_{i=1}^{\infty} |a_{ii}\xi_i|^q\right)^{1/q} \leq (\sup_{i \geq n} |a_{ii}|) \|x\|_q \leq \sup_{i \geq n} |a_{ii}|$$

and [2; Th. 2] applies. The last inequality follows from Jensen's inequality and our assumptions $p \leq q$, $\|x\|_p \leq 1$.

LEMMA 3. *Suppose X is a Banach space with a closed subspace $\mathfrak{M}$ onto which there is a bounded projection E. Let $\mathfrak{N}$ be the null-space of E. Let $\mathfrak{P}$ be any closed linear manifold of X such that if $f \in \mathfrak{P}$ then $f = g + h$, with $g \in \mathfrak{P} \cap \mathfrak{M}$ and $h \in \mathfrak{P} \cap \mathfrak{N}$. Then, given any bounded projection F onto $\mathfrak{P}$, EF is a bounded projection onto $\mathfrak{P} \cap \mathfrak{M}$ such that $\|EF\| \leq \|E\| \, \|F\|$.*

The proof is an obvious modification of [3; Lemma 1.2.1].

Let $\mathfrak{P}$ be the set of compact operators, $\mathfrak{M}$ the set of diagonal operators, E the projection of Lemma 1, and $\mathfrak{N}$ its null-space. In order to apply Lemma 3 it remains to show: given any compact operator f, Ef and $f - Ef$ are compact. Ef is compact because, if f is compact,

$$\lim_n \left\|\sum_{i=n}^{\infty} a_{ij}\psi_i\right\| = \lim_n \left(\sum_{i=n}^{\infty} |a_{ij}|^q\right)^{1/q} = 0$$

uniformly in j. This implies $\lim_n |a_{nn}| = 0$, which shows that Ef is compact. Hence $f - Ef$ is compact.

To prove the theorem for $[\ell^p, \ell^q], 1 \leq p \leq q < \infty$, assume there is a bounded projection F from $[\ell^p, \ell^q]$ onto $\mathfrak{P}$. By Lemma 3, the restriction of EF to $\mathfrak{M}$ is a bounded projection from $\mathfrak{M}$ onto $\mathfrak{M} \cap \mathfrak{P}$. By Lemma 2 there must be a corresponding bounded projection from ℓ^∞ onto c_0. This contradicts [4; Cor. 7.5]. For the remaining X and Y pairs of the main theorem, the proof is similar except that the existence of expressions for $\|A\|$ in terms of the matrix coefficients (e.g., see [5]) makes some of the work simpler.

Next we extend the theorem to $[U, V]$. Let $\tilde{E}$ be the function on $[U, V]$ defined by $\tilde{E}f = P_2 f P_1$ for all f in $[U, V]$. $\tilde{E}$ is linear and homogeneous and bounded. $\tilde{E}^2 f = P_2(P_2 f P_1)P_1 = P_2 f P_1 = \tilde{E}f$ so $\tilde{E}$ is a projection. The range of $\tilde{E}$ is the set of operators g such that $P_2 g P_1 = g$ and is isomorphic with $[X, Y]$. The null-space of $\tilde{E}$ is the set of operators h such that $P_2 h P_1 = 0$. If Q_i is the projection $I - P_i$, the decomposition $f = g + h$ is given by

$$f = (P_2 + Q_2)f(P_1 + Q_1) = \underbrace{P_2 f P_1}_{g} + \underbrace{(P_2 f Q_1 + Q_2 f P_1 + Q_2 f Q_1)}_{h} .$$

If f is compact, so are g and h. We apply Lemma 3 with $X = [U, V]$, $\mathfrak{M}$ the range of $\tilde{E}$, $\tilde{E}$ acting as the projection E of that lemma, and $\mathfrak{P}$ the set of compact operators from U to V. The conclusion is that if there were a bounded projection F from X to $\mathfrak{P}$, the restriction of $\tilde{E}F$ to $\mathfrak{M}$ would be a bounded projection from $\mathfrak{M}$ onto $\mathfrak{P} \cap \mathfrak{M}$, contradicting our result for $[X, Y]$.

REMARK. The problem of finding a bounded projection onto the compact operators is trivial when all the bounded operators are compact. This happens, for example, for $[\ell^p, \ell^q]$, $\infty > p > q \geq 1$, [2, p. 700], or $p = \infty$, $q = 1$, and for $[c_0, \ell^q]$, $[c, \ell^q]$, $\infty > q \geq 1$. Whether there exists a pair of normed spaces with a bounded proper projection from the bounded operators onto the compact operators seems to be unknown.

REFERENCES

1. S. Banach, *Theorie des opérations linéarès* Warsaw, 1932.

2. L. W. Cohen and N. Dunford, *Transformations in sequence spaces*, Duke Math. J., **3** (1937), 689–701.

3. F. J. Murray, *On complementary manifolds and projections in spaces L_p and ℓ_p*, Trans. Amer. Math. Soc., **41** (1937), 138–152.

4. R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc., **48** (1940), 516–541.

5. A. E. Taylor, *Introduction to functional analysis*, John Wiley and Co., (1958).

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA