ON THE EXTENSIONS OF A TORSION MODULE

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This paper concerns the structure of $\operatorname{Ext}(A, T) = \operatorname{Ext}^1_R(A, T)$ where A is a torsion-free and T is a torsion module over a Dedekind ring R. In the first section it is shown that for a given torsion-free module A the structure of $\operatorname{Ext}(A, T)$ is completely determined by the basic subgroup of T. If in addition T is primary the structure of $\operatorname{Ext}(A, T)$ depends on a single known invariant of T, called by Szele [4] the critical number. The rest of the paper is devoted to showing the nature of this dependence in the special case in which A is the quotient field of R and T is primary. The problem reduces to that of computing the rank of certain complete modules over a discrete valuation ring. This computation is carried out in section two and the description of $\operatorname{Ext}(A, T)$ is given in section three.

Throughout the paper R is assumed to be a Dedekind ring other than a field. A consequence of this assumption, used in section two, is that R is infinite. An exact sequence $0 \to A' \to A \to A'' \to 0$ and a module C give rise to two exact sequences. We follow S. MacLane in calling the one beginning $0 \to \operatorname{Hom}(A'', C)$ the *first exact sequence* and the one beginning $0 \to \operatorname{Hom}(C, A')$ the *second exact sequence.*

1. In this section it is shown that whenever A is torsion-free and C is a torsion module, then the structure of $\operatorname{Ext}(A, C)$ depends only on the basic submodule of C.

LEMMA 1.1. *If A, B, C are modules with A torsion-free and if there is a homomorphism of B into C with divisible cokernel, then* $\operatorname{Ext}(A, C)$ *is a direct summand of* $\operatorname{Ext}(A, B)$.

Proof. Suppose that $f: B \to C$ is a homomorphism with $\operatorname{Coker} f = C/Im f$ divisible. Let f be factored into an epimorphism g followed by a monomorphism $h: f = hg$. We get two exact sequences

$$0 \longrightarrow Im f \xrightarrow{h} C \longrightarrow \operatorname{Coker} f \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ker} f \longrightarrow B \xrightarrow{g} Im\ f \longrightarrow 0\,,$$

and the relevant parts of the associated second exact sequences are

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$$\begin{array}{l} \text{Hom}(A, \text{Coker} f) \longrightarrow \\ \text{Ext}(A, Im f) \xrightarrow{h^*} \text{Ext}(A, C) \longrightarrow \text{Ext}(A, \text{Coker} f) \longrightarrow 0 \\ \text{Ext}(A, \text{Koker} f) \longrightarrow \text{Ext}(A, B) \xrightarrow{g^*} \text{Ext}(A, Im f) \longrightarrow 0 . \end{array}$$

Since A is torsion-free all the terms with Ext in them are divisible. But the divisibility of Coker f implies that Hom $(A, \text{Coker} f)$ is also divisible. For suppose that $\varphi : A \to \text{Coker} f$ is a given homomorphism and r is any nonzero element of R. Since A is torsion-free, division by r in A is unique; hence there is a homomorphism $\psi : rA \to \text{Coker} f$ defined by $\psi(ra) = \varphi(a)$ for a in A. Since Coker f is divisible ψ can be extended to all of A. Then $r\psi(a) = \psi(ra) = \varphi(a)$ so that $r\psi = \varphi$ and φ is divisible by r.

Hence all the modules in the last two exact sequences are divisible and the images of the various homomorphisms are direct summands. In addition Ext $(A, \text{Coker} f) = 0$ because Coker f is divisible. It follows that Ext (A, C) is a direct summand of Ext $(A, Im f)$ which is in turn a direct summand of Ext (A, B).

COROLLARY 1.2. *If A is torsion-free and each of B and C has a homomorphism into the other with divisible cokernel, then*

$$\text{Ext}(A, B) \approx \text{Ext}(A, C) .$$

Proof. A divisible R-module is a direct sum of submodules each of which is isomorphic to Q or to a primary component of Q/R, the number of summands of each type being independent of the decomposition.

THEOREM 1.3. *If A is torsion-free, C is a torsion module, and B is a basic submodule of C, then*

$$\text{Ext}(A, C) \approx \text{Ext}(A, B) .$$

Proof. A basic submodule of a torsion module is a pure submodule for which the factor module is divisible and which is a direct sum of cyclic modules. Hence there is a homomorphism of B into C with divisible cokernel. On the other hand Szele has shown in [4] that B is a homomorphic image of C (Szele's proof is for primary groups but the generalization to this case is trivial). Hence the hypotheses of Corollary 1.2 are satisfied and the conclusion follows.

Suppose now that P is a prime ideal of R and that T is a P-primary module. The order ideal of an element x of T has the form $P^{e(x)}$ with $e(x)$ a nonnegative integer which we will call the *exponential order* of x. The submodule of T consisting of those elements with exponential order ≤ 1 is a vector space over the field R/P; its dimension will be called the P-rank of T and will be denoted by $r_P(T)$. If B is a basic submodule of T, the minimum of the numbers $r_P(P^n B)$ with n ranging over the non-negative integers is independent of the choice of B because the basic submodules of T are all isomorphic. This number is thus an invariant of T. We shall follow Szele in calling it the *critical number* of T.

If the basic submodule B of T is decomposed into the direct sum of cyclic modules, then $r_P(P^n B)$ is the number of summands whose generators have exponential order $> n$. Hence $r_P(P^n B)$ finite implies that the orders of the elements of B are bounded and the critical number of T is then 0. Thus the critical number of T is either 0 or infinite, and if 0, B is a direct summand of T which is therefore a direct sum of a divisible module and a module all of whose elements have bounded order.

THEOREM 1.4. *Let T be a P-primary module with critical number $\aleph$ and let A be torsion-free.*

(i) *If $\aleph = 0$, then* $\operatorname{Ext}(A, T) = 0$.

(ii) *If $\aleph$ is infinite and M is the direct sum of $\aleph$ copies of $\sum_n R/P^n$, then* $\operatorname{Ext}(A, T)$ *and* $\operatorname{Ext}(A, M)$ *are isomorphic. Thus the module structure of* $\operatorname{Ext}(A, T)$ *depends only on the critical number of T.*

Proof. Since the maximal divisible submodule of T is a direct summand of T and contributes neither to $\operatorname{Ext}(A, T)$ nor to the critical number of T, we may as well assume T reduced. In the paragraph preceding the theorem it was shown that if $\aleph = 0$, the orders of the elements of T are bounded. Any extension of T having a torsion-free factor module contains T as a pure submodule. Hence it splits and $\operatorname{Ext}(A, T) = 0$ in this case.

Suppose now that $\aleph$ is infinite and M is the direct sum of $\aleph$ copies of $\sum_n R/P^n$. By Theorem 1.3 $\operatorname{Ext}(A, T) \approx \operatorname{Ext}(A, B)$ where B is a basic submodule of T. We write $B = \sum_n B_n$ where each B_n is a direct sum of copies of R/P^n. There is a natural number m such that $\aleph = r_P(P^m B)$ and $B = B' + B''$ where B' is the sum of the B_n with $n \leq m$ and B'' is the sum of the remaining B_n. Since $P^m B' = 0$ and A is torsion-free, $\operatorname{Ext}(A, B') = 0$. Then the additivity of Ext implies that $\operatorname{Ext}(A, B) \approx \operatorname{Ext}(A, B'')$. The module B'' is the direct sum of cyclic modules and $r_P(B'') = r_P(P^m B'') = \aleph$ so that B'' is generated by $\aleph$ elements. Hence it is a homomorphic image of M. On the other hand B'' can be expressed as a direct sum $B'' = C + \sum_\gamma C_\gamma$ where the summands C_γ are $\aleph$ in number and each C_γ is the direct sum of a sequence of cyclic modules whose orders are strictly increasing. It follows that M is also a homomorphic image of B'', hence $\operatorname{Ext}(A, B'') \approx \operatorname{Ext}(A, M)$ by Corollary 1.2. This proves (ii).

2. In this section we assume that R is a discrete valuation ring with prime p. If M is an R-module for which the submodules p^nM have intersection 0 (i. e. if M has no elements of infinite height), then these submodules are a base at 0 for a topology called the *p-adic* topology. The completion of M in this topology will be denoted by M^*. The p-adic topology on M induces a topology on each submodule N which may or may not coincide with the p-adic topology on N. The two topologies will certainly coincide if N is pure in M for then $p^nN = N \cap p^nM$ for all n.

The problem to be solved in this section is that of determining the rank of M^* where M is a direct sum of copies of $\Sigma_n R/p^nR$.

A subset X of an R-module A is called independent if $r_1x_1 + \cdots + r_nx_n = 0$ implies $r_1 = \cdots = r_n = 0$ whenever $x_1, \cdots, x_n$ are distinct elements of X and $r_1, \cdots, r_n$ are elements of R. The cardinal $|X|$ of a maximal independent subset of A is an invariant of A called its *rank* (denoted by $r(A)$); the rank of A is in fact the dimension of $A \otimes_R Q$ as a vector space over Q. The rank formula

$$r(A) = r(B) + r(A/B)$$

holds for any R-modules A and B with B a submodule of A. If A is torsion-free its cardinal $|A|$ and its rank are connected by the relation

$$|A| = r(A)\,|R|\,.$$

In particular $r(A) = |A|$ wherenever A is torsion-free and $r(A) \leq |R|$. (The properties mentioned is this paragraph hold for any Dedekind ring.)

LEMMA 2.1. *If $M = \Sigma_\gamma M_\gamma$ is the direct sum of the modules M_γ, each of which is without elements of infinite height then M^* is isomorphic to the submodule of the direct product $\Pi_\gamma M_\gamma^*$ consisting of those sequences $u = (u_\gamma)$ such that (*) for each natural number n, $u_\gamma \in p^nM_\gamma^*$ for all but a finite set of indices.*

The condition (*) implies that $u_\gamma = 0$ for all but a countable set of indices.

Proof. For each index γ M_γ is pure in M which is pure in M^*. Hence M_γ is pure in M^*. By Lemma 20 of [2] the closure $M_{\bar\gamma}$ of M_γ in the p-adic topology is also pure in M^*. Therefore M^* induces the p-adic topology on $M_{\bar\gamma}$ and, since a closed subspace of a complete space is complete, $M_{\bar\gamma} = M_\gamma^*$.

We next show that the sum $\Sigma_\gamma M_\gamma^* \subseteq M^*$ is direct. Suppose $\Sigma_\gamma x_\gamma = 0$ where $x_\gamma \in M^*$ and γ belongs to a finite set σ of indices. For each natural number n and each $\gamma \in \sigma$ there is an $x_{\gamma n} \in M_\gamma$ such that $x_{\gamma n} - x_\gamma \in p^nM_\gamma^*$, hence $\Sigma_\gamma x_{\gamma n} = \Sigma_\gamma(x_{\gamma n} - x_\gamma) \in p^nM^*$. Since $\Sigma_\gamma M_\gamma$ is pure in M^* it is pure in $\Sigma_\gamma M_\lambda^*$ so that $\Sigma_\gamma x_{\gamma n} \in (\Sigma_\gamma M_\gamma) \cap p^n \Sigma_\gamma M_\gamma^* = p^n \Sigma_\gamma M_\gamma$. Then $x_{\gamma n} \in p^n M$ for each $\gamma \in \sigma$ because the sum $\Sigma_\gamma M_\gamma$ is direct. Thus for each $\gamma \in \sigma$, $x_{\gamma n} \to 0$ and $x_\gamma = 0$.

Let S be the submodule of $\Pi_\gamma M_\gamma^*$ defined by (*). We shall define an isomorphism φ of M^* onto S. Let x be any element of M^*. Since $\Sigma_\gamma M_\gamma^*$ is dense in M^* there is, for each natural number n, an element $x_n \in \Sigma_\gamma M_\gamma^*$ such that $x_n - x \in p^n M^*$. We express each x_n as a sum $x_n = \Sigma_\gamma x_{\gamma n}$ with $x_{\gamma n} \in M_\gamma^*$ where $x_{\gamma n} = 0$ for all γ not in some finite set τ_n. Since x_n converges to x, the arguments of the preceding paragraph show that, for each γ, $x_{\gamma n}$ converges to some $u_\gamma \in M^*$. It is easily shown that the elements u_γ depend only on x. We set $\varphi(x) = (u_\gamma)$.

It is necessary to show that u lies in S. Consider a fixed natural number i and assume that γ is not in τ_i so that $x_{\gamma i} = 0$. Then, for $j > i$, $x_{\gamma j} = x_{\gamma j} - x_{\gamma i} \in p^i M^* \cap M_\gamma^* = p^i M_\gamma^*$. Passing to the limit we have $u_\gamma \in p^i M_\gamma^*$ because $p^i M_\gamma^*$ is closed in M^*. Since each τ_i is finite, u_γ satisfies (*) and is in S as required.

To prove φ epimorphic suppose $u \in S$. For each n let τ_n be a finite set of indices such that $u_\gamma \in p^n M_\gamma^*$ for all γ not in τ_n and let x_n be the sum (in M^*) of the u_γ for $\gamma \in \tau_n$. The existence of τ_n is insured by (*). Since $\tau_n \subseteq \tau_m$ for $m \leq n$, $x_m - x_n \in p^n M^*$. Hence the x_n converge to an element x in M^*. Moreover $x_n - x \in p^n M^*$. An examination of the definition of φ shows that $x_{\gamma n} = u_\gamma$ if $\gamma \in \tau_n$ and $x_{\gamma n} = 0$ otherwise. Hence $\varphi(x) = u$ and φ is epimorphic.

Finally suppose that $\varphi(x) = 0$. Refering to the definition of φ we have, for fixed n and all $i > n$, $(\Sigma_{\gamma i} - x_{\gamma n}) = x_i - x_n \in p^n M^*$. Since $\Sigma_\gamma M_\gamma^*$ is pure in M^* and the sum is direct, this implies that $x_{\gamma i} - x_{\gamma n} \in p^n M_\gamma^*$ for each index γ and each $i > n$. We are assuming all $u_\gamma = 0$ so that $x_{\gamma i} \in p^n M_\gamma^*$ for large i, hence $x_{\gamma n} \in p^n M_\gamma^*$. But then $x_n = \Sigma_\gamma x_{\gamma n} \in p^n M^*$ and $x_n \to 0$, $x = 0$. This shows that φ is a monomorphism and completes the proof.

LEMMA 2.3. *If $M = \Pi_\gamma M_\gamma$ where γ ranges over a set of cardinal $\aleph$ and the M_γ are all torsion-free with the same rank, then*

$$r(M) = | M_\gamma |^{\aleph} .$$

Proof. Note first that for each γ $| M_\gamma | = r(M_\gamma) | R |$ so that all the M_γ have the same power. If we can show that $r(M) \geq | R |$, then $r(M) = | M | = | M_\gamma |^{\aleph}$ as required.

Suppose the indices are the natural numbers and that each $M_\gamma = R$. Consideration of a suitable Vandermonde determinant shows that the elements $(1, r, r^2, \cdots) \in M$ with r ranging over R are independent so that $r(M) \geq | R |$ in this case. In the general case $\aleph$ is infinite and each M_γ contains a copy of R so that M contains a countable product of copies of R, hence $r(M) \geq | R |$ in all cases.

LEMMA 2.3. *Suppose that N is a submodule of M and that, for each natural number n, M_n and N_n are copies of M and N respectively. If $\varphi: \Pi_n M_n \to M$ is a homomorphism such that $\varphi^{-1}(N) \subseteq \Pi_n N_n$, then*

$$r(M/N) = r(M/N)^{\aleph_0} .$$

Proof. Since φ maps $\varphi^{-1}(N)$ into N, it induces a monomorphism

$$(1) \qquad 0 \to \Pi_n M_n/\varphi^{-1}(N) \to M/N .$$

Since $\varphi^{-1}(N) \subseteq \Pi_n N_n$, there is an epimorphism

$$(2) \qquad \Pi_n M_n/\varphi^{-1}(N) \to \Pi_n(M_n/N_n) \to 0 .$$

Rank does not increase on passing to submodules or to homomorphic images, hence (1) and (2) imply

$$(3) \qquad r(M/N) \geq r(\Pi_n M_n/\varphi^{-1}(N)) \geq r(\Pi_n(M_n/N_n)) .$$

By the definition of rank M/N contains a free module F such that $r(F) = r(M/N)$. For each n let F_n be a copy of F in M_n/N_n. Then $\Pi_n F_n \subseteq \Pi_n(M_n/N_n)$ and Lemma 2.2 implies

$$(4) \qquad r(\Pi_n(M_n/N_n)) \geq r(\Pi_n F_n) = |F|^{\aleph_0} \geq r(F)^{\aleph_0} = r(M/N)^{\aleph_0} .$$

Thus (3) and (4) imply the conclusion of the lemma.

THEOREM 2.4. *If M is the direct sum of $\aleph$ copies of $\Sigma_n R/p^n R$, then $r(M^*) = (\aleph\,|R|)^{\aleph_0}$.*

Proof. We first consider the case $\aleph = 1$. It will be convenient to replace R/p^nR by the isomorphic module $R(p^n)$ which consists of all elements of Q/R annihilated by p^n, for then $R(p^n) \subseteq R(p^m)$ for all $m \geq n$. Each element $a \neq 0$ in $R(p^n)$ has a height $h_n(a)$ in $R(p^n)$ where $h_n(a)=i$ if $a \in p^i R(p^n)$ but a is not in $p^{i+1}R(p^n)$. The height and exponential order of a are related by $h_n(a) + e(a) = n$. We let $C = \Sigma_n R(p^n)$ and $D = \Pi_n R(p^n)$. Then C^* consists of those elements $x = (x_n) \in D$ such that $h_n(x_n)$ goes to ∞ with n.

We show first that $r(C^*) = r(D)$. The inequality $r(C^*) \leq r(D)$ holds because $C^* \subseteq D$. To prove the opposite inequality we define $\rho: D \to C^*$ by

$$\rho(x)_n = \begin{cases} 0 & \text{if} \quad n = 2k+1 , \\ x_k & \text{if} \quad n = 2k . \end{cases}$$

Since $R(p^k) \subseteq R(p^{2k})$, ρ is a homomorphism into D. Since $e(x_k) \leq k$ and $h_{2k}(x_k) + e(x_k) = 2k$, $h_{2k}(x_k) \geq k$ so that $\rho(x)$ lies in C^*. The map ρ is clearly a monomorphism so $r(D) \leq r(C^*)$ as required.

The next step is to show that

$$r(D) = r(D)^{\aleph_0} .$$

Let $\sigma_1, \sigma_2, \cdots$ be an infinite partition of the set of natural numbers into infinite subsets. For each n let D_n be a copy of D. An element $u \in \Pi_n D_n$ is a sequence $(u_1, u_2, \cdots)$ with $u_n = (u_{ni}) \in D$. We define $\xi : \Pi_n D_n \to D$ by $\xi(u)_k = u_{ni}$ if k is the ith element of σ_n; $u_{ni} \in R(p^k)$ because $k \geq i$. The hypotheses of Lemma 2.3 are satisfied with $M = D$ and $N = 0$ which shows that $r(D) = r(D)^{\aleph_0}$.

The module D can be represented as the module of all infinite sequences $(x_1, x_2, \cdots)$ of elements of R modulo the sequences of the form $(b_1 p, b_2 p^2, b_3 p^3, \cdots)$. Thus Lemma 2.2 and the fact that rank does not increase on passing to homomorphic images imply that $r(D) \leq |R|^{\aleph_0}$. We shall show that $r(D) \geq |R|$. Then $r(D) = r(D)^{\aleph_0} \geq |R|^{\aleph_0}$ and we get

$$r(D) = |R|^{\aleph_0} .$$

To show that $r(D) \geq |R|$ let $\alpha(r) = (1, r, r^2, \cdots)$ for each $r \in R$ and let $\bar{\alpha}(r)$ be the image of $\alpha(r)$ in D. We show that the elements $\bar{\alpha}(r)$ for $r \in R - (p)$ are independent. Suppose $r_1, \cdots, r_n$ are distinct elements of R not in (p), and suppose $a_1, \cdots, a_n \in R$ such that

$$a_1\bar{\alpha}(r_1) + \cdots + a_n\bar{\alpha}(r_n) = 0 .$$

Then elements $b_1, b_2, \cdots$ exist in R such that

$$a_1\alpha(r_1) + \cdots + a_n\alpha(r_n) = (b_1 p, b_2 p^2, \cdots) .$$

Hence, for each k, the a_i satisfy a system of n equations

$$\begin{array}{l} a_1 r_1^k + \cdots + a_n r_n^k = b_k p^k \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 r_1^{k+n-1} + \cdots + a_n r_n^{k+n-1} = b_{k+n-1} p^{k+n-1} . \end{array}$$

The determinant Δ of this system is $r_1^k \cdots r_n^k d$ where d is the Vandermonde determinant of $r_1, \cdots, r_n$; $d \neq 0$ because the r's are distinct. We set $d = p^m s$ with s prime to p and $t = r_1^k \cdots r_n^k s$. Then $\Delta = p^m t$ where t is prime to p because $r_1, \cdots, r_n, s \in R - (p)$. Then by Cramer's rule each a_i satisfies an equation of the form $p^m t a_i = p^k c_i$. Hence, for $k > m$, p^{k-m} divides ta_i and therefore divides a_i because it is prime to t. Since this is true for all $k > m$, $a_i = 0$ for each i. Therefore the $\bar{\alpha}(r)$ with r ranging over $R - (p)$ is an independent subset of D so $r(D) \geq |R - (p)|$. But $R - (p)$ is the disjoint union of cosets of (p) so that $|R - (p) \geq |(p)| = |R|$; hence $|R - (p)| = |R|$.

We now have $r(C^*) = r(D) = |R|^{\aleph_0}$ which completes the proof in the case $\aleph = 1$.

Now suppose $\aleph$ arbitrary, let Γ be a set with cardinal $\aleph$ and let $M = \Sigma_\gamma M_\gamma$ where, for each $\gamma \in \Gamma$, $M_\gamma = C = \Sigma_n R(p^n)$. In view of Lemma 2.1 and the remark following it M^* is contained in the submodule A of all sequences $x \in \Pi_\gamma M_\gamma^*$ with $x_\gamma = 0$ for all but a countable number of indices. Each such sequence is determined by the set σ of indices γ such that $x_\gamma \neq 0$ and a function $f: \sigma \to C^* - \{0\}$. From this it follows easily that $|A| \leq (\aleph |C^*|)^{\aleph_0}$. Since $C^* \subseteq D$ and D is a homomorphic image of the direct product of $\aleph_0$ copies of R, $|C^*| \leq |R|^{\aleph_0}$. Since $|R|^{\aleph_0} = r(C^*) \leq |C^*|$ we have $|C^*| = |R|^{\aleph_0}$. Hence

$$r(M^*) \leq r(A) \leq |A| \leq (\aleph |R|)^{\aleph_0} .$$

Using Lemma 2.1 again we have $\Sigma_\gamma M_\gamma^* \subseteq M^*$ so that

$$r(M^*) \geq r(\Sigma_\gamma M_\gamma^*) = |\Gamma| \, r(C^*) = \aleph |R|^{\aleph_0} .$$

These last two sets of inequalities combine to give

$$\aleph |R|^{\aleph_0} \leq r(M^*) \leq (\aleph |R|)^{\aleph_0}.$$

If $\aleph$ is finite this completes the proof. If $\aleph$ is infinite, the proof will be complete once we show that $r(M^*)^{\aleph_0} = r(M^*)$. To show this assume $\aleph$ infinite and partition the index set Γ into a countable sequence Γ_1, $\Gamma_2, \cdots$ of disjoint subsets such that $|\Gamma_n| = |\Gamma| = \aleph$ and set $M_n = \Sigma \{M_\gamma \mid \gamma \in \Gamma_n\}$. Then $M_n \approx M$ and $M_n^* \approx M^*$ for each n. Our purpose will be achieved if we can define a monomorphism $\varphi : \Pi_n M_n^* \to M^*$, for then $\varphi^{-1}(tM^*) = t(\Pi_n M_n^*) \subseteq \Pi_n tM_n^*$, where tM^* is the torsion submodule of M^*. Now Lemma 2.3 applies to give $r(M^*/tM^*) = r(M^*/tM^*)^{\aleph_0}$. But $r(M^*) = r(M^*/tM^*)$ so $r(M^*) = r(M^*)^{\aleph_0}$.

Earlier in the proof of this theorem we defined a monomorphism $\rho : D \to C^*$. For each k we now define a monomorphism $\psi_k : D \to D$ by

$$\psi_k(x)_i = \begin{cases} 0, & i \leq k , \\ x_{i-k}, & i > k . \end{cases}$$

For $i > k$ we have $e(x_{i-n}) \leq i - k$ so that $h_i(x_{i-k}) = i - e(x_{i-k}) \geq k$. Hence $\psi_k(D) \subseteq p^k D$ so that $\rho\psi_k$ maps D into $p^k C^*$. We define φ_k: $C^* \to p^k C^*$ to be the restriction of $\rho\psi_k$ to C^* and note that it is a monomorpoism.

We now use Lemma 2.1 to identify M^* with the submodule of $\Pi_\gamma M_\gamma^*$ described by the condition (*). An element x of $\Pi_n M_n^*$ is a sequence $(x_1, x_2, \cdots)$ where $x_n \in M_n^* \subseteq \Pi \{M_\gamma^* \mid \gamma \in \Gamma_n\}$. We define φ by $\varphi(x)_\gamma = \varphi_n(x_{n\gamma})$ for $\gamma \in \Gamma_n$. Then $\varphi : \Pi_n M_n^* \to \Pi_\gamma M_\gamma^*$ and is a monomorphism because each φ_n is one. There remains the task of showing that $\varphi(x)$ lies in M^*. Let n be a natural number. For each $k < n$ there is by Lemma 2.1 a finite subset τ_k of Γ_k such that $x_{k\gamma} \in p^n M_\gamma^*$ for $\gamma \in \Gamma_k$

but not in τ_k. By the definition of φ_k, $\varphi_k(x_{k\gamma}) \in p^n M^*_\gamma$ for all $\gamma \in \Gamma_k$ with $k \geq n$. Hence $\varphi(x)_\gamma \in p^n M^*_\gamma$ for all not in $\tau_1 \cup \cdots \cup \tau_{n-1}$ which is a finite set. Thus $\varphi(x)$ satisfies (*) of Lemma 2.1 and is in M^* as required.

3. Let R once more be an arbitrary Dedekind ring and let P be a prime ideal of R. For any R-module T, $\operatorname{Ext}(Q, T)$ is a vector space over Q and is therefore completely described by its dimension over Q or equivalently its rank over R. According to Theorem 1.4 this dimension is a function of the critical number of T if T is primary.

THEOREM 3.1. *If T is a P-primary R-module with infinite critical number $\aleph$, then the rank of* $\operatorname{Ext}(Q, T)$ *is* $(\aleph \mid R \mid)^{\aleph_0}$.

Proof. In order to make the results of section two available we change rings. The module T, being P-primary, can be considered as a module over the ring S consisting of all elements of the form a/b in Q with a and b in R and b prime to P. The theory of P-primary modules is left unchanged by the shift from R to S. In particular the critical number of T is $\aleph$ in both cases.

Since S is torsion-free as an R-module Proposition 4.1.3. of [1] applies to give a natural isomorphism

$$\operatorname{Ext}_R(Q, T) \approx \operatorname{Ext}_S(S \otimes_R Q, T) .$$

Since R and S have the same quotient field Q, $Q = S \otimes_R Q$ and

$$\operatorname{Ext}_R(Q, T) \approx \operatorname{Ext}_S(Q, T) .$$

These are both vector spaces over Q and the isomorphism is a Q-isomorphism; hence the two modules have the same dimension over Q. Let M be the direct sum of $\aleph$ copies of $\Sigma_n S/p^n S$ where p is the prime of S. According to Theorem 1.4

$$\operatorname{Ext}_S(Q, T) \approx \operatorname{Ext}_S(Q, M) .$$

Since M is a basic submodule of tM^*, Theorem 1.3 gives

$$\operatorname{Ext}_S(Q, M) \approx \operatorname{Ext}_S(Q, tM^*) .$$

By Theorem 7.4 of [3], $\operatorname{Ext}_S(Q, M^*) = 0$ because M^* is complete, while $\operatorname{Hom}_S(Q, M^*) = 0$ because M^* is reduced. Hence the second exact sequence associated with Q and $0 \to tM^* \to M^* \to M^*/tM^* \to 0$ reduces to

$$0 \to \operatorname{Hom}_S(Q, M^*/tM^*) \to \operatorname{Ext}_S(Q, tM^*) \to 0 .$$

Since M^*/tM^* is torsion-free divisible

$$\operatorname{Hom}_S(Q, M^*/tM^*) \approx M^*/tM^* .$$

It follows that $\mathrm{Ext}_R(Q, T)$ and M^*/tM^* have the same dimension over Q. This dimension is $(\aleph \mid S \mid)^{\aleph_0}$ by Theorem 2.5. Moreover $\mid R \mid = \mid S \mid$. Hence the theorem is proved.

Since the integers are the most important example of a Dedekind ring it is appropriate to interpret the last theorem for this special case. Since rank and cardinality coincide for torsion-free abelian groups of infinite rank, we can say that *if T is a p-primary abelian group with infinite critical number $\aleph$, there are $\aleph^{\aleph_0}$ inequivalent extensions of T by the rational numbers.*

REFERENCES

1. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton 1956.
2. I Kaplansky, *Infinite Abelian Groups*, Ann Arbor, 1954.
3. R. J. Nunke, *Modules of extensions over Dedekind rings*. Ill. Journ. of Math., **3**, (1959), 222–241.
4. T. Szele, *On the basic subgroups of abelian p-groups*, Acta. Math. Acad. Sci. Hung., **5** (1954), 129-141.

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