A NOTE ON ASSOCIATIVITY

DONALD A. NORTON

1. Introduction. In a groupoid with binary operation $(\cdot)$ the constraints that the groupoid be a quasigroup[1] and that it be associative are not independent. This note defines three forms of associativity in order of descending strength and shows that in a groupoid they are essentially independent while in a quasigroup (with minor limitations on the number of elements) the stronger implies the weaker. Let us define:

A groupoid is *tri-associative* if for every triple x, y, z of distinct elements

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \; ; \tag{1}$$

A groupoid is *di-associative*[2] if in (1) above, exactly two of the elements are distinct;

A groupoid is *mono-associative* if (1) is true when all three x, y and z are equal.

The next section shows that a tri-associative quasigroup Q which contains sufficient elements (seventeen are adequate) for which $Q^2 = \{q^2, \text{ all } q \in Q\}$ also contains sufficient elements (seventeen are again adequate) is di-associative. Further, any di-associative quasigroup is mono-associative. The restrictions on the minimum number of elements in Q and Q^2 are necessitated by the method of proof for which there does not seem any essential improvement but Theorem II is probably true for all quasigroups. An examination of all possibilities indicates its validity if Q contains no more than 5 elements.

The final section illustrates, by examples, the falseness of these theorems if the assumption that Q is a quasigroup is removed.

2. Associativity conditions. We shall first prove a theorem of interest in its own right but which contributes little to the main theorems-Theorems II and III.

THEOREM I. *A tri-associative quasigroup Q has a unity element.*

Before proving the theorem it is convenient to have

LEMMA. *There exists no idempotent tri-associative quasigroup Q containing at least 2 elements.*

Proof of Lemma. We shall use product as our operation in Q with

Received May 21, 1959.

[1] For definitions of groupoid and quasigroup see, for instance, [**1**, pp. 1, 8, 15].

[2] This definition differs from the one used by this author [**2**, p. 59] in which di-associativity included power-associativity, and thereby mono-associativity. Theorem III shows that, in a quasigroup, this distinction is vacuous.

the usual conventions of juxtaposition of u and x to mean the binary product of u and x and the notation $a \cdot ux$ to mean $a(ux)$.

Suppose that $q^2 = q$, all $q \in Q$. For fixed $q \in Q$ let $u \in Q$, $u \neq q$. Then if x is the solution of $q = ux$, it is true that $x \neq q, u$; for if:

(a) $x = q$, $q = uq = q^2$ implies $u = q$;

(b) $x = u$, $q = u^2$. But $u^2 = u$ implies $u = q$.

Either is a contradiction.

Now consider $q^2 = q$. Since $q = ux$, substitution yields $q \cdot ux = ux$. Since $u \neq q \neq x \neq u$, tri-associativity implies $qu \cdot x = ux$, from which $qu = u = u^2$. So $q = u$; a contradiction. We are now ready for:

Proof of Theorem I. If Q contains 1, 2, or 3 elements an examination of possibilities yields the theorem. So suppose that Q contains at least 4 elements.

Q is not idempotent by preceding lemma so there is an $a \in Q$ so that $a^2 \neq a$. Let $ae = a$ whence $e \neq a$. Now choose some $b \neq a, e$. Tri-associativity yields $a \cdot eb = ae \cdot b = ab$; and since Q is a quasigroup

(1) $$eb = b \text{ for all } b \neq a, e\,.$$

Finally choose $c \in Q$, $c \neq b, e$. As before $cb = c \cdot eb = ce \cdot b$ and

(2) $$ce = e \text{ for all } c \neq b, e\,.$$

Therefore, combining (1) and (2), we see that e is a unity except perhaps for the products ea, ee, and be. Listing the possible values of the products from (1):

$$\text{I (a)} \quad ea = a\,; \quad ee = e\,; \qquad \text{I (b)} \quad ea = e\,; \quad ee = a\,;$$

and from (2):

$$\text{II (a)} \quad be = b\,; \quad ee = e\,; \qquad \text{II (b)} \quad be = e\,; \quad ee = b\,.$$

Now I(b) and II(b) are inconsistent since $a \neq b$. Similarly II(a) and I(b) or I(a) and II(b) are inconsistent since $e \neq a$, and $e \neq b$ respectively.

This leaves I(a) and II(a), or $ea = a$

$$ee = e$$
$$be = b$$

and e is a unity element.

We can now prove

THEOREM II. *Let Q be a tri-associative quasigroup for which both Q and $Q^2 = \{q^2;\ all\ q \in Q\}$ contain a "sufficient number" of elements, then Q is di-asssociative.*

Proof. There are 3 equalities to show, where $a \neq b$:

(1) $a \cdot ab = a^2 \cdot b$;

(2) $a \cdot ba = ab \cdot a$;

(3) $b \cdot a^2 = ba \cdot a$.

Because of the symmetry of the postulates, it is necessary to prove only one of (1) and (3). We shall prove (1) and (2).

As the proof will be given, each step of it has restrictions on the elements which will be listed and considered at the end.

(1) *Proof*	*Restrictions on elements*
$a \cdot ab$	
$= xy \cdot ab$	(a) $xy = a$
$= x(y \cdot ab)$	(b) $x \neq ab \neq y \neq x$
$= x(ya \cdot b)$	(c) $y \neq a \neq b \neq y$
$= (x \cdot ya)b$	(d) $x \neq ya \neq b \neq x$
$= (xy \cdot a)b$	(e) $x \neq y \neq a \neq x$
$= a^2b.$	

Let us now consider the restrictions:

(a) Since Q is a quasigroup, given either x, or y the other can always be found.

(b) If Q contains sufficient elements it is always possible to find x and y; $x \neq ab$, $y \neq ab$.

We next note that if Q^2 contains n elements, there will be at least n or $n-1$ pairs, x, y, $x \neq y$ for which $xy = a$, (the number depending on whether or not $a \in Q^2$).

(c) Conditions $y \neq a, b$ can always be satisfied if Q contains sufficient pairs to satisfy (a) and Q^2 enough to also satisfy (b) as well.

(d) The same as (c) may be said about the conditions $ya \neq b$ and $x \neq b$. Consider now the condition $x \neq ya$. Then $x^2 \neq x \cdot ya$.

Before proceeding we can also satisfy (e) which is a condition similar to (c).

Now since $x \neq y$; $x, y \neq a$.

$$x^2 \neq x \cdot ya = xy \cdot a = a^2$$

Conversely, if

$$x^2 \neq a^2 = xy \cdot a = x \cdot ya$$

then $x \neq ya$.

So the remaining condition of (d) can be satisfied if Q^2 contains an adequate number of elements.

The proof of (2) is parallel.

$$\begin{aligned} (2)\quad & a\cdot ba \\ &= xy\cdot ba \\ &= x(y\cdot ba) \\ &= x(yb\cdot a) \\ &= (x\cdot yb)a \\ &= (xy\cdot b)a \\ &= ab\cdot a \end{aligned}
\qquad
\begin{aligned} &\text{(a)}\quad xy = a \\ &\text{(b)}\quad ba \neq x \neq y \neq ba\ . \\ &\text{(c)}\quad y \neq a \neq b \neq y\ . \\ &\text{(d)}\quad x \neq yb \neq a \neq x\ , \\ &\text{(e)}\quad x \neq y \neq b \neq x\ . \end{aligned}$$

Condition (a), (c), (e) and $a \neq x$ of (d) have already been met previously. Condition (b) is a condition similar to (b) of the previous part and can be similarly met if Q contains adequate elements. The condition

$$\begin{aligned} &x \neq yb \text{ of part (d) yields} \\ &x^2 \neq x\cdot yb \\ &x^2 \neq xy\cdot b \\ &x^2 \neq ab\ . \end{aligned}$$

Again if Q^2 contains a sufficient number of elements, this may be met.

To complete this section we shall prove

THEOREM III. *If a quasigroup Q satisfies the constraint $x\cdot xy = x^2y$ when $x \neq y$, then Q is mono-associative.*

Proof. We must show that $q\cdot q^2 = q^2\cdot q$, all $q \in Q$. Since Q is a quasigroup, $\exists\, x$ so that

$$a\cdot a^2 = a^2x\ .$$

If $x \neq a$, from the condition of the theorem

$$a\cdot ax = a^2x\ .$$

Then $a^2 = ax$ since Q is a quasigroup and $a = x$, a contradiction. So it must be that $a = x$.

COROLLARY. *A di-associative quasigroup is mono-assoiciative.*

3. Associativity conditions for groupoids.

EXAMPLE I. The groupoid whose multiplication table is displayed is trivially tri-associative since any triple of distinct elements must contain c and so the product must be c. However, it is not di-associative since

$\cdot$	a	b	c
a	b	b	c
b	a	b	c
c	c	c	c

$$ab\cdot a = ba = a \text{ while } a\cdot ba = a^2 = b\ ;$$

nor is it mono-associative since

$$a^2\cdot a = ba = a \text{ while } a\cdot a^2 = ab = b\ .$$

EXAMPLE II. The groupoid whose multiplication table is displayed is di-associative as an examination of all possible triple products containing two distinct elements will reveal but it is not mono-associative since

$$aa^2 = ab = y \text{ while } a^2a = ba = x .$$

$\cdot$	a	b	x	y
a	b	y	a	a
b	x	a	b	b
x	a	b	x	x
y	a	b	x	x

These examples illustrate that for the groupoid the "stronger" associativity assumption does not imply the weaker, while examples of power-associative and Moufang loops illustrate that, even for quasigroups the "weaker" do not imply the "stronger".

REFERENCES

1. R. H. Bruck, *A survey of binary systems*, Springer-Verlag. Berlin, 1958.
2. D. A. Norton, *Hamiltonian loops*, Proc. Amer. Math. Soc., **3** (1952), 56-65.

UNIVERSISY OF CALIFORNIA
DAVIS, CALIFORNIA