ON TERMINATING PROLONGATION PROCEDURES*

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In the classical treatments [3] of systems of differential equations there are two outstanding techniques—the Cauchy-Kowalewski theorem and completely integrable systems (the latter is really a special case of the former [1, p. 77]). In terms of systems of differential forms the Cauchy-Kowalewski theorem becomes the Cartan-Kahler theorem, and systems with independent variables which satisfy its conditions are called involutive.

Many systems are not involutive, and the central problem of prolongation theory is to construct a procedure by which one can reduce every system to an equivalent involutive system. For total prolongations Kuranishi's theorem [4, p. 44] gives a precise answer to the question of when total prolongations will lead to involutive systems. If S is the initial system in euclidean space E^n, $P^g(S)$ the g^{th} total prolongation in the space R_g, then for all points $x \in E^n$, except possibly on a proper subvariety, there is a number g_0 such that if $g \geqq g_0$ and $y \in R_g$ is a point over x, then $P^g(S)$ is involutive at y if and only if y is an ordinary integral point [4, p. 7] and the 1-forms of $P^g(S)$ do not imply any dependencies among the independent variables at integral points in a neighborhood of y. Then y is called a normal point.

The first part of this paper deals with an application of this theorem to certain types of differential systems. We show that under certain conditions the total prolongation process must result in normal points if there are to be *any* solutions. An application of this leads to a theorem often used in differential geometry [2, p. 14].

The second section is concerned with what can be done if normal points are not obtained for $P^g(S)$ as is the case with an example of Kuranishi. Here we must distinguish two cases. If $P^g(S)$ does not contain ordinary intergal points, so that its 0-forms are not a regular system of equations [4, p. 7] the Cartan-Kahler theory does not apply. Let us call such systems *singular*. We shall not consider this aspect of the problem in this paper.

If, however, the problem lies in a dependency among the independent variables implied by 1-forms of $P^g(S)$, at generic integral points, one would naturally think of restricting the system to those points where dependencies do not occur, since solutions must lie only in these points. Thus one obtains a sort of partial prolongation which could in turn be prolonged. Such a procedure was certainly what Cartan and Kuranishi had in mind. However, it is not clear that the process will ever result

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in an involutive system. One might conceivably go on obtaining non-normal systems indefinitely.

Kuranishi has recently proved a generalization of his prolongation theorem which is used to show that the above procedure does in fact ultimately stop, barring the occurrence of singular systems somewhere along the line.

The first section of this paper is part of the author's doctoral thesis at the University of California at Berkeley, written under the direction of Professor Harley Flanders to whom the author would like to record here his appreciation.

All functions, forms, and manifolds are assumed to be real analytic.

1. Kuranishi's fundamental theorem [4, p. 44] concerns a certain general type of differential system (called *normal*) which is generated by 1-forms θ^α, $\alpha = 1, \cdots, \alpha_1$. If $\omega^1, \cdots, \omega^p$ is a basis of a system of independent variables and $\pi^1, \cdots, \pi^m$ any other 1-forms to fill out a basis, then the θ^α are normal if $d\theta^\alpha$ can be expressed as

$$d\theta^\rho \equiv \sum_{i=1}^{p} \sum_{\lambda=1}^{m} A_{\varphi,i\lambda}\omega^i \wedge \pi^\lambda + \sum_{i=1}^{p} \sum_{j=1}^{p} \tfrac{1}{2} B_{\varphi,ij}\omega^i \wedge \omega^j$$

modulo (θ^α). Suppose that these are defined on E^n where $n = \alpha_1 + p + m$ of variables $x^1, \cdots, x^n$. Then R_g is the euclidean space of variables

$$x^j,\ u^\lambda_{i_1},\ u^\lambda_{i_1 i_2},\ \cdots,\ u^\lambda_{i_1 \cdots i_g}\ ,$$

where $j = 1, \cdots, n$; $i_1, \cdots, i_g = 1, \cdots, p$; $\lambda = 1, \cdots, m$, and the $u^\lambda_{ij\cdots k}$ are symmetric in the lower indicies.

Then $P^g(S)$ can be taken to be the system on R_g generated by the 1-forms

$$\pi_g \begin{cases} \theta^\alpha, \\ d\pi^\lambda - \sum_{j=1}^{n} u^\lambda_j \omega^j, \\ du^\lambda_{j_1} - \sum_{j=1}^{p} u^\lambda_{j_1 j}\omega^j, \\ \vdots \\ du^\lambda_{j_1 j_2 \cdots j_{g-1}} - \sum_{j=1}^{p} u^\lambda_{j_1 j_2 \cdots j_{g-1} j}\omega^j\ , \end{cases}$$

and certain functions

$$\Theta_{\varphi;ij;k_1\cdots k_t}, \qquad t \leqq g-1\ .$$

It turns out that for $t \leqq g-2$,

$$d\Theta_{\varphi;ij;k_1\cdots k_t} \equiv 0 \qquad \text{modulo } \pi_g\ ,$$

while

$$d\Theta_{\varphi;ij;k_1\cdots k_{g-1}} \equiv \sum_{\lambda=1}^{m}(A_{\varphi;i\lambda}du^{\lambda}_{jk_1\cdots k_{g-1}} - A_{\varphi;j\lambda}du^{\lambda}_{ik_1\cdots k_{g-1}}) + \sum_{k=1}^{p} B_{\varphi;ij;k_1\cdots k_{g-1}k}\omega^k ,$$

modulo π_g.

These B's are defined inductively by

$$B_{\varphi;ij;k_1\cdots k_t} = \sum_{\lambda=1}^{m}[(D_{k_t}A_{\varphi;i\lambda})du^{\lambda}_{jk_1\cdots k_{t-1}} - (D_{k_t}A_{\varphi;j\lambda})du^{\lambda}_{ik_1\cdots k_{t-1}}] + D_{k_t}B_{\varphi;ij,k_1\cdots k_{t-1}} ,$$

where D_kF is defined as follows.

If F is any function on R_{t-1} it can be considered to be a function on R_s for all $s \geqq t-1$. If we form dF, then modulo π_s, when $s \geqq t$, dF involves only $\omega^1, \cdots, \omega^p$:

$$dF \equiv \sum_{k=1}^{p} F_k\omega^k \qquad \text{modulo } \pi_s ,$$

and the F_k are independent of s so long as $s \geqq t$. Then one defines D_kF to be F_k. D_kF is a function on R_t.

THEOREM 1. *Let S be a normal system where*

(1) *the $A_{\varphi;i\lambda}$ are constants,*

(2) *$dB_{\varphi;ij} \equiv 0$ modulo $(\omega^i, \theta^\alpha)$.*

Then if $P^g(S)$ is non-singular for all g, there is a g_0 such that $P^g(S)$ is involutive for all $g \geqq g_0$ at ordinary integral points, or else there exist no solutions.

Proof. If an ordinary integral point $y \in R_g$ is not normal, then there must be a dependency among $\omega^1, \cdots, \omega^p$ implied by the 1-forms of $P^g(S)$ at integral points y_1 arbitrarily near y. This can happen only if there is a relation of the type

$$\Sigma\Gamma^{\varphi;ij;k_1\cdots k_{g-1}}(y_1)(d\Theta_{\varphi;ij;k_1\cdots k_{g-1}})_{y_1} \equiv 0 \text{ modulo } (\omega^i) ,$$

where the left side does not vanish identically. This can only happen if

$$0 = \Sigma\Gamma^{\varphi;ij;k_1\cdots k_{g-1}}(y_1)[A_{\varphi;i\lambda}(y_1)(du^{\lambda}_{jk_1\cdots k_{g-1}})_{y_1} - A_{\varphi;j\lambda}(y_1)(du^{\lambda}_{ik_1\cdots k_{g-1}})_{y_1}] ,$$

while for some k,

$$\Sigma\Gamma^{\varphi;ij;k_1\cdots k_{g-1}}(y_1)B_{\varphi;ij;k_1\cdots k_{g-1}k}(y_1) \neq 0 .$$

Since the A depend only on $x^1, \cdots, x^n$, we can choose the Γ to be functions of $x^1, \cdots, x^n$.

Now, the functions in $P^{g+1}(S)$ have the form

$$\Theta_{\varphi;ij;k_1\cdots k_g} = \sum_{\lambda=1}^{m} (A_{\varphi;i\lambda} u^{\lambda}_{jk_1\cdots k_g} - A_{\varphi;j\lambda} u^{\lambda}_{ik_1\cdots k_g}) + B_{\varphi;ij;k_1\cdots k_g} .$$

Hence we have in $P^{g+1}(S)$ the function which is not in $P^g(S)$,

$$\Sigma \Gamma^{\varphi;ij;k_1\cdots k_{g-1}} \Theta_{\varphi;ij;k_1\cdots k_{g-1}k} = \Sigma \Gamma^{\varphi;ij;k_1\cdots k_{g-1}} B_{\varphi;ij;k_1\cdots k_{g-1}k} .$$

Consider now these B. Since the A are constants,

$$B_{\varphi;ij;k_1\cdots k_t} = D_{k_t} B_{\varphi;ij;k_1\cdots k_{t-1}} ;$$

where

$$dB_{\varphi;ij;k_1\cdots k_{s-1}} \equiv \sum_{l=1}^{p} D_k B_{\varphi;ij;k_1\cdots k_{t-1}} \omega^k$$

modulo π_s.

By assumption (2), $dB_{\varphi;ij}$ have the form

$$dB_{\varphi;ij} = \sum_{k=1}^{p} C_{\varphi;ij;k}\omega^k + \sum_{\beta=1}^{\alpha_1} E_{\varphi;ij;\beta}\theta^\beta ,$$

$$= \sum_{k=1}^{p} C_{\varphi;ij;k}\omega^k \qquad \text{modulo } \pi_1 ,$$

hence

$$B_{\varphi;ij;k} = D_k B_{\varphi;ij} = C_{\varphi;ij;k}$$

are functions of $x^1, \cdots, x^n$ alone. Obviously $dB_{\varphi;ij;k} \equiv 0$ modulo $(\theta^\alpha, \omega^i)$ also, so the argument can be repeated to show that the functions

$$B_{\varphi;ij;k_1\cdots k_t}$$

depend only on $x^1, \cdots, x^n$. But that means that

$$\text{(I)} \qquad \Sigma \Gamma^{\varphi;ij;k_1\cdots k_{g-1}} B_{\varphi;ij;k_1\cdots k_{g-1}k}$$

is a function in $P^{g+1}(S)$, not in $P^g(S)$, and dependent only on $x^1, \cdots, x^n$.

Now, to any integral manifold I of S there corresponds a unique integral manifold I^{g+1} of $P^{g+1}(S)$, such that if ρ^{g+1} is the natural fibre bundle mapping on R_{g+1} to E^n, then $\rho^{g+1}(I^{g+1}) = I$ [4, p. 15]. I^{g+1} must annihilate the function (1). Since it is a function of $x^1, \cdots, x^n$ alone, I must itself annihilate it.

We conclude; if there exist ordinary integral points in R_g where $P^g(S)$ is not normal, then the manifold of integral points of S where solutions can occur must satisfy an additional condition to any imposed by $P^t(S)$, $t < g$. Clearly, this can happen at most $n - p$ times if there are to be solutions

Since the $A_{\varphi;i\lambda}$ are constants, every point of E^n is regular of order 0 [4, p. 36], so by Kuranishi's fundamental theorem there exists an integer g_1 such that if y is an ordinary integral point in R_g for $g \geqq g_1$, then $P^g(S)$ is involutive at y if and only if y is normal. Taking $g_0 = g_1 + (n - p)$ one obtains the theorem.

Next an application of this theorem will be made to a certain type of system of differential equations.

Let E^n be the euclidean space of variables $x^1, \cdots, x^p, z^1, \cdots, z^m$. Consider the problem of finding m functions $f^\lambda(x^1, \cdots, x^p) = z^\lambda$ which will satisfy a given set of first order partial differential equations

$$\frac{\partial z^\alpha}{\partial x^i} = \psi_i^\alpha(x, z), \qquad \alpha = 1, \cdots, m;\ i = 1, \cdots, p\ .$$

In terms of differential forms this is the problem of finding integral manifolds of the system S generated by the 1-forms

$$\theta^\alpha = dz^\alpha - \sum_{i=1}^{p} \psi_i^\alpha(x, z) dx^i$$

with independent 1-forms $dx^1, \cdots, dx^p$. Here there are no π^λ. Then

$$d\theta^\alpha \equiv \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\Big(\sum_{\beta=1}^{m}\frac{\partial\psi_i^\alpha}{\partial z^\beta}\psi_j^\beta - \sum_{\beta=1}^{m}\frac{\partial\psi_j^\alpha}{\partial z^\beta}\psi_i^B + \frac{\partial\psi_i^\alpha}{\partial x^j} - \frac{\partial\psi_j^\alpha}{\partial x^i}\Big)dx^j \wedge dx^i \qquad \text{modulo } (\theta^\alpha)\ .$$

If then

$$B_{\alpha;ij} = \sum_{\beta=1}^{m}\frac{\partial\psi_i^\alpha}{\partial z^\beta}\psi_j^\beta - \sum_{\beta=1}^{m}\frac{\partial\psi_j^\alpha}{\partial z^\beta}\psi_i^\beta + \frac{\partial\psi_i^\alpha}{\partial x^j} - \frac{\partial\psi_j^\alpha}{\partial x^i}$$

one can deduce the following theorem from the nature of the forms

$$d\Theta_{\phi;ij;k_1\cdots k_t} \equiv B_{\varphi;ij;k_1\cdots k_t k}\omega^k\ .$$

THEOREM 2. *In order that the system of differential equations*

$$\frac{\partial z^\alpha}{\partial x^i} = \psi_i^\alpha(x, z)$$

have a solution, given that the equations

$$B_{\varphi;ij;k_1\cdots k_t} = 0, \qquad t \leqq g$$

are non-singular for all g, it is necessary and sufficient that for all $\varphi, i, j, k_1, \cdots, k_g$,

$$B_{\varphi,ij,k_1\cdots k_g} \equiv 0 \text{ modulo } (B_{\theta;rs;h_1\cdots h_t} | t \leqq g - 1)$$

for some $g \leqq m - 1$. [See 2, p. 14].

2. In Theorem 1, the prolongation process had to yield an involutive system because whenever a non-normal prolongation occured, this implied additional restrictions on the original system. In general this need not happen. Kuranishi gives an example of a system in which $P^g(S)$ is not normal for any $g \geqq 1$ [4, p. 45].

Normality at integral points y of $P^g(S)$ involves two conditions; the set of 0-forms of $P^g(S)$ which define y must define a regular system of equations at y, and the 1-forms of $P^g(S)$ must imply no relations among the independent variables at integral points near y. This paper will ignore the first problem. It would seem to call for a more delicate approach to the Cartan-Kahler theorem. Let y be a non-normal integral point of $P^g(S)$ such that for all integral points y_1 near y there is a dependency of the type

$$\sum_{i=1}^{p} A_i(y_1)(\omega^i)_{y_1}$$

in $P^g(S)$. Then obviously solutions can occur only at points y_1 where $A_1(y_1) = A_2(y_1) = \cdots = A_p(y_1) = 0$. Hence a natural step to solving the system would be to add $A_1, \cdots, A_p$ as 0-forms to the system $P^g(S)$. One would obtain a system having the same solutions as $P^g(S)$.

Observe also that if $P^g(S)$ contains a 0-form which is a function on R_{g-1}, obviously any solution of $P^{g-1}(S)$ must annihilate that function; hence, adding it to $P^{g-1}(S)$ would generate a system having the same solutions as $P^{g-1}(S)$.

We introduce the following definition: let the system T in independent variables $x^1, \cdots, x^p$, and dependent variables $y^1, \cdots, y^r, z^1, \cdots, z^m$ be called *complete* if the 1-forms of T contain no forms of the type $\Sigma A_i\omega^i$, where $\omega^1, \cdots, \omega^p$ is a basis of independent variables, A_i not in T.

LEMMA. *Let S be any system with independent variables $x^1, \cdots, x^p$, and dependent variables $z^1, \cdots, z^m$. Then there exists a sequence $\{S^g\}$ of differential systems S^g, closed, on R_g such that*

(1) *S^g has the same solutions as $P^g(S)$,*

(2) *S^g is complete,*

(3) *$P(S^{g-1}) \subseteqq S^g$, and*

(4) *the 0-forms of S^g contain no functions on R_{g-1} except those in S^{g-1},*

(5) *S^g is generated by 0-forms, π_g, and their derivatives.*

Proof. Let X be the set of all sequences $\{T^g \mid g = 1, 2, \cdots\}$, where T^g is a closed differential system on R_g generated by 0-forms, π_g and their derivatives and having the same solutions as $P^g(S)$ and $P(T^{g-1}) \subseteqq T^g$. The elements of X can be partially ordered by inclusion: $\{U^g\} \geqq \{T^g\}$ if $U^g \supseteqq T^g$ for all $g = 1, 2, \cdots$ If $A = \{\{T^g_a\} \mid a \in A\}$ is a nest in X, then $\{T^g\}$, where T^g is the closed differential system generated by $U\{T^g_a | a \in A\}$, is in X and is $\geqq$ every element of A. Hence, X contains a maximal element, $\{S^g\}$. By definition, $\{S^g\}$ satisfies (1) and (3). If S^h were not complete, one could add to S^h the coefficients of forms of the type $\Sigma A_i \omega^i$ to obtain a still larger system $\bar{S}^h$, and $\{\bar{S}^g\}$, where $\bar{S}^g = S^g$ for $g < h$, and $\bar{S}^g = P^{g-h}(\bar{S}^h)$ for $g \geqq h$, would be properly greater than $\{S^g\}$. Similarly, if condition (4) did not hold for some S^h, we could enlarge S^{h-1}. Hence the lemma.

The construction of such a sequence, given S, could proceed as follows. Form $P(S)$ and complete it in the obvious way to form a system T^1. If the resulting system involves any functions on R_0 i.e., depending only on the coordinates of R_0, add these to the system S and begin again. Otherwise, form $P(T^1)$ and complete to form T^2. If T^2 contains functions on R_1, add these to T_1 and begin again at that step. Observe that the addition of new functions to any one system on, say, R_g, is limited by the dimension of R_g, since each such addition reduces the dimension of the variety of integral points, which must have at least dimension p if there are to be any solutions at all.

Granted that such a sequence $\{S^g\}$ as given in the lemma exists, it is still not clear whether any S^g is involutive. Of course, the 0-forms might not define a regular system of equations for the integral points. But barring this one can prove that for g sufficiently large, S^g is involutive. This follows from a recent extension of Kuranishi's prolongation theorem [5, Theorem III. 1], where the required conditions are precisely those of the lemma.

THEOREM 3. *Given a differential system S with independent variable $dx^1, \cdots, dx^p$, there exists a sequence $\{S^g\}$ of closed differential systems, where S^g is on R_g, $g = 1, 2, \cdots$, which have the same solutions as $P^g(S)$. Moreover, if for all $g \geqq g_0$, S^g is non-singular, then there exists a g_1 such that for $g \geqq g_1$, $P(S^{g-1}) = S^g$ and S^g is involutive.*

REFERENCES

1. E. Cartan, *Les systemes differentielles exterieurs et leurs applications geometriques*, Paris, 1945.

2. L. P. Eisenhart, *Non-Riemannian geometry*, New York, 1927.

3. E. Goursat, *Lecons sur l'integration des equations aux derivees partielles du premier order*, Paris, 1921.

4. M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math., vol. **79** (1957), pp. 1–47.

5. ———, *On the abstract approach to the local theory of continuous infinite pseudo groups*, Project Report, University of Chicago, 1957.