UNITARY OPERATORS IN C^*-ALGEBRAS

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1. Introduction. We present several results concerning unitary operators in uniformly closed self-adjoint algebras of operators on a Hilbert space (C^*-algebras). Section 2 contains these results the key one of which (Theorem 1) asserts a form of transitivity for unitary operators in an irreducible C^*-algebra (an application of [2, Theorem 1]). Section 3 consists of some applications. The first (Corollary 8) is a clarification of the relation between unitary equivalence of pure states of a C-*algebra and of the representations they induce. The most desirable situation prevails: two pure states of a C^*-algebra are unitarily equivalent (i. e. conjugate via a unitary operator in the algebra) if and only if the representations they induce are unitarily equivalent. The second application (Corollary 9) provides a sufficient condition for two pure states ρ and τ to be unitarily equivalent: viz. $\|\rho - \tau\| < 2$. The final application (Theorem 11) is to the affirmative solution of the conjecture that the * operation is isometric in B^*-algebras [1].

We use the notation $\sigma(A)$ for the spectrum of A; C for the set of complex numbers of modulus 1; $\mathscr{S}^-$ for the strong closure of the set of operators $\mathscr{S}$; and ω_x for the state, $A \rightarrow (Ax, x)$, due to the unit vector x. Our C^*-algebras all contain the identity operator I.

2. Unitary operators. The theorem which follows establishes an n-fold transitivity property for the unitary operators in an irreducible C^*-algebra. Its relation to [2, Theorem 1] is clear—it is, in fact, the multiplicative version of the self-adjoint portion of that theorem.

THEOREM 1. *If $\mathfrak{A}$ is a C^*-algebra acting irreducibly on $\mathscr{H}$ and V is a unitary operator on $\mathscr{H}$ such that $Vx_k = y_k$, $k = 1, \cdots, n$, then there is a unitary operator U in $\mathfrak{A}$ such that $Ux_k = y_k$ and $\sigma(U) \neq C$.*

Proof. Passing to an orthonormal basis for the finite-dimensional space generated by $\{x_1, \cdots, x_n\}$, we see that there is no loss in generality if we assume that $\{x_1, \cdots, x_n\}$, and hence $\{y_1, \cdots, y_n\}$, are orthonormal sets. Moreover, employing a unitary extension of the mapping carrying x_j onto y_j, $j = 1, \cdots, n$ to the space generated by $\{x_1, y_1, \cdots, x_n, y_n\}$ and a diagonalizing basis for this unitary operator; we see that it suffices to consider the case in which $Vx_j = \beta_j x_j$, $|\beta_j| = 1, j = 1, \cdots, n$.

Choose real α_j in the half-open interval $(-\pi, \pi]$ such that $\exp i\alpha_j = \beta_j$, and let A be a self-adjoint operator in $\mathfrak{A}$ such that $Ax_j = \alpha_j x_j$ (such

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an operator exists by [2, Theorem 1]). Define $g(\alpha)$ as α for α in [min $\{\alpha_j\}$, max $\{\alpha_j\}$], as min $\{\alpha_j\}$ and max $\{\alpha_j\}$, for $\alpha \leq \min\{\alpha_j\}$ and $\alpha \geq \max\{\alpha_j\}$, respectively. Then $g(A)$ is a self-adjoint operator in $\mathfrak{A}$ with spectrum in [min $\{\alpha_j\}$, max $\{\alpha_j\}$], and $g(A)x_j = \alpha_j x_j$. It follows that $\exp ig(A)$ is a unitary operator U in $\mathfrak{A}$, $\sigma(U) \neq C$, and $Ux_j = \beta_j x_j$.

Another unitary analogue of a known result which seems of some value is the following variant of Kaplansky's Density Theorem [3, Theorem 1]. It is a consequence of Kaplansky's theorem and some commutative spectral theory.

THEOREM 2. *If $\mathscr{U}(\mathfrak{A}, k)$ is the set of unitary operators in the C^*-algebra $\mathfrak{A}$ whose distance from I does not exceed k, then $\mathscr{U}(\mathfrak{A}, k)^-$ contains $\mathscr{U}(\mathfrak{A}^-, k)$.*

Proof. Note that $\| U - I \| \leq k$, for a unitary operator U, if and only if $\sigma(U)$ is contained in $\{z\colon |z - 1| \leq k, |z| = 1\}$, a closed subset S_k of the unit circle C. From spectral theory, each unitary operator is a uniform limit of unitary operators which are finite linear combinations of orthogonal spectral projections for it, and which do not have -1 in their spectra (i. e. whose distance from I is less than 2). Thus, it suffices to consider the case where $k < 2$.

Assuming $k < 2$, let arg z be that number in the open interval $(-\pi, \pi)$ such that $z = \exp[i \arg z]$, for z in S_k; and let f be a continuous extension of arg to C. If $a = 2\sin^{-1}(k/2)$ and $\mathscr{S}(\mathfrak{A}, \mathrm{a})$ denotes the set of self-adjoint operators in $\mathfrak{A}$ with norm not exceeding a, then f maps $\mathscr{U}(\mathfrak{A}, k)$ into $\mathscr{S}(\mathfrak{A}, a)$ continuously in the strong topology, $U = \exp[\mathrm{if}(U)]$, and exp maps $i\mathscr{S}(\mathfrak{A}, a)$ into $\mathscr{U}(\mathfrak{A}, k)$ continuously in the strong topology, from spectral theory, [3, Lemma 3], and [3, Lemma 2]. Thus, if U lies in $\mathscr{U}(\mathfrak{A}^-, k)$, $f(U)$ lies in $\mathscr{S}(\mathfrak{A}^-, a)$ and is a strong limit point of $\mathscr{S}(\mathfrak{A}, a)$, from [3, Theorem 1]; so that $U (= \exp[\mathrm{if}(U)])$ is a strong limit point of $\mathscr{U}(\mathfrak{A}, k)$.

In the next lemma, we make use of Mackey's concept of disjoint representations [5]. These are *-representations of self-adjoint operator algebras which have no unitarily equivalent non-zero subrepresentations (the restriction of the representation to an invariant subspace). The application in [5] is to unitary representations of groups and ours is to *-representations of algebras—the difference is slight, however; and our lemma and proof are valid for groups.

LEMMA 3. *If $\{\phi_\alpha\}$ are *-representations of the self-adjoint operator algebra $\mathfrak{A}$, then $\{\phi_\alpha\}$ consists of mutually disjoint representations if and only if $\phi(\mathfrak{A})^- = \sum \oplus (\phi_\alpha(\mathfrak{A})^-)$, where $\phi = \sum \oplus \phi_\alpha$.*

Proof. Suppose $\phi_\alpha(\mathfrak{A})$ acts on $\mathscr{H}_\alpha$, $\mathscr{H} = \sum \oplus \mathscr{H}_\alpha$, and P_α is the orthogonal projection of $\mathscr{H}$ upon $\mathscr{H}_\alpha$. If $\phi(\mathfrak{A})^- = \sum \oplus (\phi_\alpha(\mathfrak{A})^-)$, and U is a partial isometry [6] of $E_\alpha(\mathscr{H}_\alpha)$ onto $E_{\alpha'}(\mathscr{H}_{\alpha'})$, where $\alpha \neq \alpha'$ and E_α, $E_{\alpha'}$ are projections commuting with $\phi_\alpha(\mathfrak{A})$ and $\phi_{\alpha'}(\mathfrak{A})$, respectively, such that $U\phi_\alpha(A)U^* = \phi_{\alpha'}(A)E_{\alpha'}$ for each A in $\mathfrak{A}$, then U commutes with $\phi(\mathfrak{A})$. In fact, $U\phi(A) = U\phi_\alpha(A) = \phi_{\alpha'}(A)U = \phi(A)U$. Thus U commutes with $\sum \oplus \phi_\alpha(\mathfrak{A})$ and, in particular, with each P_α. But $UP_\alpha = U = P_\alpha U = 0$; so that $0 = E_\alpha = E_{\alpha'}$ and $\{\phi_\alpha\}$ consists of mutually disjoint representations.

If the ϕ_α are mutually disjoint and V is a partial isometry in the commutant of $\phi(\mathfrak{A})$ with the initial space E_α in P_α and final space $E_{\alpha'}$ in $P_{\alpha'}$ (cf. [6]); then $VE_\alpha\phi_\alpha(A)E_\alpha V^* = VV^*V\phi(A)V^*VV^* = E_{\alpha'}\phi(A)VV^*E_{\alpha'} = \phi_{\alpha'}(A)E_{\alpha'}$, for each A in $\mathfrak{A}$. Thus, by disjointness, E_α and $E_{\alpha'}$ are 0. It follows that the central carrier of P_α is orthogonal to that of each $P_{\alpha'}$, and hence to each $P_{\alpha'}$, with $\alpha' \neq \alpha$ (see [4], for example). Since $\sum_\alpha P_\alpha = I$, and the central carrier of P_α contains P_α, P_α is its own central carrier. In particular, P_α lies in the center of the commutant and therefore in $\phi(\mathfrak{A})^-$. It is immediate from this that $\phi(\mathfrak{A})^- = \sum \oplus (\phi_\alpha(\mathfrak{A})^-)$.

Since the commutant of an irreducible representation consists of scalars, two such are either unitarily equivalent or disjoint. From this and Lemma 3, we have as an immediate consequence:

COROLLARY 4. *If $\{\phi_\alpha\}$ is a family of irreducible *-representations of a self-adjoint operator algebra $\mathfrak{A}$, no two of which are unitarily equivalent, then $\phi(\mathfrak{A})^- = \sum \oplus \mathscr{B}_\alpha$, where $\phi = \sum \oplus \phi_\alpha$ and $\mathscr{B}_\alpha$ is the algebra of all bounded operators on the representation space of ϕ_α.*

We shall need a result asserting the possibility of "lifting" unitary operators from a representing algebra to the original algebra under certain circumstances.

LEMMA 5. *If ϕ is a *-representation of the C^*-algebra $\mathfrak{A}$ and U is a unitary operator in $\phi(\mathfrak{A})$ with $\sigma(U) \neq C$, there is a unitary operator U_0 in $\mathfrak{A}$ such that $\phi(U_0) = U$.*

Proof. As in Theorem 2, we can find a continuous function f on C such that $f(U)$ is self-adjoint and $\exp[if(U)] = U$. Let A be a self-adjoint operator in $\mathfrak{A}$ such that $\phi(A) = f(U)$. (Recall that $\phi(\mathfrak{A})$ is a C^*-algebra, so that $f(U)$ lies in $\phi(\mathfrak{A})$. If $\phi(B) = f(U)$ then A may be chosen as $(B^* + B)/2$.) If $U_0 = \exp iA$ then $\phi(U_0) = \exp[if(U)] = U$, by uniform continuity of ϕ.

REMARK 6. It may not be possible to lift a given unitary operator (as indicated by the condition $\sigma(U) \neq C$ in Lemma 5). In fact, illustrating this with commutative C^*-algebras, we may deal with the algebras of continuous complex-valued functions on compact Hausdorff spaces and unitary functions on them (functions with modulus 1). View C as the equator of a two-sphere S, and let α be the inclusion mapping of C into S. Then α induces a homomorphism of the function algebra of S onto that of C ("onto" by the Tietze Extension Theorem) which is, of course, the mapping that restricts a function on S to C. The identity mapping of C onto C is a unitary function on C which does not have a continuous unitary extension to S; for such an extension restricted to one hemisphere would amount to a retraction of the disk onto its boundary.

As a corollary to the foregoing considerations, we have the following extension of Theorem 1:

COROLLARY 7. *If $\{\phi_\alpha\}$ is a family of unitarily inequivalent irreducible *-representations of the C^*-algebra $\mathfrak{A}$ on Hilbert spaces $\{\mathscr{H}_\alpha\}$, ϕ is the direct sum of $\{\phi_\alpha\}$, $\mathscr{H}$ of $\{\mathscr{H}_\alpha\}$, $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are two finite sets of vectors with $\{x_1, \cdots, x_n\}$ linearly independent and each x_j and corresponding y_j in some $\mathscr{H}_\alpha$; then there is an A in $\mathfrak{A}$ such that $\phi(A)x_j = y_j$. If $Bx_j = y_j$ for some self-adjoint or unitary operator B on $\mathscr{H}$ then A may be chosen self-adjoint or unitary, respectively.*

Proof. The argument of [2, Theorem 1] applies directly to the first assertion once we note that the general constructions and norm estimates of that theorem can be performed on each $\mathscr{H}_\alpha$, since each x_j, y_j lie in some $\mathscr{H}_\alpha$; and the strong approximations are valid by virtue of Corollary 4. With B self-adjoint, each $P_\alpha B P_\alpha$ is self-adjoint and $P_\alpha B P_\alpha x_j = y_j$ (for x_j, y_j in $\mathscr{H}_\alpha$), where P_α is the projection of $\mathscr{H}$ onto $\mathscr{H}_\alpha$; so that the argument of [2, Theorem 1], in the self-adjoint case, applies to give a self-adjoint operator $\phi(A)$ such that $\phi(A)x_j = y_j$, $j = 1, \cdots, n$. Of course, A may be chosen self-adjoint in this case. If B is unitary it can be replaced by one which maps each $\mathscr{H}_\alpha$ onto itself and acts in the same way on $\{x_1, \cdots, x_n\}$ (extend the mappings of x_j onto y_j on each $\mathscr{H}_\alpha$). Having the self-adjoint result, in this case, the argument of Theorem 1 now applies to give a unitary operator $\phi(U)$ such that $\phi(U)x_j = y_j$, $j=1, \cdots, n$, and $\sigma[\phi(U)] \neq C$. From Lemma 5, U may be chosen as a unitary operator in $\mathfrak{A}$.

3. Some applications.

The next result indicates that the most favorable situation obtains with regard to the relation between pure states which give rise to unitarily equivalent representations.

COROLLARY 8. *If ρ and τ are pure states of the C^*-algebra $\mathfrak{A}$, then ρ and τ induce unitarily equivalent representations of $\mathfrak{A}$ if and only if there is a unitary operator U in $\mathfrak{A}$ such that $\rho(A) = \tau(U^*AU)$ for each A in $\mathfrak{A}$.*

Proof. If such a U exists, and ϕ_ρ and ϕ_τ are the representations due to ρ and τ with left kernels $\mathscr{J}$ and $\mathscr{K}$, respectively; then the mapping V of $\mathfrak{A}/\mathscr{J}$ onto $\mathfrak{A}/\mathscr{K}$ defined by, $V(A+\mathscr{J}) = AU + \mathscr{K}$, is an isometric mapping of a dense subset of the representation space for ρ onto a dense subset of the representation space for τ, since $\rho(A^*A) = \tau(U^*A^*AU)$. Thus V has a unitary extension mapping one representation space onto the other. Moreover, $V^{-1}\phi_\tau(B)V(A+\mathscr{J}) = V^{-1}(BAU+\mathscr{K}) = BA + \mathscr{J} = \phi_\rho(B)(A+\mathscr{J})$, whence the unitary extension of V implements a unitary equivalence between ϕ_τ and ϕ_ρ.

Suppose, now, that V implements a unitary equivalence between ϕ_ρ and ϕ_τ, that x and y are unit vectors in the representation spaces for ϕ_ρ and ϕ_τ, respectively, such that $\omega_x\phi_\rho = \rho$ and $\omega_y\phi_\tau = \tau$, and that $U_0Vy = x$, with U_0 a unitary operator in $\phi_\rho(\mathfrak{A})$ such that $\sigma(U_0) \neq C$ (cf. Theorem 1). Let U be a unitary operator in $\mathfrak{A}$ such that $\phi_\rho(U) = U_0$ (cf. Lemma 5). Then

$$\begin{aligned}\rho(A) &= \omega_x\phi_\rho(A) = (\phi_\rho(A)U_0Vy,\, U_0Vy) = (VV^{-1}\phi_\rho(U^*AU)Vy,\, Vy)\\ &= \omega_y\phi_\tau(U^*AU) = \tau(U^*AU)\,,\end{aligned}$$

for each A in $\mathfrak{A}$.

COROLLARY 9. *If ρ and τ are pure states of a C^*-algebra $\mathfrak{A}$ such that $\|\rho - \tau\| < 2$, then ρ and τ give rise to unitarily equivalent representations of $\mathfrak{A}$.*

Proof. If ϕ_ρ and ϕ_τ are unitarily inequivalent and ϕ, their direct sum, represents $\mathfrak{A}$ on the direct sum $\mathscr{H}$ of $\mathscr{H}_\rho$ and $\mathscr{H}_\tau$, then there are unit vectors x and y in $\mathscr{H}_\rho$ and $\mathscr{H}_\tau$, respectively, such that $\rho = \omega_x\phi$ and $\tau = \omega_y\phi$. According to Corollary 7, we can find U in $\mathfrak{A}$ such that $\phi(U)x = x$ and $\phi(U)y = -y$ (approximation using Theorem 2 would do). Then $|(\rho - \tau)U| = |(\phi(U)x, x) - (\phi(U)y, y)| = 2$; so that $\|\rho - \tau\| = 2$ (recall that $\|\rho\| = \|\tau\| = 1$, since ρ and τ are states).

REMARK 10. The condition $\|\rho - \tau\| < 2$ noted above is not necessary for unitary equivalence. Indeed, if x and y are orthogonal unit vectors in a Hilbert space $\mathscr{H}$, E is a projection with x in its range and y orthogonal to its range, then $(\omega_x - \omega_y)(2E - I) = 2$, so that $\|\omega_x - \omega_y\| = 2$; while ω_x and ω_y give rise to unitarily equivalent representations of the algebra of all bounded operators on $\mathscr{H}$ (both, in fact, equivalent to the given representation on $\mathscr{H}$). On the other hand, $|(\omega_x - \omega_y)(A)| \leq |(Ax, x - y)| + |(A(x - y), y)| \leq 2\|x - y\|$, when $\|A\| \leq 1$; so that there are pure states giving rise to unitarily equivalent representations the norms of whose differences are as small as we please.

Our next application is to the solution of a minor problem raised by

Gelfand and Neumark in connection with their conditions for a Banach algebra to be isomorphic (and isometric) to a C^*-algebra [1]. In [1], six conditions are listed for this to be the case—the first three being the standard algebraic conditions for a * operation defined on a Banach algebra, viz. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$; the fourth, $\| a^* a \| = \| a^* \| \cdot \| a \|$, is the critical condition relating the metric structure to the * operation; the fifth, $\| a^* \| = \| a \|$, asserts the isometric character of the * operation; and the sixth, the so-called "symmetry" condition, assumes that $a^* a + e$ has an inverse. Gelfand and Neumark conjectured that both the fifth and sixth conditions are consequences of the first four. After much preliminary work (notably by I. Kaplansky), the symmetry question was reduced to showing that the sum of two self-adjoint elements with non-negative spectrum is again such an element. This was done independently by Kelley-Vought and Fukamya (though not recognized as the missing information—Kaplansky pointed this out). We noted that this had been effected without assuming the * operation is isometric, and went on to prove that it was, accordingly, isometric on regular elements. From this, its continuity followed; and one could derive all but the isometric character of the isomorphism in the Gelfand-Neumark theorem, with a little care. During some seminar lectures, we noted, some years ago that the symmetry condition could be derived in a quite natural way in the course of the imbedding proof. The last loose end, establishing the fully isometric character of the * operation, can be tied by the results of this paper. The closing of this last gap would seem to be an appropriate occasion for presenting the finished result in its entirety. From another viewpoint, the supression of the fifth condition introduces subtle traps into these considerations —statements which are made in complete safety with operators require delicate proof in the present circumstances (e. g. despite the Gelfand-Neumark commutative result, we cannot take the commutative case as settled; for the uniform closure of the real algebra generated by a single self-adjoint element is not known *a priori* to consist entirely of self-adjoint elements, since continuity of the * operation is missing—again, the Schwarz inequality for states will not yield the fact that they have norm 1, under these circumstances).

By a B^*-algebra, we shall mean a Banach algebra with unit element e and normalized norm ($\| e \| = 1$, $\| ab \| \leqq \| a \| \cdot \| b \|$) which has a * operation satisfying the first four conditions noted above. An element a is self-adjoint, unitary, positive, or regular, when $a = a^*$, $a^* a = aa^* = e$, $a = a^*$ and the spectrum $\sigma(a)$ of a consists of non-negative real numbers, or a has an inverse, respectively. A state of a B^*-algebra is a linear functional which is 1 at e and real, non-negative on positive elements. We make use of the Hahn-Banach theorem from normed space theory and the following standard facts about complex Banach algebras with a unit and a normalized norm: the spectral radius $r(a)$ of an element a (i.e. $\sup\{|\alpha|: \alpha \in \sigma(a)\}$) is $\lim_{n\to\infty} \| a^n \|^{1/n}$ (and does not exceed $\| a \|$); $e + a$ is regular if $\| a \| < 1$; $\sigma(p(a)) = p(\sigma(a)) (= \{p(\alpha): \alpha \in \varepsilon(a)\})$ for each polynomial p; and the quotient modulo a maximal ideal is the complex numbers, for a commutative algebra.

THEOREM 11 (Gelfand-Neumark). *A B^*-algebra $\mathfrak{A}$ is isometric and $*$-isomorphic with a C^*-algebra.*

Proof. If $a^* = a$ then $\| a^2 \| = \| a \|^2$ so that $\| a^{2^n} \| = \| a \|^{2^n}$ and $\| a \| = r(a)$. Since $p(a)$ is self-adjoint for each real polynomial p, $\| p(a) \| = r(p(a)) = \sup\{| p(\alpha) |: \alpha \text{ in } \sigma(a)\}$. If p is complex then $p = p_1 + ip_2$, with p_1, p_2 real, and

$$r((p_1^2 + p_2^2)(a)) \le r((p_1 - ip_2)(a)) \cdot r((p_1 + p_2)(a)) \le \| [(p_1 + ip_2)(a)]^* \| \cdot \| (p_1 + ip_2)(a) \| = \| (p_1^2 + p_2^2)(a) \| = r((p_1^2 + p_2^2)(a)) .$$

Thus, equality holds throughout; and since

$$r((p_1 - ip_2)(a)) \le \| [(p_1 + ip_2)(a)]^* \| ,$$
$$r((p_1 + ip_2)(a)) \le \| (p_1 + ip_2)(a) \| ,$$

equality must hold in each. Hence $\| p(a) \| = r(p(a)) = \sup\{| p(\alpha) |: \alpha \in \sigma(a)\}$, for complex polynomials p and self-adjoint elements a. The mapping carrying an element $p(a)$ onto the polynomial p on $\sigma(a)$ is an isomorphism of the algebra of (complex) polynomials in a into $C(\sigma(a))$ and has an isometric isomorphism extension mapping the closure $\mathfrak{A}(a)$ onto the closure P of the polynomials in $C(\sigma(a))$.

If $\alpha \in \sigma(a)$, then the mapping $g \to g(\alpha)$ (g in P) is a linear functional of norm 1 on P which assigns 1 to the image in P of e and α to that of a. Via the isometry, this gives rise to a linear functional f_0 of norm 1 on $\mathfrak{A}(a)$, such that $f_0(e) = 1, f_0(a) = \alpha$. Let f be a norm 1 extension of f_0 to $\mathfrak{A}$. If b is self-adjoint and $f(b)$ is not real, by adding a suitable real multiple of e to b we arrive at a self-adjoint element on which f takes a non-zero imaginary value. Suppose $f(b) = i\beta$, with $\beta > 0$ (if $\beta < 0$, use $-b$). Then

$$\begin{aligned} | f(b + ine) |^2 &= \beta^2 + 2\beta n + n^2 \le \| b + ine \|^2 = [r(b + ine)]^2 \\ &= (r(b + ine))(r([b + ine]^*)) = \| b + ine \| \cdot \| b - ine \| = \| b^2 + n^2 e \| \\ &\le \| b^2 \| + n^2 , \end{aligned}$$

which is absurd for $n > (\| b^2 \| - \beta^2)/2\beta$ (note that $r(c) = r(c^*)$, for each c, since $\sigma(c^*) = \sigma(c)$). Thus f is real on each self-adjoint element. In particular, $f(a) = \alpha$ is real, and $\sigma(a)$ consists of real numbers. Hence, the algebra of complex polynomials on $\sigma(a)$ is invariant under complex conjugation, the Stone-Weierstrass theorem applies, and P is $C(\sigma(a))$. If $b \geq 0$ and $f(b) < 0$, then $\sigma(b - \|b\| e) = \sigma(b) - \|b\|$. Since $\sigma(b) \geq 0$, $r(b - \|b\| e) = \|b - \|b\| e\| \leq \|b\|$. But $|f(b - \|b\| e)| = |f(b) - \|b\|| > \|b\| \geq \|b - \|b\| e\|$, contradicting $\|f\| = 1$. Thus f is a state of $\mathfrak{A}$, $f(a) = \alpha$, and f has norm 1.

If $a_1, \cdots, a_n$ are positive and $\alpha \in \sigma(a_1 + \cdots + a_n)$ there is a state f of $\mathfrak{A}$ such that $\alpha = f(a_1 + \cdots + a_n) = f(a_1) + \cdots + f(a_n) \geq 0$, so that $a_1 + \cdots + a_n$ is positive. If b is self-adjoint and has an inverse, then 0 is not in $\sigma(b)$; so that the image of b in $C(\sigma(b))$ has an inverse; and the inverse of b lies in $\mathfrak{A}(b)$. If b is in $\mathfrak{A}(a)$, with a self-adjoint, then $\mathfrak{A}(b)$ is contained in $\mathfrak{A}(a)$ and the inverse of b lies in $\mathfrak{A}(a)$. Thus the spectrum of a self-adjoint element in $\mathfrak{A}(a)$ is the same relative to $\mathfrak{A}(a)$ and to $\mathfrak{A}$. In view of the isomorphism between $\mathfrak{A}(a)$ and $C(\sigma(a))$, this spectrum is the range of its representing function in $C(\sigma(a))$. Thus $a^2 \geq 0$ for a self-adjoint. With a and b positive, choose a state f of norm 1 such that $f(a) = r(a) = \|a\|$, then $\|a + b\| \geq f(a + b) \geq f(a) = \|a\|$.

Suppose next that $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$ which is maximal with respect to the properties of being abelian and self-adjoint (i.e. $\mathfrak{A}_0^* = \mathfrak{A}_0$). If b commutes with $\mathfrak{A}_0$ then $ba^* = a^*b$, for each a in $\mathfrak{A}_0$; so that $b^*a = ab^*$. Thus, the self-adjoint elements $b + b^*$ and $(b - b^*)/i$ commute with $\mathfrak{A}_0$, and, by maximality, lie in $\mathfrak{A}_0$. Hence $b (= (b + b^*)/2 + i(b - b^*)/2i)$ lies in $\mathfrak{A}_0$; and $\mathfrak{A}_0$ is maximal with respect to the property of being abelian. It follows that $\mathfrak{A}_0$ is closed. If b is the limit of self-adjoint elements in $\mathfrak{A}_0$ and $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint (in $\mathfrak{A}_0$ — the decomposition just noted), then

$$\begin{aligned} \|b_2^2\| &\leq \|(b_1 - a)^2 + b_2^2\| = \|b_1 + ib_2 - a\| \cdot \|b_1 - ib_2 - a\| \\ &\leq \|b - a\| \cdot (\|b^*\| + \|a\|) , \end{aligned}$$

with a self-adjoint in $\mathfrak{A}_0$. Choosing a near b, we see that $\|b_2^2\| (= \|b_2\|^2)$ is dominated by an arbitrarily small quantity, so that $b_2 = 0$. Thus b is self-adjoint, and the self-adjoint elements in $\mathfrak{A}_0$ are closed. If a is self-adjoint, the polynomials in a from a commutative self-adjoint algebra which can be imbeded in a maximal one $\mathfrak{A}_0$ (Zorn's Lemma). Since $\mathfrak{A}_0$ is closed, $\mathfrak{A}(a)$ is contained in it. Thus, the closure of the real polynomials in a (which maps onto the algebra of real functions in $C(\sigma(a))$) consists of self adjoint elements.

The isomorphism of $\mathfrak{A}(a^*a)$ with $C(\sigma(a^*a))$ establishes the existence of positive elements b and c in $\mathfrak{A}(a^*a)$ such that $a^*a = b - c$, and $bc = 0$. Thus $(ac)^*(ac) = -c^3$, which is negative, so that $(ac)(ac)^*$ is negative. (In an arbitrary ring with a unit, if c is the inverse to $e - ab$ then $e + bca$ is the inverse to $e - ba$; so that, in a Banach algebra, the spectra of ab and ba with 0 adjoined is the same set.) But with $ac = a_1 + ia_2$, a_1 and a_2 self-adjoint,

$$0 \geq (ac)(ac)^* + (ac)^*(ac) = 2(a_1^2 + a_2^2) \geq 0.$$

Thus,

$$0 = || a_1^2 + a_2^2 || \geq || a_1 ||^2, || a_2 ||^2;$$

so that $a_1 = a_2 = ac = c^3 = c = 0$, and $a^*a \geq 0$. The function representing a^*a in $C(\sigma(a^*a))$ is real and non-negative and therefore has a continuous non-negative square root. This square root corresponds to an element $(a^*a)^{1/2}$ which is a positive square root of a^*a in $\mathfrak{A}(a^*a)$. If a is regular so is a^* and $(a^*a)^{1/2}$. The element $a(a^*a)^{-1/2}(=u)$ is unitary, since $uu^* = a(a^*a)^{-1}a^* = aa^{-1}a^{*-1}a^* = e$ and $u^*u = (a^*a)^{-1/2}a^*a(a^*a)^{-1/2} = (a^*a)(a^*a)^{-1} = e$. Extend the self-adjoint abelian algebra of polynomials in u and u^* to a maximal one $\mathfrak{A}_0$; and let M be a proper maximal ideal in $\mathfrak{A}_0$. Then $b + M = b(M)e + M$, for some complex number $b(M)$(i.e. $\mathfrak{A}_0/M$ is the complex numbers), $(cb)(M) = c(M)b(M)$, $b(M)$ is in the spectrum of b relative to $\mathfrak{A}_0$, and if $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint then $b^*(M) = b_1(M) - ib_2(M) = \overline{b(M)}$, since the spectra of b_1 and b_2 are real. Thus $1 = e(M) = (u^*u)(M) = | u(M) |^2$. Now $|| u || \geq r(u) \geq 1$, and similarly, $||u^*|| \geq 1$. But $1 = || e || = || u^*u || = || u^* || \cdot || u ||$, so that $|| u || = 1$. Hence

$$|| a ||^2 = || u(a^*a)^{1/2} ||^2 \leq || (a^*a)^{1/2} ||^2 = || a^*a || = || a^* || \cdot || a ||;$$

and $|| a || \leq || a^* ||$, symmetrically, $|| a^* || \leq || a ||$, so that $|| a || = || a^* ||$ (i.e. the * operation is isometric on regular elements). If $|| b || < 1$, then $e+b$ is regular, so that $|| b^* || - 1 \leq || e + b^* || = || e + b || \leq 1 + || b || < 2$. Thus the * operation is continuous (bounded), and the self-adjoint elements in $\mathfrak{A}$ form a closed set.

If f is a state of $\mathfrak{A}$, the mapping $a, b \to f(b^*a)$ is a positive semi-definite inner product on $\mathfrak{A}$ (write (a, b) for the inner product of a and b). If a is a null vector then $(ba, ba) = (a, b^*ba) = 0$ (from the Schwarz inequality); so that ba is a null vector. Thus, the set $\mathscr{I}$ of null vectors is a left ideal in $\mathfrak{A}$ (the "left kernel" of f). The quotient vector space $\mathfrak{A}/\mathscr{I}$ has a positive definite inner product induced on it from that on $\mathfrak{A}$. Let $\mathscr{H}$ be the (Hilbert space) completion of $\mathfrak{A}/\mathscr{I}$ in this inner product. Define the operator $\phi(a)$ on $\mathfrak{A}/\mathscr{I}$ by $\phi(a)(b + \mathscr{I}) = ab + \mathscr{I}$, for each a in $\mathfrak{A}$. If $c \geq 0$, then $(b^*c^{1/2})(c^{1/2}b) \geq 0$. With $|| a^*a ||e - a^*a$ in place of c, we have $|| a^*a || b^*b \geq b^*a^*ab$; so that $3 || a ||^2 (b+\mathscr{I}, b+\mathscr{I}) \geq || a^*a || f(b^*b) \geq f(b^*a^*ab) = (\phi(a)(b + \mathscr{I}), \phi(a)(b + \mathscr{I}))$, and $|| \phi(a) || \leq 3^{1/2} || a ||$ $(|| \phi(a) || \leq || a ||$ if a is self-adjoint). Thus $\phi(a)$ has a unique extension to $\mathscr{H}$, with the same bound, which we denote again by $\phi(a)$. Since $(\phi(a)(b + \mathscr{I}), c + \mathscr{I}) = f(c^*ab) = (b + \mathscr{I}, \phi(a^*)(c + \mathscr{I}))$, $\phi(a)^* = \phi(a^*)$. It follows that ϕ is a *-representation of $\mathfrak{A}$ in the algebra of bounded operators on $\mathscr{H}$. If $\phi(a) = 0$ then $f(a) = (\phi(a)(e+\mathscr{I}), e+\mathscr{I}) = 0$.

If we perform this construction for each state of $\mathfrak{A}$, the direct sum ψ of the resulting *-representations is a *-isomorphism of $\mathfrak{A}$. In fact, if $\psi(a) = 0$, then $\psi(a^*a) = 0$, so that $f(a^*a) = 0$ for each state f of $\mathfrak{A}$. But there is a state f such that $f(a^*a) = \| a^*a \| = \| a^* \| \cdot \| a \|$. Thus $a = 0$. If b is self-adjoint $\| \psi(b) \| \leq \| b \|$, since each of the representations is norm decreasing on b. With f a state of $\mathfrak{A}$ such that $\| b \| = | f(b) | = | (\phi(b)(e + \mathscr{I}), e + \mathscr{I}) |$, however, we see that $\| \phi(b) \| \geq \| b \|$; so that $\| \psi(b) \| = \| b \|$. Since the self-adjoint elements in $\mathfrak{A}$ are closed (hence complete) they are complete (hence closed) in $\psi(\mathfrak{A})$; whence $\psi(\mathfrak{A})$ is closed (i.e. a C^*-algebra). If a is regular, $\| a \|^2 = \| a^*a \| = \| \psi(a^*a) \| = \| \psi(a) \|^2$.

Defining $||| \psi(b) |||$ to be $\| b \|$, $\psi(\mathfrak{A})$ has two norms ($||| \ |||$, and its operator norm $\| \ \|$) relative to which it is a B^*-algebra. These norms agree on self-adjoint and regular elements. If we show that they agree everywhere (i.e. that ψ is isometric) then the $*$ operation is isometric on $\mathfrak{A}$ since it is preserved by ψ and is isometric on $\psi(\mathfrak{A})$. We write $\mathfrak{A}$ in place of $\psi(\mathfrak{A})$ ($\mathfrak{A}$ is a C^*-algebra with the two B^*-norms as described). As the first step, we establish the formula $\| A \| = \sup \{ | f(UAV) | \colon U$ and V unitary operators in $\mathfrak{A}$ and f a pure state of $\mathfrak{A} \}$. Since each state of $\mathfrak{A}$ has norm 1 (from the Schwarz inequality) relative to the operator norm, $| f(UAV) | \leq \| UAV \| \leq \| A \|$. On the other hand, if ϕ is the (irreducible) representation induced by f, $| f(U^*AV) | = |(\phi(A)\phi(V)x, \phi(U)x) |$, where x is a unit vector (in fact, the special one corresponding to $I + \mathscr{I}$). In view of Theorem 1 (or Theorem 2), $\sup \{ | f(U^*AV) | \colon U$ and V unitary operators in $\mathfrak{A} \} = \sup \{ | (\phi(A)x, y) | \colon \| x \| = \| y \| = 1 \} = \| \phi(A) \|$. Now the direct sum of the $*$-representations due to each pure state of $\mathfrak{A}$ is a $*$-isomorphism and hence an isometry of $\mathfrak{A}$; so that $\sup \{ \| \phi(A) \| \colon f$ a pure state of $\mathfrak{A} \} = \| A \|$, and our formula follows.

Each state of $\mathfrak{A}$ has norm 1 relative to the norm $||| \ |||$; for if $||| B ||| < 1$ and $f(B) = | f(B) | \alpha$ (where $| \alpha | = 1$), then $\bar{\alpha}B + I$ is regular. Hence, $| f(\bar{\alpha}\beta + I) | = | f(B) | + 1 \leq \| \bar{\alpha}B + I \| = ||| \bar{\alpha}\beta + I ||| < 2$. Thus $| f(UAV) | \leq ||| UAV ||| \leq ||| A |||$; and $\| A \| \leq ||| A |||$, for each A. But $\| A^* \| \cdot \| A \| = \| A^*A \| = ||| A^*A ||| = ||| A^* ||| \cdot ||| A |||$; so that $\| A \| = ||| A |||$ for each A. The proof is complete.

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