A CLASS OF SMOOTH BUNDLES OVER A MANIFOLD

JAMES EELLS, JR.

1. Introduction. In this paper we illustrate certain constructions of importance in the geometry of smooth manifolds. First of all we prove that *a homogeneous space* B *of a connected Lie group* G *can always be represented as a homogeneous space of a contractible Lie group* E, necessarily of infinite dimension in general. In particular, that representation shows that the loop space of B can be replaced effectively by a Lie group of infinite dimension. The construction is a special case of a general theory of differentiable structures in function spaces [4]. Secondly, we examine relations between the Lie algebra of G and that of E (this latter being a Banach-Lie algebra), in case G is compact and semi-simple.

As an application we consider certain differentiable fibre bundles over a smooth (i.e., infinitely differentiable) manifold X having infinite dimensional Lie structure groups. Particular attention is given to the bundles associated with maps of X into a sphere; these bundles are important because they are in natural (Poincaré dual) correspondence with certain equivalence classes of normally framed submanifolds of X. Using a theory of smooth differential forms in function spaces, *we give explicit integral representation formulas for the characteristic classes of these bundles.* These formulas provide examples of a residue theory of differential forms with singularities [1]—and express those forms with singularities as forms without singularities in differentiable bundles over X.

2. The homogeneous spaces. (A) Let G be a connected Lie group (of finite dimension!), and let $L(G)$ denote its Lie algebra, considered as the tangent space to G at its neutral element e. If K is a closed subgroup of G, we let B denote the homogeneous space G/K of left cosets of K. The coset map $\pi: G \to B$ is an analytic fibre bundle map [9, § 7].

We now construct an *acyclic* fibre bundle over B; our construction is a variant of Serre's space of paths over B based at a point [8, Ch. IV]. For this purpose we have chosen a special class of paths on G suitable for our applications in § 5. (These path spaces are also of importance in the calculus of variations.)

(B) Let G be given a left invariant Riemann structure, determined by an inner product on $L(G)$. If $\mathscr{T}(G)$ denotes the tangent vector bundle of G with projection map $q: \mathscr{T}(G) \to G$, then $\mathscr{T}(G)$ has induced Riemann structure. If u, v are tangent vectors at a point

Received June 10, 1959. Research partially supported by the Office of Naval Research.

$m \in G$, we let $(u, v)_m$ denote their inner product, and $|v|_m$ denote the length of v.

DEFINITION. If I is the unit interval $\{t \in I: 0 \leq t \leq 1\}$, we say that a map $x: I \to G$ is an *admissible path on* G if it satisfies the following conditions:

(1) $x(0) = e$, the neutral element of G;

(2) x is absolutely continuous in the metric of G; then its tangent vector $x'(t)$ exists for almost all $t \in I$, and we require that

(3) the tangent map $x': I \to \mathscr{T}(G)$ is square integrable; i.e., the Lebesgue integral

$$\int_0^1 |x'(t)|^2_{x(t)} dt \tag{1}$$

is finite. We observe that $x(t) = q \circ x'(t)$ for each $t \in I$ for which $x'(t)$ exists.

Let $E(G)$ denote the totality of admissible paths on G. Using pointwise multiplication and metric defined analogously to (1), it is easily seen that $E(G)$ is a topological (metrizable) group. As in the case of continuous path spaces [8, p. 481], $E(G)$ is a contractible group with contraction $h: E(G) \times I \to E(G)$ given by $h(x, t)s = x(ts)$.

Let $p: E(G) \to G$ be defined by $p(x) = x(1)$. Then p is a continuous epimorphism whose kernel is the subgroup $\Omega(G) = \{x \in E(G): x(1) = e\}$ of admissible loops on G; thus we have an exact sequence

$$0 \longrightarrow \Omega(G) \longrightarrow E(G) \xrightarrow{p} G \longrightarrow 0 \tag{2}$$

of topological groups. If $E(G, K) = \{x \in E(G): x(1) \in K\}$, then $E(G, K)$ is a closed subgroup of $E(G)$, and the composition $\lambda = \pi \circ p: E(G) \to G \to B$ is a representation of B as a homogeneous space of $E(G)$, with $E(G, K)$ as fibre over $b_0 = \pi(K) \in B$.

PROPOSITION. *$\lambda: E(G) \to B$ is a principal $E(G, K)$-bundle.*

To prove that it remains (by [9, p. 30]) to show that there is a local section of $E(G)$ defined in a neighborhood of b_0; because π is a bundle map it suffices to find a neighborhood V of e in G and a section f of $E(G)$ over V. We use the Riemann structure of G to obtain a neighborhood V of e such that for any point $m \in V$ there is a unique geodesic segment $x_m: I \to V$ such that $x_m(0) = e$ and $x_m(1) = m$; then x_m is clearly an admissible path, and $f(m) = x_m$ is a continuous map of V into $E(G)$ such that $p \circ f(m) = m$ for all $m \in V$.

(C) The following result is an application of a general theory of function space manifolds [4].

THEOREM. *Let G be a connected Lie group, and $E(G)$ the space of* *its admissible paths. Then $E(G)$ is an infinite dimensional Lie group modeled on a separable Hilbert space. The map $p: E(G) \to G$ is an analytic bundle epimorphism.*

We recall the principal ideas of that construction. Given $x \in E = E(G)$, *the tangent space to E at x* is the separable Hilbert space $E(x)$ of maps $u: I \to \mathscr{T}(G)$ such that

(1) $q \circ u(t) = x(t)$ for all $t \in I$,

(2) $u(0) = 0$ (the zero in $L(G)$), and

(3) the map u is absolutely continuous with square integrable tangent vector field, and the norm $|u|_x$ induced from the inner product (3) below is finite. Thus $E(x)$ is considered as the space of *admissible variations* of the path x. The algebraic operations in $E(x)$ are defined pointwise; i.e., if $u, v \in E(x)$ and $a, b \in \boldsymbol{R}$, then $(au + bv)t = au(t) + bv(t)$, where the right member is computed in the tangent space $G(x(t))$. A symmetric, bilinear form in $E(x)$ is defined by

$$(u, v)_x = \int_0^1 (u'(t), v'(t))_{x(t)} dt \; ; \tag{3}$$

this is an inner product, for if $(u, u)_x = 0$, then $|u'(t)|_{x(t)} = 0$ for almost all $t \in I$, and the condition that u is admissible then implies $u(t) = 0$ for all $t \in I$. We emphasize that each $E(x)$ is complete (by standard L^2 theory), a property that is used in the theory of differentiation in infinite dimensional linear spaces.

Using the natural correspondence (defined locally) between geodesic segments on G emanating from a point m and tangent vectors in $G(m)$, we can find a neighborhood U_x (called a *coordinate patch*) of x in $E(G)$ which is mapped homeomorphically (by a map ϕ_x called a *coordinate system*) onto a neighborhood of 0 in $E(x)$ [4, § 3]. In overlapping coordinate patches U_x, U_y we have a map

$$\phi_{xy} : \phi_x(U_x \cap U_y) \longrightarrow \phi_y(U_x \cap U_y)$$

defined by $\phi_{xy}(u) = \phi_y \circ \phi_x^{-1}(u)$, and this map is analytic in its domain of definition. (If ϕ is a map of an open subset U of a Hilbert space E into a Hilbert space F, then *ϕ is analytic in U* if every $x \in U$ has a neighborhood in which ϕ can be expressed by the convergent power series

$$\phi(x + v) = \phi(x) + \sum_{k=1}^{\infty} P_\phi^k(x, v)/k! \; ,$$

where $P_\phi^k(x, v)$ denotes the kth iterated directional derivative of ϕ at x in the direction v.) Easy modifications of standard Lie group theory show that the group operation in $E(G)$ is analytic and that $p: E(G) \to G$ is an analytic homomorphism.

COROLLARY. *The fibration* $\lambda : E(G) \to B$ *is an analytic bundle map.*

(D) REMARK. *The inner product* (3) *is easily seen to provide an analytic Riemann structure on* $E(G)$. We note, however, that it is not left invariant on $E(G)$.

Suppose we let G *act* on $E(G)$ by $T_g(x)t = gx(t)g^{-1}$ for all $t \in I$ and $x \in E(G)$. If G is compact and semi-simple and if the inner product (3) is computed using the bi-invariant Riemann metric on G (see our § 3A), then the *Riemann structure on* $E(G)$ *is* G*-invariant.*

3. **The Lie algebra of certain path groups.** (A) Suppose that G is connected, compact, and semi-simple. Then its Killing form [**7**, §§ 6, 11] defines a bi-invariant Riemann structure on G (essentially unique); furthermore, the inner product and the bracket in $L(G)$ are related by

$$([x, y], z) = (x, [y, z]) \tag{1}$$

for all $x, y \in L(G)$. By taking a suitable real multiple of the Killing form we can suppose that the norm induced from the inner product and the bracket in $L(G)$ are related by

$$|[x, y]| \leq |x||y| \tag{2}$$

for all $x, y \in L(G)$.

(B) If e also denotes the neutral element of $E(G)$ (so that $e(t) = e$ for all $t \in I$), then the tangent space $E(e)$ consists of those admissible paths on $L(G)$ starting at 0; we introduce the bracket of u and v in $E(e)$ by

$$[u, v]t = [u(t), v(t)] \qquad \text{for all } t \in I. \tag{3}$$

We will call $E(e)$ *the Lie algebra of* $E(G)$, and henceforth will denote it by $L(E(G))$; note that $L(E(G) = E(L(G))$. Of course the exponential map exp: $L(E(G)) \to E(G)$ is defined by $(\exp u)t = \exp(u(t))$ for all $t \in I$.

If $|u|_e^2 = (u, u)_e$ in the notation of § 2 (3), then the following result shows that the bracket (3) on $L(E(G))$ is continuous.

LEMMA. *For any* $u, v \in L(E(G)$ *we have*

$$|[u, v]|_e \leq 2|u|_e|v|_e . \tag{4}$$

Proof. First of all, we note that if $m_u = \max\{|u(t)| : t \in I\}$, then $m_u \leq |u|_e$. Namely, for any $t \in I$ we apply the Schwarz inequality to obtain

$$|2u(t) - u(1)|^2 = \left|\int_0^1 \operatorname{sgn}(t-s)u'(s)ds\right|^2 \leq \int_0^1 \operatorname{sgn}(t-s)^2 ds \int_0^1 |u'(s)|^2 ds .$$

Thus

$$m_u \leq \max\{|2u(t) - u(1)| : t \in I\} \leq |u|_e .$$

By (2) and the Schwarz inequality in $L(G)$ we find that $|[u, v]|_e^2$ is bounded by

$$\begin{aligned}\int_0^1 \{|u'(t)|^2 |v(t)|^2 + 2|u'(t)||v(t)||u(t)||v'(t)| + |u(t)|^2 |v'(t)|^2\} dt \\ \leq m_v^2 \int_0^1 |u'(t)|^2 dt + 2m_u m_v \int_0^1 |u'(t)||v'(t)| dt + m_u^2 \int_0^1 |v'(t)|^2 dt \\ \leq 4|u|_e^2 |v|_e^2 .\end{aligned}$$

The inequality (4) follows.

REMARK. Unlike the finite dimensional Hilbert-Lie algebra $L(G)$, $L(E(G))$ does not satisfy a relation of the form (1). Thus the bracket in $L(E(G))$ respects its Banach space structure—i.e., *$L(E(G)$ is a Banach-Lie algebra*—rather than its structure as a Hilbert space.

(C) Let $p_* : L(E(G)) \to L(G)$ be defined by $p_*(u) = u(1)$; clearly p_* is a Lie algebra epimorphism, and the inequality

$$|u(t_2) - u(t_1)| \leq |t_1 - t_2|^{1/2} |u|_e \qquad \text{for any } t_1, t_2 \in I$$

shows that $|p_*(u)| \leq |u|_e$ for all $u \in L(E(G))$.

Our next result establishes an infinitesimal analogue of the analytic bundle over G given by Theorem 2C.

THEOREM. *If G is a connected, compact, semi-simple Lie group, then p_* is a continuous Lie epimorphism with kernel $L(\Omega(G)) = \Omega(L(G))$, the closed ideal of admissible loops on $L(G)$; i.e.,*

$$(5) \qquad 0 \longrightarrow L(\Omega(G)) \longrightarrow L(E(G)) \xrightarrow{p_*} L(G) \longrightarrow 0$$

is an exact sequence of Banach-Lie algebras. Furthermore, as Hilbert spaces (but not as Lie algebras), *p_* induces an orthogonal direct decomposition $L(E(G)) \approx L(\Omega(G)) \oplus M$, where M is a vector space isomorphic to $L(G)$.*

Proof. The first statement follows from the algebraic properties of p_* and the fact that p_* is bounded, and therefore continuous. To prove the second, we define a map j: $L(G) \to L(E(G))$ by letting $j(x)$ be the linear path $j(x)t = tx$ for each $x \in L(G)$; then j is a linear map of $L(G)$ onto a subspace M of $L(E(G))$, and $p_* \circ j$ is the identity; moreover, i is an isometry, because for any $x, y \in L(G)$,

$$(j(x), j(y))_e = \int_0^1 (x, y) dt = (x, y) .$$

Note, however, that M is not a subalgebra of $L(E(G))$.

The subspaces $L(\Omega(G))$ and M are orthogonal complements in $L(E(G))$, for if $x \in L(G)$ and $v \in L(\Omega(G))$, then

$$(j(x), v)_e = \int_0^1 (x, v'(t))dt = (x, v(1)) - (x, v(0)) = 0 .$$

COROLLARY. *The group $\Omega(G)$ of admissible loops on G forms a subgroup of $E(G)$ whose codimension (as a submanifold of $E(G)$) equals the dimension of G.*

REMARK. If K is a closed subgroup of G and if we set $\lambda_* = \pi_* \circ p_* : L(E(G)) \to L(G) \to L(G)/L(K)$, then we have an exact sequence of vector spaces

$$0 \longrightarrow L(E(G, K)) \longrightarrow L(E(G)) \xrightarrow{\lambda_*} L(G)/L(K) \longrightarrow 0 .$$

(D) PROBLEM. Consider $L(E(G))$ as a Hilbert space, and form its *topological exterior algebra* $C^*(L(E(G)))$, using the natural inner product on its pth exterior power. The inequality (4) implies that we can construct the Lie algebra cochain complex as in [7, § 3] and that the differential operator in $C^*(L(E(G)))$ is continuous. The elements $\omega \in C^p(L(E(G)))$ determine left invariant differential p-forms on $E(G)$—an important property because a version of de Rham's Theorem is valid for $E(G)$ (see § 5A). What are the relations between the derived cohomology algebras $H^*(L(E(G)))$, $H^*(L(\Omega(G)))$, and $H^*(L(G)) \approx H^*(G; \boldsymbol{R})$?

As a first step, because $L(\Omega(G))$ is a closed ideal in $L(E(G))$ we can appeal to our Theorem 3C and Theorem 4 of *Cohomology of Lie algebras*, G. Hochschild and J-P. Serre, Annals of Math. **57** (1953), 591–603, to obtain the

PROPOSITION. *The filtration of $C^*(L(E(G)))$ by the ideal $L(\Omega(G))$ determines a spectral sequence such that*

$$E_2^{p,q} = H^p(L(G); H^q(L(\Omega(G))) ,$$

and whose terminal algebra E_∞ is the graded algebra associated with $H^*(L(E(G)))$, suitably filtered.

4. The bundles over a manifold. (A) Let $B = G/K$ be the homogeneous space of § 2A. Since $E(G)$ is contractible, the fibre bundle $\lambda : E(G) \to B$ can be interpreted as a universal bundle [9, § 19] for the infinite dimensional Lie group $E(G, K)$. In particular, by the Classification Theorem for principal bundles we have the

PROPOSITION. *If X is a paracompact smooth manifold of finite* *dimension, then the isomorphism classes of smooth principal $E(G, K)$-bundles over X are in natural one-to-one correspondence with the smooth homotopy classes of maps of X into B.*

In that statement we have made use of the fact that for maps of X into B their classification by homotopy equivalence coincides with classfication by smooth homotopy equivalence.

REMARK. There is a certain uniqueness theorem for universal bundles over B, which implies that for any other contractible bundle over B with group Γ, the homotopy groups of Γ are isomorphic to those of $E(G, K)$; see [**6**, p. 284]. Of course, it follows directly from the homotopy sequence of a bundle and the 5-lemma that the homotopy groups of $E(G, K)$ are isomorphic to those of the loop space of B.

(B) Suppose that B is $(n-1)$-connected and that the nth homotopy group $\pi_n(B)$ is infinite cyclic $(n > 1)$; then the group $E(G, K)$ is $(n-2)$-connected, and the connecting homomorphism of the homotopy sequence of the universal bundle of B is an isomorphism of $\pi_n(B)$ onto $\pi_{n-1}(E(G, K))$.

Let $\mu : W \to X$ be an $E(G, K)$-bundle over X. Its *characteristic class* [**9**, p. 178] is the primary obstruction to the construction of a section of the bundle. The condition $n > 1$ insures that its structural group is 0-connected, whence the bundle $\mathscr{B}$ of local coefficients (used in defining characteristic classes in general) is simple [**9**, p. 153]. *To orient the bundle* is to choose one of the two isomorphism of $\mathscr{B}$ onto the product bundle $X \times Z$. Thus the characteristic class of an oriented $E(G, K)$-bundle over X is a cohomology class $w \in H^n(X, \boldsymbol{Z})$.

It is well known that such a characteristic class can be represented by a transgressive pair of cochains (a^n, c^{n-1}). (A *transgressive pair* in a bundle consists of a cochain of some sort c on W whose restriction to a fibre is a cocycle of $E(G, K)$, and such that its coboundary $dc = \mu^* a$ for some cocycle a of X.) Furthermore, the restriction of c^{n-1} to a fibre defines the generator of $H^{n-1}(E(G, K); \boldsymbol{Z}) \approx \boldsymbol{Z}$ which is the negative of that determined by the orientation of the bundle.

Let w_0 be the characteristic class of the universal oriented bundle $\lambda : E(G) \to B$. Suppose that $\mu : W \to X$ is induced by the smooth map $f : X \to B$, and let $g : W \to E(G)$ be a smooth bundle map covering f [**9**, § 19]. If (a_0, c_0) is a transgressive pair representing w_0, then $a = f^* a_0$, $c = g^* c_0$ is known to be a transgressive pair representing the characteristic class w of $\mu : W \to X$ [**2**, § 18].

5. Representations of the characteristic classes. (A) Let Y be any paracompact smooth manifold modeled on a Hilbert space E. A *differential r-form η on Y* assigns to each point $y \in Y$ an alternating r-linear functional (with real values) on the tangent space $Y(y)$, which is continuous simultaneously in the r variables, using the Hilbert space topology in $Y(y)$. In terms of the differentiable structure on Y we can define the exterior algebra $\mathscr{E}^*(Y)$ of smooth differential forms on Y and its derived cohomoly algebra $H^*(\mathscr{E}^*(Y))$. It is known (an extension of de Rham's Theorem [4, § 4]) *that there is a canonical isomorphism of* $H^*(\mathscr{E}^*(Y))$ *onto* $H^*(Y; \boldsymbol{R})$, *the singular real cohomology algebra of* Y.

We remark that this result uses the local Hilbert space structure of Y in two ways:

(1) the square of the norm in E is an analytic function on E, which implies that there are sufficiently many smooth functions on Y;

(2) there is a natural Hilbert space structure on the rth exterior power of E; its completeness is used essentially in the differentiability of differential forms.

We will now give examples of such forms which are transgressive pairs on $E(G, K)$-bundles over X.

(B) We have seen in Theorem 2C that the group $E(G)$ of admissible paths on a connected Lie group G is itself a Lie group modeled on a Hilbert space. Since $E(G)$ is contractible, the general existence theorem quoted in (A) insures that any smooth closed r-form ω on $E(G)$ is the exterior differential of a smooth $(r-1)$-form ξ (for $r > 0$). The following result uses a standard homotopy construction to give an explicit formula for ξ in case ω is the p^*-image of a form on G.

PROPOSITION. *Given any smooth closed r-form ω on G $(r > 0)$, consider the $(r-1)$-form on $E(G)$ defined as follows: For any $x \in E(G)$ and $r-1$ vectors $u_1, \cdots, u_{r-1}$ in the tangent space at x, set*

$$(1) \quad \xi(x) \cdot u_1 \vee \cdots \vee u_{r-1} = \int_0^1 \{\omega(x(t)) \cdot x'(t) \vee u_1(t) \vee \cdots \vee u_{r-1}(t)\} dt ,$$

where $x'(t)$ denotes the tangent vector to x at $x(t)$, and the bracket in the right member (involving the exterior product $\vee$) is computed in the tangent space $G(x(t))$. Then ξ is a smooth $(r-1)$-form on $E(G)$ and $d\xi = p^*\omega$.

Proof. The contraction $h: I \times E(G) \to E(G)$ given by $h(t, x)s = x(ts)$ is simultaneously continuous in the arguments (t, x), and is a smooth function of x for each $t \in I$. Furthermore, for each $x \in E(G)$ the differential $h_*(t, x)$ is a square integrable function of t; in particular, if e_1 denotes the unit vector of I, then $(h_*(t, x) \cdot e_1)s = sx'(ts)$ for almost all $x \in I$.

Because the homomorphism p is analytic, the induced form $\omega^* = p^*\omega$ is a smooth closed r-form on $E(G)$ for which

$$(2) \qquad \xi(x) = (k\omega^*)x = \int_0^1 h^*\omega^*(t, x) \wedge e_1 dt$$

exists (as a Lebesgue integral, where the integrand in the right member involves the interior product with e_1). The explicit formula (3) for $\xi(x)$ below shows that $\xi(x)$ is actually an $(r-1)$-covector and that ξ is smooth. Standard reasoning about homotopy operators for differential forms leads to the identity $\omega^* = dk\omega^* + kd\omega^*$, and because $d\omega = 0$, we have $d\xi = \omega^*$.

Consider the composite map $q = p \circ h : I \times E(G) \to B$. It is easily checked that $q_*(t, x)e_1 = x'(t)$ for almost all $t \in I$, and for any u in the tangent space at x (interpreted as the vector $0 \oplus u$ in the tangent space of $I \times E(G)$ at (t, x)) we have $q_*(t, x)u = u(t)$. If we take vectors $u_1, \cdots, u_{r-1}$ as in the hypotheses,

$$\begin{aligned} \xi(x) \cdot u_1 \vee \cdots \vee u_{r-1} &= \int_0^1 h^* \circ p^*\omega(t, x) \cdot e_1 \vee u_1 \vee \cdots \vee u_{r-1} dt \\ &= \int_0^1 \{\omega(x(t)) \cdot q_*(t, x)e_1 \vee q_*(t, x)u_1 \vee \cdots \vee q_*(t, x)u_{r-1}\} dt \qquad (3) \\ &= \int_0^1 \{\omega(x(t)) \cdot x'(t) \vee u_1(t) \vee \cdots \vee u_{r-1}(t)\} dt \,. \end{aligned}$$

COROLLARY. *Let $\lambda : E(G) \to B$ be the universal $E(G, K)$-bundle of* § 2B. *Then for any smooth closed r-form ω_0 on B, the formula* (1) *with ω replaced by $\pi^*\omega_0$ defines a smooth $(r-1)$-form ξ_0 on $E(G)$ such that $d\xi_0 = \lambda^*\omega_0$.*

If $i : E(G, K) \to E(G)$ is the inclusion homomorphism, then we remark that $\eta_0 = i^*\xi_0$ *is the suspension of* ω_0 in the sense of [8, p. 453]. Applying [8, Cor. 2, p. 469], we obtain the

COROLLARY. *If B is $(n-1)$-connected and $\pi_n(B)$ is infinite cyclic $(n > 1)$ and if ω_0 is a closed n-form representing a generator v of $H^n(B; \mathbf{Z})$, then (ω_0, ξ_0) is a transgressive pair representing v.*

REMARK. Suppose that $\mathscr{G}$ is connected, compact, and semi-simple. Then the bi-invariant Riemann structure on G induces an analytic G-invariant Riemann structure on B. In the preceding corollary a generator v is then represented by a unique harmonic n-form ω_0; furthermore, ω_0 is G-invariant, and $\pi^*\omega_0$ can be expressed as an exterior polynomial in (left invariant) Maurer-Cartan forms on G. Thus *the generator v is uniquely represented by a transgressive pair (ω_0, ξ_0) where ω_0 is harmonic and where ξ_0 is defined by* (1); see § 6A.

(C) We return to the oriented universal bundle $\lambda : E(G) \to B$, where B is $(n-1)$-connected and $\pi_n(B)$ is infinite cyclic $(n > 1)$. (These assumptions can be relaxed at the expense of simplicity of exposition.)

Let X be a smooth manifold of finite dimension, and let $\mu : W \to X$ be a smooth oriented $E(G, K)$-bundle over X with characteristic class w.

Suppose that bundle is induced by a smooth map f of X into B, and let g be a smooth bundle map covering f:

$$\begin{array}{ccc} W & \xrightarrow{g} & E(G) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} \\ X & \xrightarrow[f]{} & B \end{array}$$

If (ω_0, ξ_0) is a transgressive pair of forms representing the characteristic class w_0 of $\lambda : E(G) \to B$ as in (B), then $\omega = f^*\omega_0$, $\xi = g^*\xi_0$ is a transgressive pair representing w (§ 4B).

DEFINITION. An *admissible partial section* of the bundle $\mu : W \to X$ is a smooth section ϕ defined over $X - e(\phi)$, where $e(\phi)$ is a smooth polyhedral subset of X with $\dim e(\phi) \leq \dim X - n$. Admissible partial sections exist because $E(G, K)$ is $(n-2)$-connected. (For example, we can take a smooth locally finite simplicial subdivision L of X and let L_* be a dual subdivision; then standard obstruction theory provides a smooth section over a neighborhood of the $(n-1)$-skeleton $L^{(n-1)}$ of L which can be smoothly extended over $X - L_*^{(m-n)}$, where $m = \dim X$.)

The following result is an example of the general representation theorem of [**1**, § 4]; note that the present pair $(\omega, \phi^*\xi)$ satisfies the conditions of Corollary 5B of [**1**]. We will use freely the concepts and results of that paper. As usual in constructing integral formulas for characteristic classes, our method of proof follows that of the Gauss-Bonnet Theorem as given by Chern [**3**, § 2]: We first obtain a transgressive pair of forms representing the class; we then appeal to Stokes' Formula to localize and interpret the residue (i.e., the right member of (4) below.

THEOREM. *In the above notation, the characteristic class w of the oriented bundle $\mu : W \to X$ is represented by*

$$w \cdot c = \int_c \omega - \int_{\partial c} \phi^*\xi \tag{4}$$

for any admissible partial section ϕ, where c is any smooth integral n-chain on X whose boundary does not intersect $e(\phi)$.

Proof. First of all, $(\omega, \phi^*\xi)$ is an $(\boldsymbol{R}, n)$-pair on X because ϕ is admissible, and in $X - e(\phi)$ we have $d(\phi^*\xi) = \phi^* d\xi = (\mu \circ \phi)^*\omega = \omega$. Secondary, to verify (4) it suffices to do so for the n-simplexes of a simplical subdivision L of X (by Corollary 5A of [**1**]), provided that $e(\phi)$ lies on the $(m-n)$-skeleton of the dual L_*. Furthermore, in considering its obstruction cocycle we will suppose that ϕ is defined only on $L^{(n-1)}$, and then make below a (piecewise smooth) extension to $L^{(n)} - e$, where e is a discrete set of points; such an alteration will not change the obstruction class.

Let b_σ be the barycenter of the oriented n-simplex σ, and let σ_t be that simplex radially contracted toward b_σ by the ratio $1:(1-t)$, using an admissible coordinate system on X containing σ. Let h be a smooth covering homotopy of that contraction. For any $t < 1$ and x in $\partial\sigma_t$ let $r(x)$ be the radial projection x on $\partial 6$; setting $\phi(x) = h(t,\phi(r(x)))$ defines an extension of ϕ over $\sigma - b_\sigma$.

Applying Stokes' Formula to the chain $\tau_t = \sigma - \sigma_t$ we obtain

$$(5) \qquad -\int_{\partial\sigma_t} \phi^*\xi = \int_{\tau_t} \omega - \int_{\partial\sigma} \phi^*\xi .$$

As $t \to 1$ the right member approaches the right member of (4) with $c = \sigma$, because ω is defined on all σ. To complete the proof of the theorem we will show that as $t \to 1$ the left member determines the obstruction cocycle.

Since $-\xi$ defines the generator of $\mu^{-1}(b_\sigma)$ by § 4B, we see that (writing w for the obstruction cocycle)

$$w \cdot \sigma = -\int_{\partial\sigma} \phi^*\xi .$$

On the other hand, the homotopy h satisfies a Lipschitz condition locally on $\mu^{-1}(\sigma)$ (relative to any metric on W), whence there is a number M independent of t such that $t < 1$ implies

$$\left| \int_{\phi(\partial\sigma)} \xi - \int_{\phi(\partial\sigma_t)} \xi \right| \leq M|1-t| .$$

Using the transformation of integral formula, we find that

$$\left| w \cdot \sigma + \int_{\partial\sigma_t} \phi^*\xi \right| = \left| \int_{\partial\sigma} \phi^*\xi - \int_{\partial\sigma_t} \phi^*\xi \right| \leq M|1-t| .$$

This shows that as $t \to 1$ the left member of (5) approaches $w \cdot \sigma$, and formula (4) follows.

6. Spherical maps of a manifold. (A) As an example of the proceding constructions let $G = SO(n+1)$, the rotation group in its usual matrix representation in numerical space $\boldsymbol{R}_{n+1}$. Let $K = SO(n)$, considered as the subgroup of G which acts trivially on the $(n+1)$th axis of $\boldsymbol{R}_{n+1}$. The unit sphere S_n in $\boldsymbol{R}_{n+1}$ is then naturally identified with the homogeneous space G/K, and the coset map $\pi : SO(n+1) \to S_n$ represents $SO(n+1)$ as the principal $SO(n)$-bundle of orthonormal n-frames on S_n]9, § 7]. We will suppose that S_n has its usual Riemann structure and is oriented by the coordinate axes in $\boldsymbol{R}_{n+1}$. *Henceforth we denote the infinite dimensional Lie group* $E(SO(n+1), SO(n))$ *by* Λ_n.

Let ω_{ij} $(1 \leq i < j \leq n+1)$ be a base of Maurer-Cartan forms for the conjugate space of $L(SO(n+1))$; if we let $k(n)$ denote the reciprocal of the volume of S_n, then the exterior polynomial (the *Kronecker Index form*) on $SO(n+1)$ given by

$$(1) \qquad \omega_0^* = k(n)\omega_{1,n+1} \vee \cdots \vee \omega_{n,n+1}$$

is known to be S_n-basic (i.e., there is a unique $SO(n+1)$-invariant n-form ω_0 on S_n such that $\pi^*\omega_0 = \omega_0^*$), and thereby represents the harmonic generator of $H^n(S_n; \boldsymbol{Z})$.

Suppose n is even; then a crucial step in the derivation of the Gauss-Bonnet Theorem [3] for S_n establishes that ω_0 is part of a transgressive pair in the principal frame bundle of S_n. If n is odd, then ω_0 does not generally have that property. However, *for all* $n > 1$ *Proposition* 5B *gives an explicit transgressive pair in the oriented universal bundle of* S_n*, determined entirely by the Kronecker Index form.*

(B) If X is a compact, oriented, smooth Riemann manifold of dimension $n + m$, then the isomorphism classes of smooth principal Λ_n-bundles over X play an important role in its geometry, primarily because of the following construction: Let V be a closed, oriented, m-dimensional regularly imbedded submanifold of X; suppose that V admits a smooth normal n-frame in X, and let ϕ be such a frame field; we will call the pair (V, ϕ) *a normally framed submanifold of* X. These have been studied by Kervaire [5, § 1] and Thom [10, Ch. II, 4]. It is known that certain equivalence classes of normally framed m-submanifolds of X are in natural one-to-one correspondence with the homotopy classes of maps of X into S_n [5, § 1]. Combining with the Classification Theorem for Λ_n-bundles, we have the

PROPOSITION. *If X is a compact, oriented, smooth Riemann $(n + m)$-manifold, then there is a natural one-to-one correspondence between equivalence classes of normally framed m-submanifolds of X and isomorphism classes of smooth Λ_n-bundles over X.*

Let (V, ϕ) be a normally framed m-submanifold, and let $i : V \to X$ be the inclusion map; then since V is closed and oriented (the orientation on X and the frame field ϕ determine an orientation of V) we have a distinguished generator $v_0 \in H_m(V, \boldsymbol{Z})$, which determines a definite homology class $i_*(v_0) = v \in H_m(X, \boldsymbol{Z})$; Furthermore, v depends only on the equivalence class of (V, ϕ). On the other hand, applying a theorem of Thom [10, Théorème II.2], we obtain the

PROPOSITION. *In the correspondence of the above proposition, the homology class of a normally framed submanifold is the Poincaré dual of the characteristic class of the oriented Λ_n-bundle associated with it.*

(C) Let X be a smooth manifold of finite dimension. In the study of differential forms with singularities [**1**] it is important (e.g., in working with exterior products of such forms) to know when a closed $(\boldsymbol{Z}, r)$-pair is cohomologous to a pair defined in terms of a transgressive pair (as in Theorem 5C). For example, it is well known that the isomorphism classes of $SO(2)$-bundles over X are (by their characteristic classes) in natural one-to-one correspondence with the elements of $H^2(X; \boldsymbol{Z})$. An easy construction shows that *every* 2-*dimensional integral cohomology class of* X *can be represented by a transgressive pair in a canonically defined* $SO(2)$-*bundle over* X.

A cohomology class $u \in H^n(X; \boldsymbol{Z})$ is said to be *spherical* if there is a map $f: X \to S_n$ such that $u = f^*(s)$ for some $s \in H^n(S_n; \boldsymbol{Z})$. The representation theorem [**1**, § 4] of cohomology classes by forms with singularities together with our *Theorem* 5C *gives a transgressive integral representation formula for every spherical class of* X *in a* Λ_n-*bundle.* That bundle is uniquely defined by the homotopy class of $f: X \to S_n$, but is not generally determined by u.

EXAMPLE. Suppose that X has dimension n. The Hopf Classification Theorem then implies that the isomorphism classes of smooth Λ_n-bundles over X are in natural one-to-one correspondence with the elements of $H^n(X; \boldsymbol{Z})$, the correspondence assigning to each isomorphism class its characteristic class. *Theorem* 5C *gives a transgressive integral representation formula for each element* v *of* $H^n(X; \boldsymbol{Z})$ *in a bundle canonically associated with* v. Of course that fact is significant only for compact manifolds, because $H^n(X; \boldsymbol{Z}) = 0$ if X is open. On the other hand, it is particularly useful for non-orientable compact manifolds, because then $H^n(X; \boldsymbol{Z})$ has torsion, in which case the singularity of a $(\boldsymbol{Z}, n)$-pair representing v plays an essential role.

If X is orientable and if its Euler characteristic $\chi(X) \neq 0$, then the Gauss-Bonnet Theorem provides a transgressive integral formula for the elements of $H^n(X; \boldsymbol{Z})$ in a finite dimensional bundle over X. In general (and for lower dimensional spherical classes) it appears necessary to use infinite dimensional smooth bundles to obtain such a formula.

BIBLIOGRAPHY

1. C. B. Allendoerfer and J. Eells, *On the cohomology of a manifold*, Comm. Math. Helv. **32** (1958), 165–179.

2. A. Borel, Selected topics in the homology theory of fibre bundles. Mimeographed Notes. Univ. of Chicago (1954).

3. S. S. Chern, *On the curvatura integra in a Riemannian manifold*, Annals of Math. **46** (1945), 674–684.

4. J. Eells, *On the geometry of function spaces*, Sym. Inter. de Top. Alg. Mexico (1958), 303–308.

5. M. Kervaire, *Courbure intégrale généralisée et homotopie,* Math. Ann. **131** (1956), 219-252.
6. J. Milnor, *Construction of universal bundles I,* Ann. of Math. **63** (1956), 272–284.
7. Séminaire S. Lie; E.N.S. 1954–5.
8. J-P. Serre, *Homologie singulière des espaces fibrés,* Ann. of Math. **54** (1951), 425–505.
9. N. E. Steenrod, The Topology of Fibre Bundles, Princeton, 1951.
10. R. Thom, *Quelques propriétés globales des variétés différentiables,* Comm. Math. Helv **28** (1954), 17–86.

COLUMBIA UNIVERSITY