UNIONS OF CELL PAIRS IN E^3

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In [4] it is shown that there are pairs of cells of all dimensions possible in euclidean 3-space, E^3, which are tame separately, but which have a wild set as their union. Such pairs can be constructed when the individual cells intersect in a single point. The present paper gives conditions that unions of some such pairs be tame sets as well as a number of other results.

LEMMA 1. *Let D_1 be a disk which is polyhedral and which lies on the boundary, ∂T, of a tetrahedron T in E^3. If D_2 is a disk in E^3 which has a polygonal boundary and is locally polyhedral mod ∂D_2 while $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc, then $D_1 \cup D_2$ is a tame disk.*

Proof. Let P_1 and P_2 be polyhedral disks in ∂T, $P_1 \cap P_2 = \square$ and $(P_1 \cup P_2) \cap D_1 = \square$. Then $\overline{\partial T \backslash (P_1 \cup P_2)}$ is a polyhedral annulus, A_1. If Q is a polyhedral disk in $D_2 \backslash \partial D_2$, then $\overline{D_2 \backslash Q}$ is an annulus A_2 which is locally polyhedral mod ∂D_2. By applying Lemma 5.1 of [8] to A_1 and A_2 one obtains a space homeomorphism h carrying E^3 onto E^3 while $h(D_1 \cup D_2)$ is a polyhedral set. This completes the proof of Lemma 1.

LEMMA 2. *Let D_1 be the disk of Lemma 1 while D_2 is a tame disk in E^3 such that $D_2 \cap T = D_2 \cap D_1 = \partial D_2 \cap \partial D_1 = J$, an arc. Then $\partial T \cup \partial D_2$ is tame.*

Proof. By Theorem 2 of [3] $\partial D_1 \cup \partial D_2$ is locally tame and hence tame by [1] or [8]. Let a be a point of ∂J and J' be an interval of ∂D_1 having a as an end point and $J' \cap \partial D_2 = a$. We choose a polygonal disk M on ∂T with $(J'/\partial J')$ in its interior while $\partial D_1 \cap M = J'$. By a swelling [5] of M toward the component of $E^3 \backslash \partial T$ which meets ∂D_2 we obtain a disk M' which is locally polyhedral mod ∂M and $M' \cap \partial T = \partial M = \partial M'$. The sphere $S = M' \cup (\partial T \backslash M)$ is tame by [8] and S is pierced at a by a tame arc lying on $\partial(D_1 \cup D_2)$. Hence by [7] $\partial D_2 \cup S$ is locally tame at a. We select an arc P in $(S \backslash M') \cup a$ which is locally polyhedral except at the point a. There is an arc A on ∂D_2 which lies in the exterior of S except for its end point a. The arc $A \cup P$ is tame since $S \cup \partial D_2$ is tame. Let the arc P be swollen into a 3-cell C^3 with P in its interior such that C^3 is locally polyhedral mod a, $C^3 \cap S$ is a disk while $C^3 \cap M = a$. Then ∂C^3 is pierced at a by $A \cup P$ and so $A \cup P \cup \partial C^3$ is tame by [7]. Evidently there is an arc P' on ∂C^3 so

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that $A \cup P'$ pierces ∂T at a. Again by [7] $\partial D_2 \cup \partial T$ is locally tame at a. A similar argument applies to the other end point of ∂J. Hence $\partial D_2 \cup \partial T$ is tame. This proves Lemma 2.

THEOREM 1. *Let D_1 and D_2 be two tame disks in E^3 such that $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = J$, an arc. Then $D_1 \cup D_2$ is a tame disk.*

Proof. Since D_1 is tame there is a homeomorphism h_1 of E^3 onto E^3 such that $h_1(D_1)$ is a plane triangle. The disk $h_1(D_1)$ is to be swollen so that a 3-cell e^3 is formed such that

(i) $h_1(D_1) \subset \partial e^3$,

(ii) e^3 is tame,

(iii) and $e^3 \cap h_1(D_2) = h_1(J)$.

That such a cell e^3 exists follows from Lemma 5.1 of [5] and Theorem 9.3 of [8].

There is a homeomorphism h_2 of E^3 onto E^3 which carries ∂e^3 and $h_1(D_1)$ onto the boundary of a tetrahedron and a polyhedral disk, respectively. By Lemma 2 $h_2(e^3) \cup h_2h_1(\partial D_2)$ is a tame set. By Theorem 2 of [6] we can insist that $h_2h_1(D_2)$ be locally polyhedral mod $h_2h_1(\partial D_2)$, while $h_2h_1(\partial D_2)$ is polygonal. Hence by Lemma 1 $h_2h_1(D_1 \cup D_2)$ is tame and so $D_1 \cup D_2$ is tame.

The following result gives a characterization of tame 1-dimensional complexes in E^3. By *a* 1_n-star we mean a homeomorphic image of a 1-dimensional simplicial complex K with a vertex x whose star is K and x is the common end point of the n segments meeting only in x.

THEOREM 2. *If N is a 1_n-star in E^3 such that $(n-1)$ of the branches of N lie on a disk D which meets the remaining branch J at x only and if each arc in N is tame, then N is tame.*

Proof. By [2] we may assume that D is locally polyhedral mod N. An application of the method in Theorem 1 of [3] makes it possible to select a subset D' of D which is a disk consisting of $(n-1)$ tame disks which contain arcs with x as an end point of all branches of N except J. An argument almost identical with that of Theorem 2 of [3] suffices to show that $J \cup D'$ is tame and hence N is tame by [1] or [8].

COROLLARY 1. *Let G be a graph in E^3 such that the star of each vertex of G meets the conditions of Theorem 2, then G is tame. The conditions are evidently necessary as well.*

COROLLARY 2. *Let D be a tame disk and J a tame arc in E^3. If $D \cap J = \partial D \cap J = p$, an end point of J, and if $\partial D \cup J$ is tame, then $D \cup J$ is tame.*

Proof. Since D is tame there is a space homeomorphism h which carries D onto a face of a tetrahedron T, $[h(J)\backslash h(p)] \subset E^3\backslash T$. Let P be a segment on $h(\partial D)$ with $h(p)$ as an end point. We enclose P in a polyhedral disk M in ∂T such that P spans M and $h(\partial D) \cap M = P$. We swell M as in Lemma 2 to obtain a tame disk M' such that $\partial M' = \partial M$, and $M'\backslash\partial M' \subset E^3\backslash T$. Then $h(J) \cup h(\partial D)$ contains a tame arc which pierces the tame sphere [8] $S = M' \cup (\partial T\backslash M)$ at $h(p)$ and so $S \cup h(J)$ is tame by [7]. The construction of an arc P' as in Lemma 2 completes the proof.

In Example 1.4 of [4] an arc A which is the union of two tame arcs is shown. Although A has an open 3-cell complement in compactified E^3, it is nevertheless wild. A similar example can be obtained from Example 1.4 of two tame disks which meet at a point on the boundary of each and which have a wild union. In this connection we give the following result.

THEOREM 3. *Let D_1 and D_2 be disks in E^3 such that each arc in D_1 and D_2 is tame and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = J$, an arc. Then $D_1 \cup D_2$ is a disk such that each arc in $D_1 \cup D_2$ is tame.*

Proof. Let J' be an arc in $D_1 \cup D_2$. If $\partial J'$ does not lie in $\partial D_1 \cup \partial D_2$ we extend J' so that this is the case, obtaining $J'' \supset J'$, $\partial J'' \subset \partial D_1 \cup \partial D_2$ and $J'' \subset D_1 \cup D_2$. By [2] there is a disk D such that $\partial D = \partial(D_1 \cup D_2)$, $J \cup J'' \subset D$ and D is locally polyhedral mod $J \cup J'' \cup \partial D$. The arc J in D is the intersection of two disks in D, D_1' and D_2', such that $D_1' \cup D_2' = D$. Consider any point x of J'' in $D_1'\backslash\partial D_1'$. In [3] a method is given for enclosing x in the interior of a tame subdisk of D_1'. Hence D_1' is locally tame at each of its interior points and $\partial D_1'$ is tame. By [8] D_1' is tame. A similar argument can be applied to D_2'. Hence $D_1' \cup D_2'$ is a tame disk by Theorem 2. Then J'' is tame and so J' is tame. Since J' was arbitrarily chosen $D_1 \cup D_2$ is a disk in which each arc is tame.

COROLLARY 1. *Let L_1 and L_2 be tame disks which intersect in a single point on the boundary of each. If $L_1 \cup L_2$ lies on a disk in which each arc is tame, then $L_1 \cup L_2$ is tame.*

Proof. Let $L_1 \cup L_2$ lie on a disk D such that each arc in D is tame. By Theorem 2 $\partial L_1 \cup \partial L_2$ is tame. There is a disk D' in D with a tame boundary such that $D' \cap (L_1 \cup L_2) \subset \partial L_1 \cup \partial L_2$ while $D' \cup L_1 \cup L_2$ is a disk. Then by [2] there is a disk D'' such that $\partial D'' = \partial D'$, D'' is locally polyhedral mod $\partial D''$ and $\partial D'' \cap (L_1 \cup L_2) = \partial D' \cap (L_1 \cup L_2)$. Now D'' is tame by [8] and so $D'' \cup L_1 \cup L_2$ is tame by Theorem 2. It follows that $L_1 \cup L_2$ is tame.

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