HÖLDER CONTINUITY OF N-DIMENSIONAL QUASI-CONFORMAL MAPPINGS

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1. Introduction and main results. This paper is an extension of previous work on the Hölder continuity of two-dimensional mappings. We shall use the approach of Finn and Serrin[1] and prove analogous results in n dimensions. A two-dimensional quasi-conformal mapping is one which carries infinitesimal circles into infinitesimal ellipses of bounded eccentricity. An n-dimensional quasi-conformal mapping carries infinitesimal spheres into infinitesimal ellipsoids of bounded eccentricity. Finn and Serrin gave an elementary proof that a quasi-conformal mapping is uniformly Hölder continuous in compact subdomains and obtained the best possible Hölder exponent. Their proof makes extensive use of the Dirichlet integral. We obtain similar results in n dimensions using a modified Dirichlet integral suggested by C. Loewner. It is not clear whether the n-dimensional exponent is the best possible one.

Let $u(x, y)$ and $v(x, y)$ be continuously differentiable functions in a domain D of the complex z-plane. Then the function $w(z) = u + iv$ represents a quasi-conformal mapping if there exists a constant K such that

$$|\nabla w|^2 \equiv u_x^2 + u_y^2 + v_x^2 + v_y^2 \leq 2K(u_x v_y - u_y v_x) , \tag{1}$$

for all points of the domain of definition of w. If $K < 1$, the mapping functions are constant; if $K = 1$, they are conformal. The only case of interest is $K \geq 1$. Geometrically, (1) implies that infinitesimal circles map into infinitesimal ellipses for which the ratio of minor to major axis $\geq K - \sqrt{K^2 - 1}$.

Let $f = (u_1, \cdots, u_n)$ be an n-dimensional mapping of a domain A of E_n into E_n such that f is continuously differentiable, the Jacobian, J, of the transformation is non-negative and

$$|\nabla f|^2 \equiv \sum_{i,j=1}^{n} u_{i,j}^2 \leq nKJ^{2/n} , \qquad \text{where } u_{i,j} = \partial u_i/\partial x_j \tag{2}$$

and K is a constant holding for all points of the domain A of definition.

If $K < 1$, the mapping functions are constant, if $K = 1$, the mappings are the conformal mappings of space. Geometrically the mapping $x \rightarrow f(x)$ is sense preserving and infinitesimal spheres map onto infinitesimal ellipsoids. In this paper the norm used is the usual one for E_n

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[1] "On the Hölder Continuity of Quasi-Conformal and Elliptic Mappings." *Transactions of American Mathematical Society*, Vol. 89, No. 1 (1958), pp. 1–15. See this paper for a bibliography of previous work.

and is denoted by $|x|$.

Finn and Serrin treat a class of mappings which they call elliptic mappings. This generalization of the notion of quasi-conformal mapping is due to L. Nirenberg. $w(z)$ is an ellistic mapping if it satisfied the conditions for a quasi-conformal mapping except that condition (1) is replaced by

$$|\nabla w|^2 \leq 2KJ + K_1 , \tag{3}$$

where K and K_1 are constant, $K \geq 1$ and $K_1 \geq 0$. A generalization of two-dimensional elliptic mappings is obtained by replacing condition (2) in the definition of n-dimensional quasi-conformal mappings by

$$|\nabla f|^n \leq (nK)^{n/2} J + K_1 , \tag{4}$$

where K and K_1 are constants, $K \geq 1$, and $K_1 \geq 0$. Such mappings we shall call near quasi-conformal mappings.

In two dimensions many important estimates are given in terms of the Dirichlet integral

$$D(r) = \iint_{C_r} |\nabla w|^2 \, dxdy ,$$

where C_r is a circle of radius r. We shall find that the appropriate n-dimensional analog of this integral is

$$D(r) = \int_{S_r} \left\{ \sum_{i,j=1}^{n} u_{i,j}^2 \right\}^{n/2} dV , \tag{5}$$

where S_r is an n-dimensional hypersphere of radius r. This integral was suggested by C. Loewner in a paper that will appear shortly in the Journal of Mathematics and Mechanics.

The proofs of Finn and Serrin make use of Morrey's lemma, which is based on the usual Dirichlet integral. By means of the modified Dirichlet integral, an analogous lemma is proved in n dimensions.

For the n-dimensional quasi-conformal mappings and the near quasi-conformal mappings the following two theorems are proved.

THEOREM 1. *Let f be a quasi-conformal mapping defined in a domain A of E_n. Assume $|f| \leq 1$. Then in any compact subregion B of A,*

$$|f(x_1) - f(x_2)| \leq C \frac{|x_1 - x_2|^{\mu}}{d^{\mu}} , \tag{6}$$

where d is the distance from B to the boundary of A; $\mu = \mu(n, K)$ and $0 < \mu \leq 1$; and $C = C(n, K)$, a constant depending only on the dimension of the space and K. (See equation (12) for definition of μ.)

THEOREM 2. *Let f be a near quasi-conformal mapping defined on a domain A of E_n. Let $|f| \leq 1$. Then in any compact subregion B of A*

$$|f(x_1) - f(x_2)| \leq H|x_1 - x_2|^{\mu}, \tag{7}$$

where H is a constant depending on n, K, K_1, and d (d is the distance from B to boundary of A) *and $\mu = \mu(n, K)$, $0 < \mu \leq 1$. μ is the same constant that appears in Theorem* 1.

2. Preliminary lemmas. To generalize the proofs of Finn and Serrin to n dimensions, several lemmas are needed. They are listed below and the more difficult proofs are given.

LEMMA 1. *The weak Maximum Principle holds for quasi-conformal mappings, i.e., if f is quasi-conformal in a bounded region A and continuous in $\bar{A}$, then the maximum of the norm* (and of the components) *is attained on the boundary $\dot{A}$ of A. The minimum of the components is also attained on $\dot{A}$.* (The proof is the same as in two dimensions.)

LEMMA 2. *Let u be a function defined in some domain A. If $u = 0$ on $\dot{S}_r$ where $\dot{S}_r$ is the surface of a sphere of radius r in A and n is the dimension of the space, then*

$$\int_{\dot{S}_r} |u|^n dA \leq Cr^n \int_{\dot{S}_r} |u_t|^n dA, \tag{8}$$

where u_t is the tangential component of the gradient of u on S_r and C is a constant depending only on the dimension of the space.

LEMMA 3. *For all $a, b \geq 0$, $\lambda > 0$ and $n \geq 2$,*

$$\frac{n}{(n-1)^{\frac{n-1}{n}}} a^{1/n} b^{\frac{n-1}{n}} \leq \frac{a}{\lambda^{n-1}} + \lambda b, \tag{9}$$

and the constant of this inequality cannot be improved.

LEMMA 4. *Let u be a function defined in a domain A and let $\omega \equiv \omega(\dot{S}_r)$ be the oscillation of u on the surface of sphere of radius r in A where n is the dimension of the space. Then there exists a constant C depending only on the dimension of the space such that*

$$\frac{\omega^n(\dot{S}_r)}{r} \leq C \int_{\dot{S}_r} |u_t|^n dA. \tag{10}$$

LEMMA 5. *Let (a_{ij}) be an $n \times n$ matrix with real coefficients. Then*

$$(11) \qquad |\det(a_{ij})| \leq \frac{1}{n^{n/2}}\Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{n/2} .$$

The constant in the inequality cannot be approved. (This lemma follow immediately from the proof of Hadamard's inequality.)

Morrey's lemma in n dimensions. Let B be a closed subregion of D and let $d =$ distance $(\dot{B}, \dot{D})$. Suppose there exist constants L, μ, r_0, where $0 < \mu$ and $r_0 \leq d$, such that for all spheres S_r with center in B, $r \leq r_0$,

$$D(r) \equiv \int_{S_r} |\nabla f|^n dV \leq Lr^{n\mu} ,$$

Then f satisfies a Hölder condition in B:

$$|f(x_1) - f(x_2)| \leq C_2 |x_1 - x_2|^{\mu} ,$$

where

$$C_2 = \frac{1}{\pi^{n-1}}\Big(\frac{nL}{n-1}\Big)^{1/n}\Big(\frac{2\pi C_1(n-1)}{\mu}\Big)^{\frac{n-1}{n}}$$

and $C_1 = C_1(n)$.

Proof of Lemma 2. Let $n \geq 3$. Choose the coordinates such that $u = 0$ at the north pole. For given $(\theta_2, \cdots, \theta_{n-1})$, let $u_m = u_m(\theta_2, \cdots, \theta_{n-1})$ be the maximum of $|u|$ for $0 \leq \theta_1 \leq \pi$. We have $u = \int_0^\theta u_\theta d\theta$, which implies that

$$u_m \leq \int_0^\pi |u_\theta|\, d\theta \leq \Big[\int_0^\pi |u_\theta|^n r^{n-1} \sin^{n-2}\theta_1 d\theta_1\Big]^{1/n}\Big[\int_0^\pi \frac{d\theta_1}{r \sin^{\frac{n-2}{n-1}}\theta_1}\Big]^{\frac{n-1}{n}}$$

by Hölder's inequality. Let

$$C = \int_0^\pi \frac{d\theta_1}{\sin^{\frac{n-2}{n-1}}\theta_1} .$$

$C < \infty$. Hence

$$\begin{aligned} u_m^n r^{n-1} &\leq C^{n-1}\int_0^\pi |u_\theta|^n r^{n-1} \sin^{n-2}\theta_1 d\theta_1 \\ &\leq r^n C^{n-1}\int_0^\pi |u_t|^n r^{n-1} \sin^{n-2}\theta_1 d\theta_1 . \end{aligned}$$

Now

$$\int_{\dot{S}_r} |u|^n dA = r^{n-1}\int_0^{2\pi}\underbrace{\int_0^\pi \cdots \int_0^\pi}_{n-2} |u|^n \sin^{n-2}\theta_1 d\theta_1 d\omega_{n-1}$$

$$\leq r^{n-1}\int u_m^n \sin^{n-2}\theta_1 d\theta_1 d\omega_{n-1}$$

$$\leq r^{n-1}\sqrt{\pi}\frac{\Gamma\{(n-1)/2\}}{\Gamma(n/2)}\int u_m^n d\omega_{n-1} .$$

Combining the above results

$$\int_{\dot{S}_r} | u^n | dA \leq \sqrt{\pi}\frac{\Gamma\{(n-1)/2\}}{\Gamma(n/2)} C^{n-1} r^n \int_{\dot{S}_r} | u_t |^n dA .$$

Proof of Morrey's lemma. Denote the points x_1, x_2 by P and 0, respectively. Let $| x_1 - x_2 | \leq r_0$ and let $r = | x_1 - x_2 |$. Let M be a perpendicular bisector of $\overline{PQ}$. Select a point S on M such that $\overline{PS} = \overline{QS} \leq \overline{PQ} \leq r_0$. Then

$$f(P) - f(Q) = \int_{PS} f_r dr - \int_{QS} f_r dr$$

which implies

$$| f(P) - f(Q) | \leq \int_{PS} | f_r | dr + \int_{QS} | f_r | dr .$$

Hence

$$\int_0^{2\pi}\underbrace{\int_0^{\pi}\cdots\int_0^{\pi}}_{n-3}\int^{\pi/3} | f(P) - f(Q) | d\theta_1 \cdots d\theta_{r-1}$$

$$\leq 2\int_0^R\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int^{\pi/3} | \nabla f | d\theta_1 \cdots d\theta_{n-1} dr .$$

So

$$| f(P) - f(Q) | \leq \frac{3}{\pi^{n-1}}\int_{S_r} | \nabla f | d\theta_1 \cdots d\theta_{n-1} dr$$

$$\leq \frac{3}{\pi^{n-1}}\left[\int_{S_r} | \nabla f |^n r^{n-1-\mu} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1}\right]^{1/n} \cdot I^{\frac{n-1}{n}} ,$$

where

$$I = \int_{S_r} r^{-1+\frac{\mu}{n-1}} \sin^{-\left(\frac{n-2}{n-1}\right)}\theta_1 \sin^{-\left(\frac{n-3}{n-1}\right)}\theta_2 \cdots \sin^{-\frac{1}{n-1}}\theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1} .$$

$$I = r^{\frac{\mu}{n-1}}\left(\frac{n-1}{\mu}\right) 2\pi C_1$$

where

$$C_1 = \int_0^{\pi}\cdots\int_0^{\pi} \sin^{-\left(\frac{n-2}{n-1}\right)}\theta_1 \sin^{-\left(\frac{n-3}{n-2}\right)}\theta_2 \cdots \sin^{-\frac{1}{n-1}}\theta_{n-2} d\theta_1 \cdots d\theta_{n-2} < \infty .$$

$$\int_{S_r} |\nabla f|^n r^{n-1-\mu} d\omega_n = r^{-\mu} D(r) ,$$

$$u\int_0^r D(r) r^{-\mu-1} dr \leq r^{-\mu} L r^{n\mu} + \mu L \int_0^r r^{\mu(n-1)-1} dr = \frac{n}{n-1} L r^{(n-1)\mu} ,$$

since by hypothesis

$$D(r) \leq L r^{n\mu} .$$

Combining

$$|f(P) - f(Q)| \leq \frac{3}{\pi^{n-1}} \left(\frac{n}{n-1} L r^{(n-1)\mu} \right)^{1/n} \left\{ r^{\frac{\mu}{n-1}} \left(\frac{n-1}{\mu} \right) (2\pi C_1) \right\}^{\frac{n-1}{n}} = C_2 r^{\mu} ,$$

where

$$C_2 = \frac{3}{\pi^{n-1}} \left(\frac{nL}{n-1} \right)^{1/n} \left(\frac{2\pi C_1 (n-1)}{\mu} \right)^{\frac{n-1}{n}} .$$

Proof of Lemma 4.[2] The surface of the n dimensional hypersphere of radius r can be mapped onto a $n-1$ dimensional hyperplane by a stereographic mapping. Under such a transformation

$$\int_{\dot{S}_r} |u_t|^{n-1} dS = \int_V |\nabla u|^{n-1} dV ,$$

and

$$\int_{\dot{S}_r} (1 - \cos\theta_1) |u_t|^n dS = \int_V |\nabla u|^n dV ,$$

where the variables on the surface of the sphere are $(\theta_1, \theta_2, \cdots, \theta_{n-1})$, on the hyperplane are $(\rho, \theta_2, \cdots, \theta_{n-1})$, and domains of integration are mapped onto one another. Hence

$$\int_V |\nabla u|^n dV \leq \int_{\dot{S}_r} |u_t|^n dS .$$

In the hyperplane

$$\begin{aligned}\int_{|x|\leq\rho} |\nabla u|^{n-1} dV &\leq \left[\int |\nabla u|^n dV \right]^{\frac{n-1}{n}} \left[\int dV \right]^{1/n} \\ &\leq C_1 \left[\rho \int_{\dot{S}_r} |u_t|^n dS \right]^{\frac{n-1}{n}} ,\end{aligned}$$

where latter integration is taken over the whole surface of the n dimensional hypersphere and

[2] The author is indebted to R. Finn for suggesting this proof which strengthens and simplifies the author's original proof.

$$c_1^n = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin^{n-3}\theta_2 \sin^{n-4}\theta_3 \cdots \sin\theta_{n-2}\, d\theta_1 \cdots d\theta_{n-1} .$$

Hence by Morrey's lemma applied in the $n-1$ dimensional hyperplane

$$\omega^n \leq C\rho \int_{\dot{S}_r} |u_t|^n dS ,$$

where C is a constant depending only on n. It follows immediately that

$$\frac{\omega^n(\dot{S}_r)}{r} \leq 2C \int_{\dot{S}_r} |u_t|^n dS .$$

3. Proof of Theorems 1 and 2. The proof of Theorem 2 will be given before that of Theorem 1, and Theorem 1 will follow as an immediate corollary of Theorem 2. Then an alternate method of proof for Theorem 1 will be given. This second proof uses a modulus of continuity instead of Morrey's lemma.

Proof of Theorem 2. It must be shown that it f is a near quasiconformal mapping, then $D(r) \leq \text{constant} \cdot r^{n\mu}$ for r sufficiently small. Then the conclusion will follow by Morrey's n dimensional lemma. By (4)

$$D(r) \leq (nK)^{n/2} \int_{S_r} J dV + \omega_n r^n K_1 .$$

$$\int_{S_r} J dV = \int_{\dot{S}_r} u_1 du_2 \cdots du_n = \int (u_1 - \bar{u}_1) du_2 \cdots du_n$$
$$= \int_{\dot{S}_r} (u_1 - \bar{u}_1) \frac{\partial(u_2 \cdots u_n)}{\partial(s_2 \cdots s_n)} ds_2 \cdots ds_n ,$$

where $\bar{u}_1$ is the mean value of u_1 over S_r, $ds_2 = r d\theta_1$, $ds_3 = s \sin\theta_1 d\theta_2$, $ds_4 = r \sin\theta_1 \sin\theta_2 d\theta_3, \cdots$, and $ds_n = r \sin\theta_1 \cdots \sin\theta_{n-2} d\theta_{n-1}$. Hence by Lemma 5, Lemma 3, and Lemma 2

$$\int_{S_r} J dV \leq \frac{1}{(n-1)^{\frac{n-1}{2}}}$$
$$\times \int_{\dot{S}_r} |u - \bar{u}_1| [u^2_{2,s_2} + \cdots + u^2_{2,s_n} + u^2_{3,s_2} + \cdots + u^2_{n,s_n}]^{\frac{n-1}{2}} dA$$
$$\leq \frac{(n-1)^{\frac{n-1}{n} - \frac{n-1}{2}}}{n} \left\{ \int_{\dot{S}_r} \frac{|u_1 - \bar{u}_1|^n}{r^{n-1}} dA + r \int_{\dot{S}_r} [u^2_{2,s_2} + \cdots + u^2_{n,s_n}]^{n/2} dA \right\}$$
$$\leq \frac{(n-1)^{\frac{n-1}{n} - \frac{n-1}{2}}}{n} r \left\{ C \int_{\dot{S}_r} |u_{1,t}|^n dA + \int_{\dot{S}_r} [u^2_{2,s_2} + \cdots + u^2_{n,s_n}]^{n/2} dA \right\} ,$$

where $C = C(n)$ is the constant of Lemma 2. Hence

$$\int_{S_r} J dV \leq C' r \int_{\dot{S}_r} |f_t|^n dA ,$$

where

$$C' = \frac{(n-1)^{\frac{n-1}{n} - \frac{n-1}{2}}}{n} C ,$$

and finally

$$\int_{S_r} J dV \leq C' r \frac{dD}{dr} .$$

The Hölder exponent μ is defined by the equation

$$\frac{1}{\mu} = C n^{n/2} K^{n/2} (n-1)^{-\frac{n^2-n+2}{2n}} , \tag{12}$$

where C is the constant of Lemma 2.

Combining above results

$$D(r) \leq \frac{r}{n\mu} \frac{dD}{dr} + \omega_n r^n K_1 , \tag{13}$$

where ω_n is the area of the unit sphere in n dimensions.

Let B be a closed subregion of A, and let d be the distance from B to $\dot{A}$. Let S_r be a sphere whose center is in B. For such a sphere

$$D(r) \leq \frac{r}{n\mu} \frac{dD}{dr} + \omega_n r^n K_1 , \qquad \text{for } 0 < r < d.$$

Hence

$$-\frac{d}{dr}(r^{-n\mu} D) \leq n\mu K_1 \omega_n r^{n-1-n\mu} ,$$

and integrating

$$D(\rho) \leq \{D(t) + K_2\} \left(\frac{\rho}{t}\right)^{n\mu} , \qquad \rho \leq t \leq d \tag{14}$$

where

$$K_2 = \frac{\pi K_1}{1-\mu} \omega_n t^n .$$

We now wish to estimate $D(t)$. We know

$$D(t) \leq \frac{(nK)^{n/2}}{(n-1)^{\frac{n-1}{2}}} \int_{\dot{S}_r} |u_1| [u^2_{2,s_2} + \cdots + u^2_{n,s_n}]^{\frac{n-1}{2}} dA + r^n \omega_n K_1$$

$$\leq K_3\Big[\int_{\dot{S}_r} |u_1|^n dA\Big]^{1/n}\Big[\int_{\dot{S}_r} |f_t|^n dA\Big]^{\frac{n-1}{n}} + r^n\omega_n K_1$$
$$\leq K_4\Big[r\int_{\dot{S}_r} |f_t|^n dA\Big]^{\frac{n-1}{n}} + r^n\omega_n K_1$$
$$\leq K_4\Big[r\frac{dD}{dr}\Big]^{\frac{n-1}{n}} r^n\omega_n K_1 ,$$

where

$$K_3 = \frac{(nK)^{n/2}}{(n-1)^{\frac{n-1}{2}}} \quad \text{and} \quad K_4 = K_3\omega_n^{1/n} .$$

We have also used the fact that $|f| < 1$. The immediately preceeding result implies

$$(D(r) - K_5)^{\frac{n}{n-1}} \leq K_4^{\frac{n}{n-1}} r\frac{dD}{dr} ,$$

for $r < d$ and where $K_5 = d^n\omega_n K_1$.

Now suppose $D(t) > K_5$ for some t. Then $D(r) > K_5$ for all $r \geq t$. Hence

$$(D(t) - K_5)^{\frac{1}{n-1}} \leq \frac{K_4^{\frac{n}{n-1}}(n-1)}{\log(d/t)} ,$$

or

$$D(t) - K_5 \leq \frac{K_6}{\{\log(d/t)\}^{n-1}} ,$$

where

$$K_6 = K_4^n(n-1)^n .$$

So

$$D(t) \leq \frac{K_6}{\{\log(d/t)\}^{n-1}} + d^n\omega_n K_1 . \tag{15}$$

This inequality also holds if $D(r) \leq K_5$.

Now let $t = de^{-\nu}$ where $\nu = 1/n\mu$. Combining (14) and (15) we obtain $D(\rho) \leq H\rho^{n\mu}$ where H is a constant depending only on n, K, K_1, and d. We can now conclude that for $x_1, x_2 \in D$ and $|x_1 - x_2| \leq de^{-\nu}$ that $|f(x_1) - f(x_2)| \leq H|x_1 - x_2|^{\mu}$. Because of the bound on $|f|$, we get a similar result when $|x_1 - x_2| > de^{-\nu}$.

Alternate proof of Theorem 1. Here we do not use Morrey's lemma, instead a modulus of continuity on f is obtained in terms of $D(r)$.

Proof. For any point set $T \subset A$, let $\omega(t) = \text{l.u.b.}\,|f(x_1) - f(x_2)|$ where $x_1 x_2 \in T$. Since f is quasi-conformal, it satisfies the weak maximum principle. Let S_r be a sphere of radius r such that its center is at least a distance ρ from A. Then $\omega(S_r) = \omega(\dot{S}_r)$. By Lemma 4,

$$\frac{\omega^n(S_s)}{r} \leq C\frac{dD}{dr} \qquad \text{for } s \leq r.$$

Hence

$$\omega^n(S_s)\log\frac{\rho}{s} \leq CD(\rho) \qquad \text{for } s < \rho.$$

This implies

$$\omega(S_s) \leq \left[\frac{CD(\rho)}{\log(\rho/s)}\right]^{1/n},$$

where C depends only on the dimension of the space.

$D(\rho)$ can be estimated by the technique used in the proof of Theorem 2.

$$D(\rho) \leq e(n\mu)^{n-1}(nK)^{n^2/2}\omega_n(n-1)^{\frac{n+1}{2}}\left(\frac{\rho}{d^*}\right)^{n\mu},$$

where e is the base of the natural logs, μ is defined as in proof of Theorem 2 and $\rho \leq d^* e^{-\nu}$. This is valid for all spheres of radius ρ whose centers are at least a distance d^* from $\dot{A}$.

Let x_1 and x_2 be two points in B such that $|x_1 - x_2| = 2s < de^{-n\nu} = de^{-1/\mu}$. The midpoint of the line segment $\overline{x_1 x_2}$ is at least a distance $d^* = d/2$ from $\dot{A}$. Consequently

$$|f(x_1) - f(x_2)| \leq \omega(S_s) \leq \frac{C(n, K)}{[\log(\rho/s)]^{1/n}}\left(\frac{\rho}{d^*}\right)^{\mu}$$

for $s \leq \rho \leq d^* e^{-\mu}$.

Let

$$\rho = se^{\nu}.$$

Then

$$|f(x_1) - f(x_2)| \leq C(n, K)\frac{|x_1 - x_2|^{\mu}}{d^{\mu}}.$$

On the other hand, if $|x_1 - x_2| \leq de^{-\nu\mu}$, we again get a Hölder estimate since $|f| \leq 1$.

4. Additional results. Theorems 3 and 4 are on removable singularities. The final theorem is concerned with one-to-one quasi-conformal mappings.

THEOREM 3. *Let $f(x)$ satisfy the hypothesis of Theorem 1 or 2 for all points x in the domain A except on a set T of isolated points in A. Then f can be defined at the points of T such that the resulting function is continuous in A and satisfies the conclusion of Theorems 1 or 2.*

Proof. To prove Theorem 3 it is sufficient to show that $D(r)$ exists and satisfies

$$D(r) \leq (nK)^{n/2}\int_{\dot{S}_r} u_1 du_2 \cdots du_n + \omega_n r^n K_1 ,$$

for all spheres whose surface contains no points of T. Then all the previous statements are valid and hence f satisfies a Hölder condition in $B - T$. Finally f can be defined on T such that resulting function is continuous in A and satisfies a Hölder condition throughout A.

Let S be a sphere of radius r. Let S_r contain exactly one point x_0 of T. Let S^σ be a sphere of radius σ with center x_0.

$$D(\sigma, r) \equiv \int_{S_r - S_\sigma} |\nabla f|^n dV .$$

Hence

$$D(\sigma, r) \leq -(nK)^{n/2}\int_{\dot{S}_r} u_1 du_2 \cdots du_n + I ,$$

when

$$I = (nK)^{n/2}\int_{\dot{S}_r} u_1 du_2 \cdots du_n + \omega_n r^n K_1 . \qquad |f| \leq 1.$$

Hence

$$(D(\sigma, r) - I) \leq K_4\left[-\sigma \frac{dD}{d\sigma}\right]^{\frac{n-1}{n}} ,$$

which implies

$$(D(\sigma, r) - I)^{\frac{n}{n-1}} \leq K_4^{\frac{n}{n-1}}\left(-\sigma \frac{dD(\sigma, r)}{d\sigma}\right) .$$

Suppose $D > I$ for some value of σ, say $\sigma = \sigma_2$. Then $D > I$ for all $\sigma < \sigma_2$. There we may integrate from σ_1 to σ_2 and obtain

$$\log \frac{\sigma_2}{\sigma_1} \leq \frac{C(n, K)}{(D(\sigma_2, r) - I)^{\frac{1}{n-1}}} .$$

Let σ_1 approach zero. A contradiction is then obtained. Therefore $D(\sigma, r) \leq I$. Let σ approach zero, and we obtain $D(r) \leq I$.

Since there at most a finite number of points of T in any compact subset of A, the desired result can be obtained.

THEOREM 4. *Let f be a continuously differentiable function defined in the region $0 < |x| \leq 1$. Suppose that*

$$|\nabla f|^n \leq (nK)^{n/2}J + K_1 |x|^{-n\lambda} ,$$

where K, K_1, and d are constants such that $K \geq 1$, $K_1 \geq 0$ and $0 \leq \lambda > 1$. Also assume $u_1 = o(|x|^{-\mu})$ as $x \to 0$ where $\mu = \mu(n, K)$ as defined in Theorems 1 and 2. Then w can be defined at $x = 0$ such that the resulting function is continuous in $0 \leq |x| \leq 1$, and in any closed subregion of $|x| < 1$, f satisfies a uniform Hölder condition with exponent μ.

Proof. If S_r is any sphere in $|x| \leq 1$ whose surface does not contain the origin, the $D(r)$ exists and satisfies

$$D(r) \leq (nK)^{n/2}\int_{\dot{S}_r} u_1 du_2 \cdots du_n + K_1\int_{S_r} ||x||^{-n\lambda} dV . \tag{16}$$

If S_r does contain the origin, then let S_σ denote a sphere of radius σ and center $x = 0$.

Then as in proof of Lemma 3,

$$D(\sigma, r) \leq -(nK)^{n/2}\int_{\dot{S}_\sigma} u_1 du_2 \cdots du_n + B , \tag{17}$$

where B denotes the right hand side of (16).

By hypotheses

$$|u_1|^n \leq \varepsilon(|x|)|x|^{-n\mu} ,$$

where

$$\varepsilon(|x|) \to 0 \quad \text{as} \quad |x| \to 0 .$$

Without loss of generality we may assume the $\varepsilon(|x|)$ is monotonically increasing.

$$\begin{aligned}(D(\sigma, r) - J)^{\frac{n}{n-1}} &\leq (nK)^{\frac{n^2}{2(n-1)}}\left[\int_{\dot{S}_\sigma} u_1 du_2 \cdots dn_n\right]^{\frac{n}{n-1}} \\ &\leq -C(n, K)\varepsilon(\sigma)\sigma^{1-n\mu}\frac{dD}{d\sigma} .\end{aligned}$$

Now suppose $D > J$ for $\sigma = \sigma_0$. Then $D > J$ for all $\sigma < \sigma_0$. Hence

$$\sigma^{n\mu-1} \leq -\frac{C(n, K)\varepsilon(\sigma)}{(D - J)^{\frac{n}{n-1}}}\frac{dD}{d\sigma} \qquad \text{for } \sigma < \sigma_0.$$

Integrate from σ_1 to σ_2 where $\sigma_1 < \sigma_2 < \sigma_0$,

$$\frac{1}{n\mu}[\sigma_2^{n\mu} - \sigma_1^{n\mu}] \le C(n, K)\varepsilon(\sigma_2)(D - J)^{-\frac{1}{n-1}} .$$

Let σ_1 approach zero. Hence

$$(D - B)^{\frac{1}{n-1}} \le C(n, K)\varepsilon(\sigma)\sigma^{-n\mu} . \tag{18}$$

As in the proof of Theorem 2, the inequality of the hypothesis implies

$$D - B \le -\frac{\sigma}{n\mu}\frac{dD}{d\sigma} + \frac{\omega_n K_1}{n - n\lambda}\sigma^{n-n\lambda} .$$

It follows that

$$-\frac{d}{d\sigma}[\sigma^{-n\mu}(D(\sigma, r) - B(r)] \le C(n, K, \lambda)\sigma^{-n\lambda-n\mu+n-1} .$$

Hence, for $\sigma_1 < \sigma$,

$$\sigma_1^{-n\mu}(D(\sigma_1, r) - B) \le \sigma^{-n\mu}(D(\sigma, r) - B) + C(n, K, \lambda)\sigma^{n(1-\mu-\lambda)} ,$$

and finally

$$D(\sigma, r) - B \ge [D(\sigma_1, r) - B - C\sigma^{n(1-\mu-\lambda)}\sigma_1^{n\mu}]\frac{\sigma^{n\mu}}{\sigma_1^{n\mu}} .$$

Let $\sigma < \sigma_0$. For fixed σ, σ_1 may be chosen small enough such that

$$D(\sigma_1, r) - B - C\sigma^{n(1-\mu-\lambda)}\sigma_1^{n\mu} > 0 .$$

For small enough σ this contradicts (16). Hence $D(\sigma, r) \le B$ which implies $D(r) \le B$.

Now proceed as in the proof of Theorem 2. Let B be an arbitrary compact subregion of $|x| < 1$ and let $d =$ distance from B to $|x| = 1$. For any sphere with center in B,

$$D(r) \le \frac{r}{n\mu}\frac{dD}{dr} + K_1\int_{S_r}\rho^{-n\lambda}dV .$$

This implies

$$-\frac{d}{dr}(r^{-n\mu}D) \le K_1 r^{-n\mu-1}\int_{S_r}\rho^{-n\lambda}dV = \frac{K_1\omega_n}{n - n\lambda}r^{-1+n(1-\mu-\lambda)} .$$

Integrating from ρ to d,

$$\rho^{-n\mu}D(\rho) \le d^{-n\mu}D(d) + Cd^{n(1-\mu-\lambda)} .$$

Note that $D(\rho)$ is bounded by

$$(nK)^{n/2}\int_{S_1}JdV + K_1\int_{S_1}\rho^{-n\lambda}dV < \infty .$$

So $D(\rho) \leq$ constant $\rho^{n\mu}$.

By Morrey's n-dimensional lemma, f is uniformly Hölder continuous on B with exponent μ.

THEOREM 5. *Let $f(x)$ be a one-to-one quasi-conformal mapping of $|x| < 1$ onto $|f| < 1$ and such that $f(0) = 0$. The f can be extended to a one-to-one continuous mapping of $|x| \leq 1$ onto $|f| \leq 1$ satisfying $|f(x_1) - f(x_2)| \leq H|x_1 - x_2|^{\mu}$ where $H = H(n, K)$ and $\mu = \mu(n, K)$. $0 < \mu < 1$.*

The proof of this theorem is an immediate generalization of the proof of the 2-dimensional theorem of the Finn and Serrin paper. All new ideas have already been introduced. Hence the proof will not be given.

5. Weakened differentiability requirements. The previous theorems remain true if instead of $f \in C^1$ and $|\varDelta f|^2 \leq nKJ^{2/n}$, f satisfies

(i) $f \in C$ in A, $f = (u_1, u_2, \cdots, u_n)$,

(ii) u_i is absolutely continuous in x_j for almost all values of the other $n-1$ variables $i, j = 1, \cdots, n$,

(iii) the derivatives $u_{i,j}$ (which exist almost everywhere by (ii) should be nth integrable,

(iv) $|\nabla f|^2 \leq nKJ^{2/n}$ almost everywhere or $|\nabla f|^n \leq (nK)^{n/2}J + K_1$ almost everywhere.

To prove the above theorems it suffices to show that the following inequalities on the growth of the modified Dirichlet integral of f remain valid under the weakened hypotheses

$$D(\rho) \leq \{D(t) + K_2\}\left(\frac{\rho}{t}\right)^{n\mu}, \tag{19}$$

for $\rho \leq t \leq d$ and $K_2 = \dfrac{\mu K_1}{1-\mu}\omega_n t^n$.

$$D(t) \leq \frac{K_6}{\{\log (d/t)\}^{n-1}} + d^n \omega_n K_1 , \tag{20}$$

for $t < d$ and where

$$K_6 = \omega_n (nK)^{n/2}(n-1)^{\frac{n+1}{2}} .$$

We shall prove (19) in the case where $|\nabla f|^2 \leq nKJ^{2/n}$. The other statements are proved in a similar manner.

Let f be approximated in the nth integral norm of its derivative by a sequence of functions $f^{(h)} \in C^1$. Thus $\int_A |\nabla(f - f^{(h)})|^n dV$ and $\sup_A |f - f^{(h)}|$ approach zero as h approaches zero. For $f^{(h)}$, (let $J^{(h)}$ be its Jacobian), $Q^{(h)}$ is defined to be $\int_S J^{(h)} dV$.

$$\int_{S_r} J^{(h)} dV \leq \frac{1}{(n-1)^{\frac{n-1}{2}}} \int_{\dot{S}_r} |u_1| [u^2_{2,s_2} + \cdots + u^2_{n,s_n}]^{\frac{n-1}{2}} dA$$
$$\leq \frac{\omega_n^{1/n}(1+\varepsilon)}{(n-1)^{\frac{n-1}{2}}} \left[r\frac{dD^{(h)}}{dr} \right]^{\frac{n-1}{n}},$$

since $|f| \leq 1$. ε approaches zero as h approaches infinity, and

$$D^{(h)}(r) = \int_{S_r} |\nabla f^{(h)}|^n dV .$$

Hence

$$\int_r^{r+\lambda} [Q^{(h)}(\rho)]^{\frac{n-1}{n}} d\rho \leq \left(\frac{1+\varepsilon}{(n-1)^{\frac{n-1}{2}}} \right) \omega_n (r+\lambda)[D^{(h)}(r+\lambda) - D^{(h)}(r)] .$$

Let h approach infinity. Thus

$$\int_r^{r+\lambda} [Q(\rho)]^{\frac{n}{n-1}} d\rho \leq \frac{\omega_n}{(n-1)^{n/2}} (r+\lambda)[D(r+\lambda) - D(r)] ,$$

where

$$Q(\rho) = \int_{S_\rho} J dV .$$

We know $|\nabla f|^2 \leq nKJ^{2/n}$ almost everywhere. Hence

$$D(r) \leq (nK)^{n/2} Q(r) .$$

Therefore

$$\int_r^{r+\lambda} [D(\rho)]^{\frac{n}{n-1}} d\rho \leq \frac{\omega_n (nK)^{\frac{n^2}{2(n-1)}}}{(n-1)^{n-2}} [D(r+\lambda) - D(r)] .$$

Let

$$F(r) = \int_r^{r+\lambda} D(\rho) d\rho .$$

Then

$$F(r) \leq \left[\int_r^{r+\lambda} [D(\rho)]^{\frac{n}{n-1}} d\rho \right]^{\frac{n-1}{n}} \left[\int_r^{r+\lambda} d\rho \right]^{1/n} ,$$

which implies

$$[F(r)]^{\frac{n}{n-1}} \leq \lambda^{\frac{1}{n-1}} \int_r^{r+\lambda} [D(\rho)]^{\frac{n}{n-1}} d\rho .$$

So

$$[F(r)]^{\frac{n}{n-1}} \leq C(n, K)^{\frac{1}{n-1}} (r+\lambda) F'(r) ,$$

which implies

$$\frac{dr}{(r+\lambda)} \leq \lambda^{\frac{1}{n-1}} C(n, K) \frac{dF}{[F(r)]^{1+\frac{1}{n-1}}} .$$

Hence

$$\log \frac{R+\lambda}{r+\lambda} \leq \lambda^{\frac{1}{n-1}} C(n, K)\left[\frac{1}{F(r)^{\frac{1}{n-1}}} - \frac{1}{F(R)^{\frac{1}{n-1}}}\right]$$
$$\leq \frac{\lambda^{\frac{1}{n-1}} C(n, K)}{(\lambda D(r))^{\frac{1}{n-1}}} = \frac{C(n, K)}{D(r)^{\frac{1}{n-1}}} .$$

Let λ approach zero and we obtain the desired inequality.

6. Improvement on Hölder exponent.

LEMMA 6. *If*

$$|\nabla f|^2 \leq nKJ^{2/n} ,$$

then

$$|f_{x_1}|^n \leq C^{n/2} |\nabla f|^n ,$$

where

$$C = \frac{K(n-1)^{2/n} - \lambda}{(1-\lambda)^n K(n-1)^{2/n}} ,$$

for any such that $0 < \lambda < 1$.

Proof.

$$J \leq |f_{x_1}| \left(\frac{n-1}{(n-1)^{\frac{n-1}{2}}}\right)\left[\sum_{\substack{i=1,\cdots,n \\ j=2,\cdots,n}} u_{i,j}^2\right]^{\frac{n-1}{2}} ,$$

since

$$|\det(a_{ij})| \leq \frac{1}{n^{n/2}}\left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{n/2} .$$

Because

$$\frac{n}{(n-1)^{\frac{n}{n-1}}} a^{1/n} b^{\frac{n-1}{n}} \leq \lambda^{n-1} a + \lambda b \quad \text{for} \quad 0 < \lambda < 1 ,$$

$$|\nabla f|^2 \leq nKJ^{2/n} \leq nK |f_{x_1}|^{2/n} (n-1)^{\frac{-1-n}{n}} \left[\sum_{\substack{i=1 \\ j=2}}^{n} u_{i,j}^2\right]^{\frac{n-1}{n}}$$
$$\leq K(n-1)^{2/n}\left[\lambda^{n-1} |f_{x_1}|^2 + \lambda \sum_{\substack{i=1 \\ j=2}}^{n} u_{i,j}^2\right] .$$

Hence

$$|f_{x_1}|^2 \leq \frac{(K(n-1)^{2/n} - \lambda)}{(1-\lambda^n)K(n-1)^{2/n}} |\nabla f|^2 .$$

A simple calculation shows that there is exactly one value of λ between 0 and 1 which will minimize $C(\lambda)$. To find the value of λ, solve the equation

$$(n-1)\lambda^n - nK(n-1)^{2/n}\lambda^{n-1} + 1 = 0 .$$

The Hölder exponent μ of Theorem 1 and Theorem 2 is not the largest exponent that can be obtained. In the proofs of Theorem 1 and Theorem 2 if Lemma 6 were used, the size of μ would be increased.

The constant of Lemma 2 also determines the size of μ. We conjecture that the best constant for this lemma is 1, i.e.,

$$\int_{\dot{S}_r} |u|^n dA \leq r^n \int_{\dot{S}_r} |u_t|^n dA \quad \text{if} \quad \int_{\dot{S}_r} u dA = 0 . \tag{21}$$

This is true if $n = 2$ for then the inequality is Wirtinger's inequality.

If (21) is true, then μ could be defined by the equation

$$\frac{1}{\mu} = \left[\frac{K(n-1)^{2/n} - \lambda}{1-\lambda^n}\right]^{n/2} n^{n/2}(n-1)^{1-1/n-n/2} , \tag{22}$$

where λ is the root between 0 and 1 of the equation

$$(n-1)\lambda^n - nK(n-1)^{2/n}\lambda^{n/1} + 1 = 0 .$$

We further conjecture that this value of μ will be the "best" that can be obtained for given K.

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