ON THE SUMMABILITY OF DERIVED FOURIER SERIES

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1. Introduction. Bosanquet ([1] and [2]) has shown that the $(C, \alpha + r)$, $\alpha \geq 0$, summability of the rth derived Fourier series of a Lebesgue integrable function $f(x)$ is equivalent to the (C, α) summability at $t = 0$ of the Fourier series of another function $\omega(t)$ (see (4), §2) integrable in the Cesaro-Lebesgue (CL) sense. This result suggests the following question: Is there a class of functions, integrable in a sense more general than that of Lebesgue, which permits such a characterization for the summability of rth derived Fourier series and which is large enough to contain $\omega(t)$ also?

In this paper it will be shown that such a characterization is possible within the class of Cesaro-Perron (CP) integrable functions for a summability scale more general than the Cesaro scale (Theorems 1 and 2, §4). Theorem 3 provides sufficient conditions for the summability of the Fourier series of $\omega(t)$ in terms of the Cesaro behavior of $\omega(t)$ at $t=0$.

Integrals are to be taken in the CP sense and of integral order, the order depending on the integrand.[1] It will be convenient to define the $C_{-1}P$ integral as the Lebesgue integral.

2. Definitions. A series Σu_ν is said to be summable (α, β) to S if

$$\lim_{n\to\infty} B\sum_{\nu<n} (1 - \nu/n)^\alpha \log^{-\beta}\left(\frac{1}{1 - \nu/n}\right) u_\nu = S$$

for C sufficiently large, where $B = \log^\beta C$ and $C > 1$. (It is sufficient to say for every $C > 1$.[2])

The function $\lambda_{\alpha,\beta}(x)$ is defined by the equation:

$$\lambda_{\alpha,\beta}(x) + i\tilde{\lambda}_{\alpha,\beta}(x) = \frac{B}{\pi}\int_0^1 (1 - u)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1 - u}\right) e^{ixu} du \,. \tag{1}$$

$$\varphi(t) \equiv \varphi(t, r, x) = \tfrac{1}{2}[f(x + t) + (-1)^r f(x - t)] \,. \tag{2}$$

$$P(t) \equiv P(t, r) = \sum_{i=0}^{[r/2]} \frac{a_{r-2i}}{(r - 2i)!} t^{r-2i} \,. \tag{3}$$

$$\omega(t) = t^{-r}[\varphi(t) - P(t)] \,, \tag{4}$$

[1] Many properties of CP integration have been given by Burkill ([4], [5] and [6]) and by Sargent [7]. Other properties used in this paper can easily be verified by induction.

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[2] Bosanquet and Linfoot [3]. They have also shown the consistency of this scale for $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$.

for $-\pi < t < \pi$ and is of period 2π.

The rth derived Fourier series of $f(t)$ at $t = x$ will be denoted by $D_rFSf(x)$, and the nth mean of order (α, β) of $D_rFSf(x)$ by $S_{\alpha,\beta}(f, x, n)$. The kth iterated integral of $f(x)$ will be written $F_k(t)$ or $[f(t)]_k$.

3. Lemmas. The following result is due to Bosanquet and Linfoot [3]:

LEMMA 1. *For $r \geq 0$ and $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$,*

$$\lambda^{(r)}_{1+\alpha,\beta}(x) = 0(|x|^{-1-\alpha} \log^{-\beta} |x|) + |x|^{-r-2}) \text{ as } |x| \to \infty .$$

LEMMA 2. *For $\alpha \geq 0, \beta \geq 0$ and $r \geq 0$,*

$$x^r\lambda^{(r)}_{1+\alpha+r,\beta}(x) = \sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta)\lambda_{1+\alpha+r-i,\beta+j}(x),$$

where the $B^r_{ij}(\alpha, \beta)$ are independent from x and have the properties:

(i) $B^r_{ij}(\alpha, 0) = 0$ *for* $j \geq 1$;
(ii) $\beta^r_{r0}(\alpha, \beta) \neq 0$.

Proof. Let us put $\gamma_{1+\alpha,\beta}(x) = \lambda_{1+\alpha,\beta}(x) + i\bar{\lambda}_{1+\alpha,\beta}(x)$. For $r = 0$ we take $B^0_{00}(\alpha, \beta) = 1$. For $r \geq 1$ an integration by parts and the identity $u^r = -u^{r-1}(1 - u) + u^{r-1}$ yield the following recursion:

$$\begin{aligned} x^r\gamma^{(r)}_{1+\alpha+r,\beta}(x) &= -(\alpha + 2r)x^{r-1}\gamma^{(r-1)}_{1+\alpha+r,\beta}(x) - \frac{\beta}{\log C}x^{r-1}\gamma^{(r-1)}_{1+\alpha+r,\beta+1}(x) \\ &\quad + (\alpha + r)x^{r-1}\gamma^{(r-1)}_{1+\alpha+r-1,\beta}(x) + \frac{\beta}{\log C}x^{r-1}\gamma^{(r-1)}_{1+\alpha+r-1,\beta+1}(x) . \end{aligned} \tag{5}$$

The lemma follows easily from successive applications of equation (5).

LEMMA 3. *For $n > 0$ and $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$,*

$$\left(\frac{d}{dt}\right)^r \left\{\frac{1}{2\pi} + \frac{B}{\pi}\sum_{\nu \leq n} \left(1 - \frac{\nu}{n}\right)^\alpha \log^{-\beta}\left(\frac{C}{1 - \frac{\nu}{n}}\right) \cos \nu t\right\}$$
$$= n^{r+1} \sum_{k=-\infty}^{\infty} \lambda^{(r)}_{1+\alpha,\beta}[n(t + 2k\pi)] ,$$

for $r = 0, 1, 2, \cdots$.

Proof. Smith ([8], Lemma, 3.1) has shown that for every even periodic, Lebesgue integrable function $Z(t)$,

$$2n\int_0^\infty Z(t)\lambda_{1+\alpha,\beta}(nt)dt = S_{\alpha,\beta}(Z, 0, n) . \tag{6}$$

Using Lemma 1 and the properties of $Z(t)$, one can show in a straightforward manner that

$$(7)\qquad \int_0^\infty Z(t)\lambda_{1+\alpha,\beta}(nt)dt = \int_0^\pi Z(t) \sum_{k=-\infty}^{\infty} \lambda_{1+\alpha,\beta}[n(t+2k\pi)]dt .$$

Let us define $Z(t) = \begin{Bmatrix} 1 \text{ for } |t| \leq x \\ 0 \text{ for } x < |t| \leq \pi \end{Bmatrix}$. Equations (6) and (7) imply that for every $x, 0 \leq x \leq \pi$,

$$(8)\qquad \int_0^x n \sum_{k=-\infty}^{\infty} \lambda_{1+\alpha,\beta}[n(t+2k\pi)]dt = \int_0^x \left\{ \frac{1}{2\pi} + \frac{B}{\pi} \sum_{\nu \leq n} \left(1 - \frac{\nu}{n}\right)^\alpha \cdot \log^{-\beta}\left(\frac{C}{1-\frac{\nu}{n}}\right) \cos \nu t \right\} dt .$$

Since the integrands in (8) are continuous, even and periodic, the lemma is proven for $k = 0$.

To prove the lemma for $k \geq 1$, we need only to observe that the derived series are uniformly convergent in every closed interval by Lemma 1.

LEMMA 4. *Let $f(x) \in CP[-\pi, \pi]$ and be of period 2π. Then for $n > 0$ and $\alpha = 0, \beta > 1$ or $\alpha > 0, \beta \geq 0$,*

$$S^r_{\alpha,\beta}(f, x, n) = 2(-1)^r n^{r+1} \int_0^\pi \varphi(t) \sum_{k=-\infty}^{\infty} \lambda^{(r)}_{1+\alpha,\beta}[n(t+2k\pi)]dt .$$

Proof. This result can be verified by direct calculation using Lemma 3 and the properties of CP integration.

When $f(x)$ is Lebesgue integrable, Lemma 4 is equivalent to a slightly different representation given by Smith [8].

LEMMA 5. *Let $f(x) \in C_\mu P[-\pi, \pi]$ and be of period 2π. Let $\xi, 0 \leq \xi \leq \mu + 1$, be an integer for which $\varphi_\xi(t) \in L[0, \pi]$. Then, for $r \geq 0$ and $\alpha = \xi, \beta > 1$ or $\alpha > \xi, \beta \geq 0$,*

$$S^r_{\alpha+r,\beta}(f, x, n) - a_r = 2(-1)^r n^{r+1} \int_0^\pi [\varphi(t) - P(t)] \lambda^{(r)}_{1+\alpha+r,\beta}(nt)dt + o(1) .$$

Proof. From Lemmas 1 and 4 we see that

$$\begin{aligned} S^r_{\alpha+r,\beta}(P, 0, n) &= 2(-1)^r n^{r+1} \int_0^\pi P(t) \sum_{k=-\infty}^{\infty} \lambda^{(r)}_{1+\alpha+r,\beta}[n(t+2k\pi)]dt \\ &= 2(-1)^r n^{r+1} \int_0^\pi P(t) \lambda^{(r)}_{1+\alpha+r,\beta}(nt)dt + o(1) \text{ as } n \to \infty . \end{aligned}$$

Since $(d/dt)^r P(t) = a_r$, then $S^r_{\alpha+r,\beta}(P, 0, n) \to a_r$ for $\alpha = 0$, $\beta > 1$ or $\alpha > 0, \beta \geq 0$.[3]

It remains to be shown that

$$(9) \qquad S^r_{\alpha+r,}(f, x, n) = 2(-1)^r n^{r+1}\int_0^\pi \varphi(t)\lambda^{(r)}_{1+\alpha+r,\beta}(nt)dt + o(1)\ .$$

Successive integrations by parts give

$$(10) \qquad n^{r+1}\int_0^\pi \varphi(t)\sum_{k=-\infty}^{\infty}{}' \lambda^{(r)}_{1+\alpha+r,\beta}[n(t+2k\pi)]dt = \sum_{j=0}^{\xi-1}(-1)^j n^{r+1+j}\Phi_{j+1}(\pi)$$
$$\cdot \sum_{k=-\infty}^{\infty}{}' \lambda^{(r+j)}_{1+\alpha+r,\beta}[\pi n(2k+1)] + (-1)^\xi n^{r+1+\xi}\int_0^\pi \Phi_\xi(t)\sum_{k=-\infty}^{\infty}{}'$$
$$\cdot \lambda^{(r+\xi)}_{1+\alpha+r,\beta}[n(t+2k\pi)]dt$$

By Lemma 1 each of the integrated terms on the right side of (10) is $o(1)$ as $n \to \infty$, and

$$n^{r+1+\xi}\sum_{k=-\infty}^{\infty}{}' \lambda^{(r+\xi)}_{1+\alpha+r,\beta}[n(t+2k\pi)] = o(1)$$

uniformly in t, $0 \leq t \leq \pi$. Since $\Phi_\xi(t)$ is Lebesgue integrable, it follows that the left side of (10) is $o(1)$. This result and Lemma 4 prove (9) and complete the proof of the lemma.

It can be shown that Lemma 5 holds if $\int_0^\pi$ is replaced by $\int_0^\delta$, $\delta > 0$. Thus, for the values of α and β under consideration, the summability of $D_r FSf(x)$ is a local property of $f(x)$.

Having found an expression for $S^r_{\alpha,\beta}(f, x, n)$, let us estimate the integer ξ in the preceding lemma.

LEMMA 6. *If $h(t) \in C_\mu P[0, a]$ and $t^r h(t) \in C_\lambda P[0, a]$, then*

$$H_{1+\xi}(t) \in L[0, a],\ \textit{where}\ \xi = \min[\mu, \max(\lambda, r)]\ .$$

Proof. The case $\mu = -1$ is trivial by definition of $C_{-1}P$. Therefore, let us assume $\mu \geq 0$. We may also assume, by the consistency of CP integration, that $\lambda \geq r$.

It will be convenient to use the "integration by parts" formula:

$$(11) \qquad [t^r h(t)]_k = \sum_{j=0}^{r} C_j(k, r)t^{r-j}H_{k+j}(t), \qquad k = 1, 2, \cdots,$$

where the $C_j(k, r)$ do not depend on t or the function h.

By the Cesaro continuity and consistency of CP integration, there exists an integer $k \geq \lambda + 1$ such that for $j \geq 0$,

$$(12) \qquad H_{k+1+j}(t) = o(t^{k+j-r}) \text{ as } t \to 0\ .$$

[3] Smith [8], Theorem 3.1.

Since $k \geq \lambda + 1$, equations (11) and (12) imply

$$[t^r h(t)]_k = o(t^{k-1}) = t^r H_k(t) + \sum_{j=1}^{r} t^{r-j} o(t^{k+j-1-r}) ;$$

hence, $H_k(t) = o(t^{k-1-r})$. This result and (12) yield

$$H_{k+j}(t) = 0\,(t^{k-1+j-r}) \text{ as } t \to 0 \text{ for } j \geq 0 . \tag{13}$$

Since (13) is merely (12) with k replaced by $k-1$, this inductive process terminates with $H_{\lambda+1}(t) = o(t^{\lambda-r})$. Therefore, $H_{\lambda+1}(t) = o(1)$ as $t \to 0$ if $\lambda \geq r$.

But for $\eta > 0$, $h(t) \in C_\lambda P[\eta, a]$. Therefore, $H_{1+\xi}(t) \in L[0, a]$.

Lemmas 5 and 6 may be combined to give the following:

LEMMA 7. *Let $f(x) \in C_\lambda P[-\pi, \pi]$ and be of period 2π. If $\omega(t) \in C_\mu P[0, \pi]$, then for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$,*

$$S_{\alpha,\beta}(\omega, 0, n) = 2n\int_0^\pi \omega(t)\lambda_{1+\alpha,\beta}(nt)dt + o(1), \text{ where } \xi$$
$$= \min\,[\mu, \max\,(\lambda, r)] .$$

This section is concluded with two results of Tauberian nature.

LEMMA 8. *If $\alpha \geq 0, \beta > 0$, $\{b_i\}_{i=0}^{k}$* and *$\{a_\nu\}_{\nu=0}^{\infty}$ are sequences of real numbers with $b_0 \neq 0$, and if*

$$F_{\alpha,\beta}(n) = \sum_{i=0}^{k} b_i \sum_{\nu \leq n} \left(1 - \frac{\nu}{n}\right)^\alpha \log^{-(\beta+i)} \left(\frac{C}{1 - \frac{\nu}{n}}\right) a_\nu = o(1) \text{ as } n \to \infty ,$$

then $\sum_{\nu=0}^{\infty} a_\nu = o(\alpha, \beta)$.

The proof of this result is too long to be given here. In general, however, this method is similar to one employed by Bosanquet and Linfoot.[4]

LEMMA 9. *Let $S_{\alpha,\beta}(u, n)$ denote the* nth *mean of order (α, β) of the series Σu_ν. For α, β and $r \geq 0$ and $i, j = 0, 1, \cdots, r$, let us assume that*

(i) *The constants $C_{ij}(\alpha, \beta, r)$ have properties* (i) *and* (ii) *of the $B_{ij}^r(\alpha, \beta)$ in Lemma 2;*

(ii) $\sum_{i,j=0}^{r} C_{ij}(k + \alpha, \beta, r) S_{k+\alpha+r-1,\beta+j}(u, n) = o(1), \qquad k = 0, 1, 2, \cdots ;$

(iii) $\sum_{\nu=0}^{\infty} u_\nu = 0(C)$.

[4] Bosanquet and Linfoot [**3**], Theorem 3.1.

Then $\sum_{\nu=0}^{\infty} u_\nu = 0(\alpha, \beta)$.

Proof. Let us consider the case $\beta > 0$. By (iii) of the lemma and the consistency of (α, β) summability, there exists an integer $K \geq 1$ such that $S_{\alpha+K+i, \beta+j}(u, n) = o(1)$ as $n \to \infty$ for $i, j = 0, 1, 2, \cdots$. Putting $k = K - 1$ in (ii) above, we see that

$$\sum_{j=0}^{r} C_{rj}(K - 1 + \alpha, \beta, r) S_{K-1+\alpha, \beta+j}(u, n) + o(1) = o(1) .$$

Therefore, from (i) above and Lemma 8, $S_{K-1+\alpha, \beta}(u, n) = o(1)$. That is, for $K \geq 1$, $\sum_{\nu=0}^{\infty} u_\nu = o(\alpha + K, \beta)$ implies $\sum_{\nu=0}^{\infty} u_\nu = o(\alpha + K - 1, \beta)$. It follows immediately that $\sum_{\nu=0}^{\infty} u_\nu = o(\alpha, \beta)$.

The case $\beta = 0$, in which we deal with linear combinations of Riesz means, is proved similarly.

4. Theorems.

THEOREM 1. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *If there exist constants* a_{r-2i}, $i = 0, 1, \cdots, [r/2]$, *such that*

(i) $\omega(t) \in C_\mu P]0, \pi]$ *for some integer* μ;

(ii) $FS\omega(0) = 0(\alpha, \beta)$ *for* $\alpha = 1 + \xi, \beta > 1$ *or* $\alpha > 1 + \xi, \beta \geq 0$, *where* $\xi = \min [\mu, \max (\lambda, r)]$,

then $D_r FSf(x) = a_r(\alpha + r, \beta)$.

THEOREM 2. *Let* $f(x) \in C_\lambda P[-\pi, \pi]$ *and be of period* 2π. *If* $D_r FSf(x) = a_r(\alpha + r, \beta)$ *for* $\alpha = 1 + \lambda, \beta > 1$ *or* $\alpha > 1 + \lambda, \beta \geq 0$, *then there exist constants* a_{r-2i}, $i = 0, 1, \cdots, [r/2]$, *such that*

(i) $\omega(t) \in C_\mu P[0, \pi]$ *for some integer* μ;

(ii) $FS\omega(0) = o(\alpha', \beta')$, *where*

$$\left.\begin{cases} \alpha' = 1 + \xi, \beta' > 1 \ \textit{if} \ 1 + \lambda \leq \alpha < 1 + \xi \ \textit{or} \ \alpha = 1 + \xi, \beta \leq 1 \\ \alpha' = \alpha, \beta' = \beta \ \textit{if} \ \alpha = 1 + \xi, \beta > 1 \ \textit{or} \ \alpha > 1 + \xi, \beta \geq 0 \end{cases}\right\} \text{and}$$

$\xi = \min [\mu, \max (\lambda, r)]$.

Before proving these theorems, let us observe that the existence of the a_{r-2i} in the theorems implies their uniqueness from the definition of $\omega(t)$. In fact, somewhat more is true. Observe that $\omega(t) = \omega(t, r) \in CP[0, \pi]$ implies $\omega(t, r - 2i) = o(1)(C)$ as $t \to 0$. Therefore, if $\omega(t, r) \in CP[0, \pi]$ and $FS\omega(0) = 0(C)$, then assuming the truth of Theorems 1 and 2, it is clear that the a_{r-2i} are given by the formula:

$$D_{r-2i} FSf(x) = a_{r-2i}(C), \; i = 0, 1, \cdots [r/2] .$$ [5]

Proof of Theorem 1. Lemma 7 and the consistency of (α, β) sum-

[5] Compare Bosanquet [**2**], eqn. 5.2, for $f(x) \in L[\pi, \pi]$.

mability give the relations:

$$2n\int_0^\pi \omega(t)\lambda_{1+\alpha+r-i,\beta+j}(nt)dt = S_{\alpha+r-i,\beta+j}(\omega, 0, n) + o(1) = o(1) ,$$

for $i, j = 0, 1, 2, \cdots, r$. Therefore,

$$2n\int_0^\pi \omega(t) \sum_{i,j=0}^{r} B_{ij}^r(\alpha, \beta)\lambda_{1+\alpha+r-i,\beta+j}(nt)dt = o(1) ,$$

which by Lemma 2 becomes

$$(14) \qquad 2n^{r+1}\int_0^\pi \omega(t)t^r\lambda_{1+\alpha+r,\beta}^{(r)}(nt)dt = o(1) .$$

Since $t^r\omega(t) = \varphi(t) - P(t)$, relation (14) and Lemma 5 imply that $S_{\alpha+r,\beta}^r(f, x, n) - a_r = o(1)$, i.e., $D_rFSf(x) = a_r(\alpha + r, \beta)$.

Proof of Theorem 2. Let us first prove part (i). Putting $P(t) \equiv 0$ in Lemma 5, we obtain

$$(15) \qquad 2(-1)^r n^{r+1}\int_0^\pi \varphi(t)\lambda_{1+\alpha+r,\beta}^{(r)}(nt)dt = S_{\alpha+r,\beta}(f, x, n) + o(1) .$$

If the left side of (15) is integrated by parts $\lambda + 1$ times, the integrated part is $o(1)$ as $n \to \infty$ by Lemma 1, and (15) becomes

$$(16) \qquad 2(-1)^{r+\lambda+1}n^{r+\lambda+2}\int_0^\pi \Phi_{\lambda+1}(t)\lambda_{1+\alpha+r,\beta}^{(r+\lambda+1)}(nt)dt = S_{\alpha+r,\beta}^r(f, x, n) + o(1)$$

Let us define $\Phi_{\lambda+1}(t)$ for $-\pi < t < 0$ to be an odd (even) function if $r + \lambda + 1$ is odd (even). Then (16) may be written

$$S_{\alpha+r,\beta}^{r+\lambda+1}(\Phi_{\lambda+1}, 0, n) = S_{\alpha+r,\beta}^r(f, x, n) + o(1) .$$

It follows that $D_{r+\lambda+1}FS\Phi_{\lambda+1}(0) = a_r(C)$.

Since $\Phi_{\lambda+1}(t) \in L[-\pi, \pi]$, a theorem of Bosanquet establishes the following result.[6] There exist constants $a^{r+\lambda+1-2i}$, $i = 0, 1, \cdots, [(r+\lambda+1)/2]$, with $a^{r+\lambda+1} = a_r$, such that

$$(17) \qquad \gamma(t) \equiv \{\Phi_{\lambda+1}(t) - P_*(t)\}t^{-(r+\lambda+1)} \in CL[0, \pi] \text{ and } FS\gamma(0) = 0(C) ,$$

where $P_*(t) = \sum_{i=0}^{[(r+\lambda+1)/2]} [a^{r+\lambda+1-2i}/(r + \lambda + 1 - 2i)!]t^{r+\lambda+1-2i}$.

For $\lambda = -1$, put $a^{r-2i} = a_{r-2i}$ in (17). Then (17) states that $\omega(t) \in CP[0, \pi]$ and $FS\omega(0) = 0(C)$.

Let us consider the case $\lambda \geq 0$, and define $h(u, m + 1) \equiv \{\Phi_{m+1}(u) - P_*^{(\lambda-m)}(u)\}u^{-(r+m+1)}$, $m = -1, 0, 1, \cdots, \lambda$. Then for $0 < \eta < t \leq \pi$, an integration by parts yields

[6] Bosanquet [2], Theorem 2. The superscript notation has been used here to distinguish these constants from those whose existence is to be proven.

$$\int_\eta^t h(u, m)du = uh(u, m+1)\Big]_\eta^t + (r+m)\int_\eta^t h(u, m+1)du\ . \tag{18}$$

Let us assume for the moment that for some integer m, $0 \le m \le \lambda$,

$$h(u, m+1) \in C_k P[0, t],\ k \ge \lambda + 1. \tag{19}$$

From (19) and a result due to Sargent[7], it follows that

$$\int_\eta^t h(u, m+1)du \in C_k P[0, t] \text{ and } (C, k+1) \lim_{\eta \to 0} \int_\eta^t h(u, m+1)du$$
$$= \int_0^t h(u, m+1)du\ .$$

Since $\eta h(\eta, m+1) \in C_k P[0, t]$ and is $o(1)(C, k+1)$ as $\eta \to 0$, the right side of (18) has a limit $(C, k+1)$ as $\eta \to 0$. Sargent's result (*ibid.*) and equation (18) imply

$$h(u, m) \in C_{k+1} P[0, t]\ . \tag{20}$$

We infer from the recursive behavior of (19) and (20) that whenever (19) is true, then $h(u, 0) \in CP[0, t]$. But (19) is true for $m = \lambda$ by (17). Therefore,

$$h(t, 0) = \{\varphi(t) - P_*^{(\lambda+1)}(t)\} t^{-r} \in C_\mu P[0, \pi] \text{ for some } \mu\ . \tag{21}$$

In the course of the argument above, it has also been shown that by taking C-limits of (18) we obtain

$$\int_0^t h(u, m)du = th(t, m+1) + (r+m)\int_0^t h(u, m+1)du \tag{22}$$

for $m = 0, 1, \cdots, \lambda$.

If we now define $a_{r-2i} = a^{r+\lambda+1-2i}$, $i = 0, 1, \cdots, [r/2]$, it is easily verified that $P_*^{(\lambda+1)}(t) = P(t)$ and $h(t, 0) = \omega(t)$. Part (i) of the theorem follows immediately from (21).

Next it will be shown that $FS\omega(0) = 0(C)$ for $\lambda \ge 0$, the case $\lambda = -1$ having been settled already.

From equations (11) and (22), it is seen that

$$[h(t, m)]_{k+1} = t[h(t, m+1)]_k + (r + m - k)[h(t, m+1)]_{k+1}\ . \tag{23}$$

If for some integer m, $0 \le m \le \lambda$, the statement

$$h(u, m+1) = o(1)(C, k) \text{ for some integer } k \tag{24}$$

is true, then (24) is also true when $m+1$ and k are replaced by m and $k+1$, respectively, by (23). In this manner we arrive at the conclusion that $h(t, 0) = \omega(t) = o(1)(C)$ as $t \to 0$, which ensures that $FS\omega(0) =$

[7] Sargent [**7**], Lemma 1.

$0(C)$. However, $h(u, \lambda + 1) = \gamma(t)$ and $FS\gamma(0) = 0(C)$ from (17). Therefore, $\gamma(t) = o(1)(C)$[8], so that (24) is true for $m = \lambda$.

It remains only to prove the order relations in part (ii).

Having determined the polynomial $P(t)$, we may state, with the aid of Lemmas 2 and 5, that

$$(25) \qquad S^r_{\alpha+r,\beta}(f, x, n) - a_r = (-1)^r \sum_{i,j=0}^{r} B^r_{ij}(\alpha, \beta) \left\{2n\int_0^\pi \omega(t)\lambda_{1+\alpha+r-i,\beta+j}(nt)dt\right\} + o(1) .$$

If $\lambda + 1 \leq \alpha < 1 + \xi$ or $\alpha = 1 + \xi, \beta \leq 1$, then for $\beta^* > 1$ and $k = 0, 1, 2, \cdots, S^r_{r+1+\xi+k,\beta^*}(f, x, n) - a_r = o(1)$. Equation (25) then implies

$$(26) \qquad \sum_{i,j=0}^{r} B^r_{ij}(1 + \xi + k, \beta^*)\left\{2n\int_0^\pi \omega(t)\lambda_{2+\xi+k+r-i,\beta^*+j}(nt)dt\right\} = o(1)$$

Similarly, for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$, it can be shown that

$$(27) \qquad \sum_{i,j=0}^{r} B^r_{ij}(\alpha + k, \beta)\left\{2n\int_0^\pi \omega(t)\lambda_{1+\alpha+k+r-i,\beta+j}(nt)dt\right\} = o(1) .$$

With the definition of (α', β') and by means of Lemma 7, both (26) and (27) may be combined into the single equation:

$$(28) \qquad \sum_{i,j=0}^{r} B^r_{ij}(\alpha' + k, \beta')S_{\alpha'+k+r-i,\beta'+j}(\omega, o, n) = o(1), k = 0, 1, 2, \cdots .$$

Since $FS\omega(0) = 0(C)$, Lemma 9 and (28) yield part (ii) of the theorem at once.

These two theorems may be combined in several ways to give generalizations to known results. In what follows it is assumed that $f(x) \in C_\lambda P[-\pi, \pi]$ and is of period 2π, $\xi = \min[\mu, \zeta]$ and $\zeta = \max(r, \lambda)$.

COROLLARY 1. *If $\omega(t) \in C_\mu P[0, \pi]$, then for $\alpha = 1 + \xi, \beta > 1$ or $\alpha > 1 + \xi, \beta \geq 0$, $D_r FSf(x) = a_r(\alpha + r, \beta)$ if and only if $FS\omega(0) = 0(\alpha, \beta)$.*[9]

COROLLARY 2. *For $\alpha = 1 + \zeta, \beta > 1$ or $\alpha > 1 + \zeta, \beta \geq 0$, $D_r FSf(x) = a_r(\alpha + r, \beta)$ if and only if $\omega(t) \in CP[0, \pi]$ and $FS\omega(0) = 0(\alpha, \beta)$.*[10]

From Corollary 2 it follows that $D_r FSf(x) = a_r(C)$ if and only if $\omega(t) \in CP[0, \pi]$ and $FS\omega(0) = 0(C)$. Along with a result by Sargent[11]

[8] That $FSg(0) = 0(C)$ if and only if $g(t) = o(1)(C)$ as $t \to 0$ has been shown by Sargent [**7**], Theorem 6.

[9] For $\mu = -1$ compare Wang [**9**].

[10] For $\alpha \geqq r + 1$ and $\lambda = -1$ compare Bosanquet [**2**].

[11] Sargent [**7**], Theorem 6.

this gives a solution, in the sense of Hardy and Littlewood, to the Cesaro summability problem for $D_r FSf(x)$ within the class of CP integrable functions.

The last theorem of this section sharpens a well known sufficient condition for the summability of $FS\omega(0)$ without, however, destroying the CP integrablity of $\omega(t)$.

THEOREM 3. *Let* $\omega(t) \in C_\mu P[-\pi, \pi]$ *and be an even function of period* 2π. *For* $k \geq \mu$, *sufficient conditions that* $FS\omega(0) = 0(1 + k, \beta)$, $\beta > 1$, *are*

(i) $\omega(t) = 0(1)(C, k + 1)$ and
(ii) $\omega(t) = o(1)(C, k + 2)$.

Proof. The proof of this theorem is similar to the proof of the analogous theorem for Riesz summability when $\omega(t)$ is Lebesgue integrable. Starting with Lemma 7 and $k + 1$ integrations by parts, one obtains

$$S_{1+k,\beta}(\omega, 0, n) = (-1)^{k+1} 2n^{k+2} \int_0^\pi \Omega_{k+1}(t) \lambda_{2+k,\beta}^{(k+1)}(nt)dt + o(1) .$$

Writing $\int_0^\pi = \int_0^{K/n} + \int_{K/n}^\delta + \int_\delta^\pi$, it can be shown by straightforward calculations that for arbitrary $\varepsilon > 0$ and $K > e$,
$|S_{1+k,\beta}(\omega, 0, n)| \leq M_1(K) \cdot \varepsilon + M_2 \int_K^\infty (X^{-1} \log^{-\beta} X + X^{-2})dX + o(1)$, where M_2 is independent from ε, K and n. The theorem follows from the last inequality by letting $n \to \infty$, $\varepsilon \to 0$ and $K \to \infty$ in that order.

The theorems of this section can be illustrated by means of the following CP integrable functions:
$t^{-m} \sin t^{-1}$ and $t^{-m} \cos t^{-1}$, $m = 0, 1, 2, \cdots$. For example, from Theorems 1 and 3, $FS[t^{-1} \sin t^{-1}]_{t=0} = 0(1, \beta)$ and $D_1 FS[\sin t^{-1}]_{t=0} = 0(2, \beta)$ for $\beta > 1$.

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