THE SIMPLICITY OF CERTAIN GROUPS

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The purpose of this note is to give a proof of the simplicity of certain "Lie groups" considered in [2]. The main feature of the present development is the proof of Lemma 2 below: it is superior to the corresponding proof given in [2], because no assumption on the number of elements of the base field is required, and is very much shorter than the one given by Chevalley [1] for the direct analogues, over arbitrary fields, of the simple (complex) Lie groups. Thus it turns out that the groups $E_6^1(q^2)$ with $q \leqq 4$, and $D_4^2(q^3)$ with $q \leqq 3$, to which the proof in (2) is not applicable, are simple.

Assuming the notations of [1] and [2] to be in effect, we shall prove:

1. THEOREM. *If* $\hat{G}$ *is one of the groups of type* G^1, G^2 *or* G^3, *defined in* [2], *and the rank* l *of the corresponding Lie algebra is at least* 3, *then* $\hat{G}$ *is simple.*

It will be noticed that the case A_2^1 is excluded by the assumption on l. This is of necessity, since the simplicity of A_2^1 is not universal, but depends on the base field. The same is true of groups of type A_1.

2. MAIN LEMMA. *Let* $\hat{G}$ *be a group of type* G, *that is, one of the direct analogues of the ordinary simple Lie groups, or a group of type* G^1, G^2 or G^3, *but assume* $\hat{G}$ *is not of type* A_1 *or* A_2^1. *Let* $\hat{\mathfrak{U}}$ *be the nilpotent subgroup of* $\hat{G}$ *corresponding to the positive roots of the underlying Lie algebra. Let* H *be a normal subgroup of* $\hat{G}$ *such that* $|H| > 1$. *Then* $|H \cap \hat{\mathfrak{U}}| > 1$.

Proof. Assume first that G is of type G^1. By 7.2 of [2], there is $x = uh\omega(w) \in H$ with $u \in \mathfrak{U}^1$, $h \in \mathfrak{H}^1$.

If $w = 1$, then [2, Lemma 8.5] yields the required conclusion.

If $w \neq 1$, consider first the case in which $w = w_S$ with S a fundamental element of Π^1. Then there is a fundamental $A \in \Pi^1$ such that $B = wA > 0$ and $wA \neq A$ (because A_1 and A_2^1 are excluded). Choose $y \in \mathfrak{U}_A^1$ so that $y \neq 1$ and $y \notin \mathfrak{U}_2$, the subgroup of $\mathfrak{U}$ generated by those $\mathfrak{X}_r$ for which $ht\ r \geqq 2$. Then we assert that the commutator $z = (x, y)$ is in $H \cap \mathfrak{U}^1$ and that $z \neq 1$. In fact, $z = uh\omega(w)y\omega(w)^{-1}h^{-1}u^{-1}y^{-1} = utu^{-1}y^{-1}$ with $t \in \mathfrak{U}_B^1$; hence $z \in H \cap \mathfrak{U}^1$, and, since $\mathfrak{U}/\mathfrak{U}_2$ is Abelian, we have $z \equiv ty^{-1} \not\equiv 1 \bmod \mathfrak{U}_2$, by 4.3 of [2], whence $z \neq 1$.

Finally, consider the general case in which $w \neq 1$. Choose $R \in \Pi^1$

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so that $-wR = S$ is fundamental in Π^1, and then $y \in \mathfrak{U}_S^1$ so that $y \neq 1$. Again form $z = (x, y)$. In the present case, $\omega(w)y\omega(w)^{-1} \in \mathfrak{U}_S^1\mathfrak{H}^1\omega(w_S)\mathfrak{U}_S^1$ by 7.3 of [2], so that z is conjugate to an element x_1 of the form $u_1h_1\omega(w_S)$ with $u_1 \in \mathfrak{U}^1$, $h_1 \in \mathfrak{H}^1$. Clearly $x_1 \neq 1$ and $x_1 \in H$. Thus the situation is that at the beginning of the preceding paragraph, and Lemma 2 is proved for groups of type G^1.

Now to get a proof for groups of type other than G^1, we need only delete all superscripts or replace them all by 2 or all by 3, depending on the group under consideration.

From this point on, we assume that $\hat{G}$ is of type G^1, but not of type A_l^1 (l even), and the ensuing discussion refers explicitely to this case. For groups of type A_l^1 (l even), G^2 or G^3, the changes to be made are quite clear: a prototype for these changes is the replacement of (*) below by an appropriate analogue. For groups of type G, the rest of the proof of Theorem 1 is given in [**1**].

3. Lemma. *If G^1 is not of type A_l^1 (l even) and H is a normal subgroup of G^1 such that $|H| > 1$, then, for some $R \in \Pi^1$, $|H \cap \mathfrak{U}_R^1| > 1$.*

It is convenient to precede the proof of this lemma by some preparatory results.

4. Lemma. *If $s, a, s+a$ and t are roots such that $\bar{a} \neq a$ and $s + a = t + \bar{a}$, then $t = \bar{s}$.*

Proof. We have $s(a) < 0$ and $s(\bar{a}) = (s+a)(\bar{a}) > 0$. Hence $\bar{s} \neq s$, and a simple calculation shows that $t - \bar{s} = s + a - \bar{s} - \bar{a}$ has length 0, since all roots have the same length and the only possible angles are the multiples of $\pi/3$ and $\pi/2$. Hence $t = \bar{s}$.

Let us recall that, for each positive integer m, $\mathfrak{U}_m$ denotes the subgroup of $\mathfrak{U}$ generated by those $\mathfrak{X}_r$ for which $ht\ r \geqq m$.

5. Lemma. *Let s be a positive root, a a fundamental root, and S and A the elements of Π^1 which contain them. Assume $s(a) < 0$, $x \in \mathfrak{U}_S^1$, $y \in \mathfrak{U}_A^1$, and set $ht\ s = n$. Then*

(a) *(x, y) is congruent, $\bmod \mathfrak{U}_{n+2}$, to an element of $\mathfrak{U}^1$ whose representation 4.3 of [2] has all components other than those from $\mathfrak{X}_{s+a}$ and $\mathfrak{X}_{\bar{s}+\bar{a}}$ equal to 1, and*

(b) *if x is given and $x \neq 1$, then y can be chosen so that the $\mathfrak{X}_{s+a}$ component is not 1.*

Proof. Assume first $|S| = |A| = 2$. Then $(s, a) < 0$, whence $(s, \bar{a}) \geqq 0$, because the contrary assumption yields the false conclusion that $s + \bar{s} + a + \bar{a}$ has length 0. Thus $\mathfrak{X}_s$ and $\mathfrak{X}_a$ commute elementwise with $\mathfrak{X}_{\bar{s}}$ and $\mathfrak{X}_{\bar{a}}$, and 4.1 of [2] yields

$$(*)\qquad (x_s(k)x_{\bar{s}}(\bar{k}),\, x_a(l)x_{\bar{a}}(\bar{l})) = x_{s+a}(N_{sa}kl)x_{\bar{s}+\bar{a}}(N_{sa}\overline{kl})\ .$$

Thus (a) is true. If $k \neq 0$, we can choose l so that $kl + \overline{kl} \neq 0$, and then coalesce the terms on the right of (*) if $\bar{s} + \bar{a} = s + a$. Thus (b) is also true. If $|S| = 1$ or $|A| = 1$, we replace (*) in the above argument by an appropriate analogue (see 4.1 and 8.8 of [2]).

Let us recall that a root d is dominant if $d(a) \geqq 0$ for each fundamental root a. Since these inequalities define a fundamental region for W, and all roots are congruent under W in the present case, it follows that there is a unique dominant root d. If s is any other root, then $(s, a) < 0$ for some fundamental root a, and then $s + a$ is also a root. Thus the dominant root d may also be described as the unique root of maximum height; and one has $\bar{d} = d$ and $d > s$ for each root $s \neq d$.

We now turn to the proof of Lemma 3. Among all $x \in H \cap \mathfrak{U}^1$ for which $x \neq 1$, choose one which maximizes the minimum $S \in \Pi^1$ for which $x_S \neq 1$ in the representation 4.5 of [2]. If this minimum is R, we show $x = x_R$. Assuming the contrary, one can write $x = x_R x_T \cdots$ with $x_T \neq 1$. Set $ht\ R = n$. If $r \in R$, then r is not dominant, since $R < T$. Thus $r(a) < 0$ for some fundamental root a, and $r + a$ is a root. If $a \in A \in \Pi^1$, we conclude from Lemma 5 that there is $y \in \mathfrak{U}^1_A$ such that (x_R, y) is congruent, mod $\mathfrak{U}_{n+2}$, to an element of $\mathfrak{U}^1$ with the $\mathfrak{X}_{r+a}$ component not 1. Since $z = (x, y) \in H \cap \mathfrak{U}_{n+1}$, and $>$ respects heights, we need only show $z \neq 1$ to reach a contradiction. We have $(x, y) = (x_R, y)(x_T, y) \cdots \bmod \mathfrak{U}_{n+2}$. Here the elements on the right are in $\mathfrak{U}_{n+1}$. By choice of y, the $\mathfrak{X}_{r+a}$ component of (x_R, y) is not 1, and by Lemmas 4 and 5, the $\mathfrak{X}_{r+a}$ component of each of $(x_T, y) \cdots$ is 1. Thus we conclude from 4.3 of [2] and the fact that $\mathfrak{U}_{n+1}/\mathfrak{U}_{n+2}$ is Abelian that $(x, y) \not\equiv 1 \bmod \mathfrak{U}_{n+2}$. Therefore $(x, y) \neq 1$, and Lemma 3 is proved.

The proof of Theorem 1 can now be completed, just as in [2].

REFERENCES

1. C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2) **7** (1955), p. 14.
2. R. Steinberg, *Variations on a theme of Chevalley*, Pacific J. Math. **9** (1959), p. 875.

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