IRREDUCIBLE CONGRUENCE RELATIONS ON LATTICES

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1. Introduction. The structure of the lattice L is known to depend upon properties of the distributive lattice $\vartheta(L)$ of all congruence relations on L; for example,

(1.1) (Birkhoff [1]) L is a subdirect union of a finite number of simple lattices if and only if $\vartheta(L)$ is a finite Boolean algebra,

(1.2) (Dilworth [2]) L is a direct union of a finite number of simple lattices if and only if $\vartheta(L)$ is a finite Boolean algebra in which all the elements permute.

In the early development of structure theory for lattices, L was assumed to be modular, and the notion of projectivity was used to study congruence relations. For non-modular lattices a more general concept was needed; accordingly, Dilworth [2] devised the notion of weak projectivity and showed that complementation has a strong influence on structure. He proved:

(1.3) Every relatively complemented lattice satisfying the ascending chain condition is the direct union of a finite number of simple relatively complemented lattices;

(1.4) Every finite dimensional locally relatively complemented lattice is a subdirect union of a finite union of simple, locally relatively complemented lattices;

(1.5) A relatively complemented lattice which satisfies a chain condition is simple if and only if all prime quotients are projective.

More recently these results have been developed and generalized by Tanaka [7], Maeda [6], and Hashimoto [4].

It is interesting to observe for the lattices described in (1.3), (1.4), and (1.5), weak projectivity of prime quotients reduces to projectivity. The present paper studies the relationship between weak projectivity and projectivity of prime quotients. It is shown that if L satisfies the descending chain condition and if each join irreducible element of L covers some element, then the corresponding irreducible congruence relations generate $\vartheta(L)$ and provide simple criteria for the structure of L.

2. Definitions. This section contains definitions of the basic terms which are used; terminology generally conforms to that given in Birkhoff [1].

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(a) *Quotients.* If $a \supseteq b$ in L, the *quotient* $\{x \in L \mid b \subseteq x \subseteq a\}$ is denoted by a/b. If a covers b ($a \succ b$), then a/b is called a *prime quotient.* The quotient c/d is *contained in* a/b if and only if $b \subseteq d \subseteq c \subseteq a$.

(b) *Weak projectivity.* A quotient a/b is said to be *weakly projective into* a quotient c/d (a/b **WP** c/d) whenever there exists a finite sequence of quotients $a/b = x_0/y_0, x_1/y_1, \cdots, x_k/y_k = c/d$ such that x_{i-1}/y_{i-1} is contained in a transpose of x_i/y_i. Weak projectivity of quotients is a reflexive and transitive relation, but, unlike projectivity, is not symmetric.

(c) *Congruence relations.* A *congruence relation* θ on L is an equivalence relation which is preserved by the two basic lattice operations. Congruence relations are partially ordered by writing $\theta \subseteq \phi$ if and only if $a \equiv b$ (θ) implies $a \equiv b$ (ϕ). Under this ordering the set of all congruence relations on L is a complete lattice $\vartheta(L)$ in which the operations are defined by

$a \equiv b$ $(\cup_S \theta_\alpha)$ means $a = x_0, x_1, \cdots, x_k = b$ exist such that $x_{i-1} \equiv x_i$ (θ_α) for some $\theta_\alpha \in S$,

$a \equiv b$ $(\cap_S \theta_\alpha)$ means $a \equiv b$ (θ_α) for all $\theta_\alpha \in S$.

Furthermore, $\vartheta(L)$ is distributive (Funayama and Nakayama [3]). Two congruence relations are said to *permute* whenever $a \equiv c$ (θ) and $c \equiv b$ (ϕ) imply that d exists such that $a \equiv d$ (ϕ) and $d \equiv b$ (θ). The *center* $\Gamma(L)$ of $\vartheta(L)$ is the set of all $\phi \in \vartheta(L)$ which permute with all $\theta \in \vartheta(L)$. The *trivial congruence relations* ι and ω are the unit and null elements of $\vartheta(L)$. The quotient a/b is said to be *collapsed by* θ if and only if $a \equiv b$ (θ). Clearly $x \equiv y$ (θ) if and only if $(x \cup y)/(x \cap y)$ is collapsed by θ. Then every quotient is collapsed by ι, and no proper quotient is collapsed by ω.

(d) *Structural properties.* L is *simple* if and only if the congruence relations on L are trivial. L is *irreducible* if and only if there exist distinct elements a and b such that $a \equiv b$ (θ) for every $\theta \neq \omega$. Simplicity implies irreducibility, but not conversely.

(e) *Dimensionality.* For the methods of this paper it is necessary to impose on L the condition

(δ) L satisfies the descending chain condition, and each join irreducible element covers some element.

Any such lattice will be called a *δ-lattice.*

(f) *Quotient ideals.* Given a congruence relation θ on a lattice L, let $N(\theta)$ denote the set of all quotients collapsed by θ. Then $N(\theta)$ is a *quotient ideal* as defined by Maeda [5]; that is, $N(\theta)$ satisfies

(2.1) $a/a \in N(\theta)$,
(2.2) if $a/b \in N(\theta)$ and $c/d \subseteq a/b$, then $c/d \in N(\theta)$,
(2.3) if $a/b \in N(\theta)$ and a/b $\boldsymbol{P}$ c/d, then $c/d \in N(\theta)$,
(2.4) if $a/b \in N(\theta)$ and $b/c \in N(\theta)$, then $a/c \in N(\theta)$.

Conversely, given any quotient ideal N, a congruence relation $\theta(N)$ is defined by writing $a \equiv b$ $(\theta(N))$ if and only if $(a \cup b)/(a \cap b) \in N$. It follows that $N(\theta(N)) = N$. Furthermore,

(2.5) $N(\theta) \subseteq N(\phi)$ if and only if $\theta \subseteq \phi$.

The connection between quotient ideals and weak projectivity is established as follows: let S be any set of quotients of L, and denote by $N(S)$ the set of all quotients a/b for which there exists a chain $a = x_0 \supseteq x_1 \supseteq \cdots \supseteq x_k = b$ such that x_{i-1}/x_i is weakly projective into a quotient of S, for $i = 1, \cdots, k$. Then $N(S)$ is the minimal quotient ideal containing S. In this way S determines a congruence relation $\theta(S) = \theta(N(S))$ which is the minimal congruence relation which collapses the quotients of S.

This paper is concerned primarily with the case in which S consists of a single *irreducible prime quotient* q/c_q, where q is join irreducible and $q \succ c_q$; the corresponding congruence relation will be denoted by θ_q. Such a relation will be called an *irreducible congruence relation.* This terminology is justified by the fact that precisely these congruence relations are the join irreducible elements of $\vartheta(L)$.

3. **Irreducible congruence relations.**

Let L be a δ-lattice, and let Q be the set of its join irreducible elements. For each $a \in L$ define

$$Q(a) = \{q \in Q \mid q \subseteq a\} .$$

Since the descending chain condition holds, each $a \in L$ is the union of a finite subset of $Q(a)$.

LEMMA 3.1. *If $a \supset b$, then q/c_q **WP** a/b for every $q \in Q(a) - Q(b)$, and q/c_q **P** $(q \cup b)/b$ for every q which is minimal in $Q(a) - Q(b)$.*

Proof. If $q \in Q(a) - Q(b)$, then $q/c_q \subseteq q/(q \cap b)$ $\boldsymbol{T}$ $(q \cup b)/b \subseteq a/b$, so q/c_q $\boldsymbol{WP}$ a/b. If q is minimal in $Q(a) - Q(b)$, then $q \succ q \cap b$, and the second statement holds.

For each congruence relation θ on L let $W(\theta)$ denote the set of all irreducibles q for which q/c_q is collapsed by θ; that is,

$$W(\theta) = \{q \in Q \mid q \equiv c_q(\theta)\} = \{q \in Q \mid q/c_q \in N(\theta)\} .$$

Likewise for any quotient a/b, let $\theta(a/b)$ be the congruence relation generated by collapsing a/b; it follows that

$$W(\theta(a/b)) = \{q \in Q \mid q/c_q \ \boldsymbol{WP}\ a/b\} ,$$

and we write $W(a/b)$ in place of $W(\theta(a/b))$. Using similar notation for projectivity, let

$$P(a/b) = \{q \in Q \mid q/c_q \ \boldsymbol{P} \ a/b\} .$$

The following statements are easy consequences of these definitions, the properties of projectivity and weak projectivity, and Lemma 3.1.

(3.1) The sets $P(q/c_q)$ for $q \in Q$ form a partition of Q.

(3.2) $P(a/b) \subseteq W(a/b)$.

(3.3) If a/b is prime and if q is minimal in $Q(a) - Q(b)$, then $P(a/b) = P(q/c_q)$ and $W(a/b) = W(q/c_q)$.

(3.4) If $q \in W(a/b)$, then $P(q/c_q) \subseteq W(q/c_q) \subseteq W(a/b)$.

(3.5) $W(a/b) = \vee_{W(a/b)} P(q/c_q)$.

The remainder of this section is devoted to proving a sequence of lemmas concerning these sets, weak projectivity, and congruence relations to demonstrate the role of irreducible congruence relations in generating $\vartheta(L)$.

LEMMA 3.2. *If $a \supseteq b$, then $a \equiv b(\theta)$ if and only if $\theta(a/b) \subseteq \theta$.*

Proof. Let $a \equiv b$ (θ) and $x \equiv y$ $(\theta(a/b))$. A chain, $x \cup y = a_0 \supseteq a_1 \supseteq \cdots \supseteq a_k = x \cap y$, exists for which a_{i-1}/a_i $\boldsymbol{WP}$ $a/b \in N(\theta)$. By (2.2), (2.3), and the definition of weak projectivity, $a_{i-1}/a_i \in N(\theta)$. Then $(x \cup y)/(x \cap y) \in N(\theta)$ by (2.4), and $N(\theta(a/b)) \subseteq N(\theta)$. The lemma follows from (2.5), the reverse implication being trivial.

COROLLARY. *$q \in W(\theta)$ if and only if $\theta_q \subseteq \theta$.*

The next lemma is of fundamental importance, since it reveals that the collapse of any quotient can be accomplished by the collapse of a finite number of irreducible prime quotients; hence any congruence relation is a finite union of irreducible congruence relations.

LEMMA 3.3. *If $a \supset b$, there exists a finite set $S \subseteq Q(a) - Q(b)$ such that $a \equiv b(\cup_S \theta_q)$.*

Proof. By the descending chain condition it may be assumed that every element properly contained in a has the property asserted in the lemma. Let $S_1 \subseteq Q(a) - Q(b)$ be chosen so that $q_1 \in S_1$ is not redundant in the representation $a = b \cup \bigcup_{S_1} q$. Let $S_2 = S_1 - q_1$, and let $a_2 = b \cup \bigcup_{S_2} q$; then $a \supset a_2 \supseteq b$. A finite set $S_3 \subseteq Q(a_2) - Q(b)$ exists such that $a_2 \equiv b \cup (\bigcup_{S_3} \theta_q)$. Also a finite set $S_4 \subseteq Q(q_1) - Q(a_2 \cap q_1)$ exists such that $q_1 \equiv a_2 \cap q_1(\bigcup_{S_4} \theta_q)$. Then $S = S_3 \vee S_4$ is a finite subset of $Q(a) - Q(b)$ for which $a \equiv b(\bigcup_S \theta_q)$.

LEMMA 3.4. *If $a \supset b$, there exists a finite set $S \subseteq Q(a) - Q(b)$ such that $\theta(a/b) = \bigcup_S \theta_q$.*

Proof. Lemmas 3.2 and 3.3 imply $\theta(a/b) \subseteq \bigcup_S \theta_q$. Conversely, Lemma 3.1, the definition of $W(\theta)$, and the Corollary imply $\theta_q \subseteq \theta(a/b)$ for every $q \in Q(a) - Q(b)$.

LEMMA 3.5. *For some finite set* $S \subseteq W(\theta)$, $\theta = \bigcup_S \theta_q$.

Proof. From the Corollary, $\bigcup_{W(\theta)} \theta_q \subseteq \theta$. Conversely, let $x \equiv y(\theta)$, $x \neq y$; by Lemma 3.3 $x \cup y \equiv x \cap y$ $(\bigcup_S \theta_q)$ for some finite $S \subseteq Q(x \cup y) - Q(x \cap y)$. Hence $\theta \subseteq \bigcup_S \theta_q$. But $q/c_q \in N(x \cup y/x \cap y) \subseteq N(\theta)$ for every $q \in S$, so $S \subseteq W(\theta)$.

LEMMA 3.6. $W(\theta) \subseteq W(\phi)$ *if and only if* $\theta \subseteq \phi$.

Proof. The direct implication follows from Lemma 3.5, while the reverse implication follows from (2.5) and the definition of $W(\theta)$. Observe that if equality holds in either relation, it holds in both.

LEMMA 3.7. θ *is completely join irreducible in* $\vartheta(L)$ *if and only if* $\theta = \theta_q$ *for some* $q \in Q$.

Proof. From Lemma 3.5 it is clear that any completely join irreducible θ must be of the form θ_q for some $q \in W(\theta) \subseteq Q$. Conversely, for any $q \in Q$ suppose $\theta_q = \bigcup_{\alpha \in A} \theta_\alpha$. Then $q \equiv c_q(\bigcup_{\alpha \in A} \theta_\alpha)$, so there exists a finite sequence

$$q = x_0, x_1, \cdots, x_k = c_q$$

such that

$$x_{i-1} \equiv x_i(\theta_{\alpha_i}), \text{ for some } \alpha_i \in A ,$$

for $i = 1, 2, \cdots, k$. Then

$$(x_{i-1} \cap q) \cup c_q \equiv (x_i \cap q) \cup c_q(\theta_{\alpha_i}) .$$

But

$$q \supseteq (x_i \cap q) \cup c_q \supseteq c_q ,$$

so for each $i = 0, 1, \cdots, k$, $(x_i \cap q) \cup c_q$ equals q or c_q. Since $q = (x_0 \cap q) \cup c_q$ and $c_q = (x_k \cap q) \cup c_q$, there exists an index j, $1 \leq j \leq k$, for which

$$q = (x_{j-1} \cap q) \cup c_q \equiv (x_j \cap q) \cup c_q = c_q(\theta_{\alpha_j}) .$$

By Lemma 3.2, $\theta_q \subseteq \theta_{\alpha_j}$; also the reverse relation holds by hypothesis, so $\theta_q = \theta_{\alpha_j}$.

Thus any completely irreducible element of $\vartheta(L)$ is an irreducible congruence relation, θ_q, generated by collapsing an irreducible prime quotient of L. It follows from (3.3) and Lemma 3.3 that the collapse of any prime quotient generates an irreducible congruence relation. Clearly the number of distinct completely irreducible elements of $\vartheta(L)$ cannot exceed the number of distinct irreducibles in L. Two additional remarks concerning weak projectivity conclude this section.

LEMMA 3.8. *If a/b **WP** c/d, then for each $q \in Q(a) - Q(b)$ there exists $\tilde{q} \in Q(c) - Q(d)$ such that q/c_q **WP** $\tilde{q}/c_{\tilde{q}}$.*

Proof. Let $q \in Q(a) - Q(b)$, where a/b **WP** c/d. Then $q \in W(c/d)$, and $\theta_q \subseteq \theta(c/d) = \bigcup_S \theta_{\tilde{q}}$, where $S \subseteq Q(c) - Q(d)$. But in a distributive lattice if an irreducible is contained in the join of elements, it is contained in one of those elements. Hence $\theta_q \subseteq \theta_{\tilde{q}}$ for some $\tilde{q} \in S$. By Lemma 3.6, q/c_q **WP** $\tilde{q}/c_{\tilde{q}}$.

LEMMA 3.9. *If a/b **WP** c/d, then $\theta(a/b) \subseteq \theta(c/d)$; the converse holds if a/b is prime.*

Proof. If a/b **WP** c/d, then $W(a/b) \subseteq W(c/d)$ since weak projectivity is transitive. By Lemma 3.6, $\theta(a/b) \subseteq \theta(c/d)$. Conversely, if a/b is prime, then a/b **T** q/c_q for any minimal $q \in Q(a) - Q(b) \subseteq W(a/b)$. If also $\theta(a/b) \subseteq \theta(c/d)$, $q \in W(c/d)$, so a/b **WP** c/d.

4. Structure theorems. We now consider the role of irreducible congruence relations in determining the structure of L. From the theorems quoted in the introduction, it is clear that complementation in $\vartheta(L)$, permutability in $\vartheta(L)$, and the relation between weak projectivity of prime quotients have important effects on the structure of L.

THEOREM 4.1. *In any δ-lattice the following statements are equivalent:*

(a) *$\vartheta(L)$ is a Boolean algebra,*

(b) *for every $q \in Q$, $\theta_q \succ \omega$,*

(c) *the relation of weak projectivity is symmetric on the set of all irreducible prime quotients.*

Proof. Any join irreducible element of a Boolean algebra must be a point, so (a) implies (b). Let θ_q be a point, and suppose $\tilde{q}/c_{\tilde{q}}$ **WP** q/c_q. By Lemma 3.9, $\theta_{\tilde{q}} \subseteq \theta_q$, and equality must hold. Then q/c_q **WP** $\tilde{q}/c_{\tilde{q}}$, again by Lemma 3.9, so (b) implies (c). If weak projectivity is symmetric for all irreducible prime quotients, the sets $W(q/c_q)$ partition Q. For arbitrary θ, let $\theta' = \bigcup_{W'(\theta)} \theta_q$, where $W'(\theta) = Q - W(\theta)$. Then θ' is a complement of θ in $\vartheta(L)$, which therefore is a Boolean algebra.

It follows from the preceding argument that $\vartheta(L)$ is a Boolean algebra if and only if the sets $W(q/c_q)$ partition Q. But also the sets

$P(q/c_q)$ partition Q, and $P(q/c_q) \subseteq W(q/c_q)$. Thus if $\vartheta(L)$ is a Boolean algebra, the partition of Q imposed by projectivity is a refinement of the partition imposed by weak projectivity. These two partitions can be distinct, even when L is simple. However, weak projectivity of prime quotients does reduce to projectivity for a wide class of lattices —for example, modular lattices and the lattices described in (1.3), (1.4), and (1.5). In this connection the following theorem underlies corresponding results obtained by Dilworth [2] and Hashimoto [4] for relatively complemented lattices.

THEOREM 4.2. *Let a δ-lattice L satisfy the condition that if two irreducible prime quotients are mutually weakly projective, then they are projective. Then L is simple if and only if all prime quotients are projective.*

Proof. Hashimoto uses the term *uniserial* to describe lattices in which all prime quotients are projective. Clearly any uniserial lattice is simple, because the collapse of any quotient collapses all of L. Conversely, if L is simple, let a/b and c/d be prime quotients. By (3.3) and Lemma 3.6, $\theta(a/b) = \theta(q/c_q) = \theta(c/d) = \theta(\tilde{q}/c_{\tilde{q}})$ where the middle equality holds since L is simple, and where q and $\tilde{q}$ can be chosen to be minimal, respectively, in $Q(a) - Q(b)$ and $Q(c) - Q(d)$. Then $a/b \ \boldsymbol{T}\ q/c_q \ \boldsymbol{WP}\ \tilde{q}/c_{\tilde{q}} \ \boldsymbol{T}\ c/d$, and $\tilde{q}/c_{\tilde{q}} \ \boldsymbol{WP}\ q/c_q$. Hence $q/c_q \ \boldsymbol{P}\ \tilde{q}/c_{\tilde{q}}$, and $a/b \ \boldsymbol{P}\ c/d$.

THEOREM 4.3. *If L is a δ-lattice for which $\vartheta(L)$ is a Boolean algebra, then L is simple if and only if L is irreducible.*

Proof. Let $a \neq b$ be elements which establish the irreducibility of L; $a \equiv b(\theta)$ for all $\theta \neq \omega$. For all $q \in Q$, $a \equiv b(\theta_q)$; thus $\theta(a \cup b/a \cap b) \subseteq \theta_q$. Therefore, for some $\tilde{q}$ and all q, $\theta(a \cup b/a \cap b) = \theta_{\tilde{q}} \subseteq \theta_q$, so $\vartheta(L)$ has $\theta_{\tilde{q}}$ as its only point. But if $\vartheta(L)$ is also a Boolean algebra, $\theta_{\tilde{q}} = \theta_q$ for all q, and therefore L is simple. The converse is well known.

COROLLARY. *A δ-lattice L is irreducible if and only if $\vartheta(L)$ has a single point.*

Proof. The preceding proof shows that $\vartheta(L)$ has a unique point if L is irreducible. But if θ_q is the only point of $\vartheta(L)$, then $\theta \neq \omega$ implies $\theta_q \subseteq \theta$, and thus $q \equiv c_q(\theta)$. Therefore q and c_q satisfy the condition of irreducibility for L.

Our remaining remarks concern complementation and permutability of irreducible congruence relations. The investigation of these properties arises naturally because any direct decomposition of L determines a congruence relation θ which has a complement and which permutes with all congruence relations. A congruence relation with these two properties is called a *decomposition congruence relation*, and the set of all decomposition congruence relations forms a Boolean sublattice of $\vartheta(L)$.

LEMMA 4.1. *θ_q has a complement in $\vartheta(L)$ if and only if θ_q satisfies the condition that $\tilde{q} \in Q$ and $\theta_q \cap \theta_{\tilde{q}} \neq \omega$ imply $\theta_{\tilde{q}} \subseteq \theta_q$.*[1]

Proof. Let θ^* be a complement of θ_q; it is easily verified that $W(\theta^*)$ is the complement of $W(\theta_q)$ in Q. Let $\theta_q \cap \theta_{\tilde{q}} = \theta \supset \omega$. Then $W(\theta_q) \supseteq W(\theta)$ so $W(\theta) \wedge W(\theta^*)$ is void. Also $W(\theta) \subseteq W(\theta_{\tilde{q}})$, so if $\tilde{q} \in W(\theta^*)$, then by the Corollary following Lemma 3.2, $W(\theta) \subseteq W(\theta_{\tilde{q}}) \subseteq W(\theta^*)$, which is a contradiction. Hence $\tilde{q} \in W(\theta_q)$ and $\theta_{\tilde{q}} \subseteq \theta_q$. Conversely, suppose θ_q satisfies the condition stated in the lemma. Let $W^* = Q - W(\theta_q)$, and let $\theta^* = \bigcup_{W^*} \theta_{\tilde{q}}$. Then $W(\theta^*) \supseteq W^*$, so $\theta_q \cup \theta^* = \iota$; and $\theta_q \cap \theta^* = \bigcup_{W^*} (\theta_q \cap \theta_{\tilde{q}})$. But $\tilde{q} \in W^*$ implies $\theta_{\tilde{q}} \not\subseteq \theta_q$, so $\theta_q \cap \theta_{\tilde{q}} = \omega$ for all $\tilde{q} \in W^*$. Thus $\theta_q \cap \theta^* = \omega$.

THEOREM 4.4. *If L is a δ-lattice, then $\vartheta(L)$ is a Boolean algebra if and only if θ_q has a complement for every $q \in Q$.*

Proof. The condition is trivially necessary. Suppose each θ_q has a complement; then Lemma 4.1 implies that if $\theta_{\bar{q}} \cap \theta_{\tilde{q}} \neq \omega$, then $\theta_{\bar{q}} \subseteq \theta_{\tilde{q}}$ and $\theta_{\tilde{q}} \subseteq \theta_{\bar{q}}$. Hence for each $q \in Q$, θ_q must be a point, so by Theorem 4.1 $\vartheta(L)$ is a Boolean algebra.

THEOREM 4.5. *If L is a δ-lattice, then $\Gamma(L) = \vartheta(L)$ if and only if θ_q and $\theta_{\tilde{q}}$ permute for all $q, \tilde{q} \in Q$.*

Proof. The necessity is trivial; the sufficiency follows from Lemma 3.5 and the fact that if θ permutes with each member of a set of congruence relations, then θ permutes with any union of them.

Combining Theorems 4.4 and 4.5 with (1.1) and (1.2), we see that under suitable dimensionality conditions, L is a subdirect union of simple lattices if and only if each θ_q has a complement, while L is a direct union of simple lattices if and only if each θ_q is a decomposition congruence relation.

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[1] As the referee has pointed out, Lemma 4.1 expresses a property of the join irreducibles of any complete, distributive lattice which satisfies the descending chain condition.

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