ON THE REPRESENTATION OF OPERATORS BY CONVOLUTION INTEGRALS

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1. Introduction.

Let $\mathfrak{X}$ be the complex vector space consisting of all complex-valued functions of a non-negative real variable. For each positive number u, let the *shift operator* I_u be the mapping of $\mathfrak{X}$ into itself defined by the formula

$$I_u x(t) = \begin{cases} 0 & (0 \leqq t < u) \\ x(t-u) & (t \geqq u) \end{cases}$$

Evidently, $I_{u+v} = I_u I_v$, for any positive numbers u and v.

A linear operator A which maps a subspace $\mathfrak{D}$ of $\mathfrak{X}$ into itself will here be called a *V-operator* (after Volterra) if

(1.1) for each x in $\mathfrak{D}$, the conjugate function x^* belongs to $\mathfrak{D}$,

(1.2) both $\mathfrak{D}$ and $\mathfrak{X}\backslash\mathfrak{D}$ are invariant under the shift operators,

(1.3) every shift operator commutes with A.

Many operators that occur in mathematical physics are of this type. If $\mathfrak{D}$ is any subspace of $\mathfrak{X}$ having the properties (1.1) and (1.2), the restriction to $\mathfrak{D}$ of each shift operator is an example of a V-operator. All 'perfect operators' (of which a definition may be found in [5][1]) are V-operators, on the space of perfect functions.

In this paper we obtain a representation theorem for V-operators which are continuous in a certain sense. This result leads to characterizations of two related classes of perfect operators, one of which has been considered from a different point of view in [5]. The main representation theorem (Theorem 4) is similar to a result obtained by R. E. Edwards [2] for V-operators which are continuous in another sense; and it closely resembles a theorem given recently by König and Meixner ([3], Satz 3).

2. Elementary properties of V-operators.

An important property of V-operators is given by

THEOREM 1. *Let A be a V-operator, and let x_1 and x_2 be two of its operands such that, for some positive number t_0, $x_1(t) = x_2(t)$ whenever $0 \leqq t \leqq t_0$. Then $Ax_1(t) = Ax_2(t)$ whenever $0 \leqq t \leqq t_0$.*

Proof. Let $x = x_1 - x_2$. Then, since $x(t) = 0$ if $0 \leqq t \leqq t_0$, there is

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[1] And in § 4 below.

a function y such that $x = I_{t_0}y$; and y is an operand of A, by virtue of the property (1.2). Consequently, by virtue of (1.3), $Ax = I_{t_0}Ay$; so that $Ax(t) = 0$ whenever $0 \leqq t \leqq t_0$. But $Ax = Ax_1 - Ax_2$, since A is linear: hence the conclusion of the theorem.

With products and linear combinations defined in the usual way, the V-operators on a given space $\mathfrak{D}$ constitute a linear algebra $\mathfrak{A}(\mathfrak{D})$. If A belongs to $\mathfrak{A}(\mathfrak{D})$ then so does the operator A^* defined by

$$A^*x = (Ax^*)^* ,$$

where x is any function in $\mathfrak{D}$. We therefore have the unique decomposition

$$A = B + iC ,$$

where B and C belong to $\mathfrak{A}(\mathfrak{D})$ and are 'real' in the sense that Bx and Cx are real for every real function x in $\mathfrak{D}$. (The property (1.1) ensures that every function x in $\mathfrak{D}$ can be uniquely expressed as $x_1 + ix_2$, where x_1 and x_2 are real functions in $\mathfrak{D}$.)

If A is a linear combination of shift operators, we have

$$A = \sum_{j=1}^{n} \alpha_j I_{u_j} = I_u \sum_{j=1}^{n} \alpha_j I_{u_j - u} ,$$

where $\alpha_1, \cdots, \alpha_n$ are complex numbers, u is the least of the positive numbers $u_1, \cdots, u_n$, and I_0 is the unit operator (to be denoted henceforth by 'I'). From this it is apparent that A has no reciprocal in the algebra $\mathfrak{A}(\mathfrak{X})$; however, $I - A$ has a reciprocal in $\mathfrak{A}(\mathfrak{X})$, as the following result shows.

THEOREM 2. *Let A be a V-operator on a space $\mathfrak{D}$, and let u be any positive number. Then the formula*

$$Bx(t) = x(t) + \sum_{n=1}^{\infty} I_{nu} A^n x(t) ,$$

where x is any function in $\mathfrak{D}$, and $t \geqq 0$, defines a linear transformation B, of $\mathfrak{D}$ into $\mathfrak{X}$, which commutes with every shift operator and is such that $B(I - I_uA)x = x$ for every x in $\mathfrak{D}$ and $(I - I_uA)Bx = x$ if Bx is in $\mathfrak{D}$.

Proof. The series defining B certainly converges (pointwise): in fact, if $t_0 \geqq 0$ and m is a positive integer such that $mu \geqq t_0$, then, for any x in $\mathfrak{D}$,

$$Bx(t) = x(t) + \sum_{n=1}^{m} I_{nu} A^n x(t)$$

whenever $0 \leqq t \leqq t_0$. Hence if Bx is in $\mathfrak{D}$ then, by Theorem 1,

$$(I - I_u A)Bx(t) = x(t) - I_{(m+1)u}A^{m+1}x(t) = x(t)$$

whenever $0 \leqq t \leqq t_0$; so that $(I - I_u A)Bx = x$, since t_0 is arbitrary. Also, if x is in $\mathfrak{D}$ then $(I - I_u A)x$ is in $\mathfrak{D}$, so that

$$\begin{aligned} B(I - I_u A)x(t) &= (I - I_u A)x(t) + \sum_{n=1}^{m} I_{nu}A^n(I - I_u A)x(t) \\ &= x(t) - I_{(m+1)u}A^{m+1}x(t) = x(t) \end{aligned}$$

whenever $0 \leqq t \leqq t_0$. Thus $B(I - I_u A)x = x$. It can be verified in a similar way that B commutes with the shift operators and is linear.

If the transformation B of Theorem 2 maps $\mathfrak{D}$ into itself, then $I - I_u A$ has a reciprocal in $\mathfrak{A}(\mathfrak{D})$, namely B. This is certainly the case if $\mathfrak{D}$ consists of all the functions x that have some purely local property (for example, continuity, with $x(0) = 0$, or differentiability, with $x(0) = x'(0) = 0$, or local integrability).[2] It is also the case with certain other choices of $\mathfrak{D}$, provided that A is restricted to be a linear combination of shift operators; for example, if $\mathfrak{D}$ consists of the perfect functions, then an operator of the form

$$\alpha_0 I + \alpha_1 I_{u_1} + \cdots + \alpha_n I_{u_n} \tag{2.1}$$

has a reciprocal in $\mathfrak{A}(\mathfrak{D})$ if $\alpha_0 \neq 0$ (this can be seen at once on taking Laplace transforms and using Theorem 6 of [5]).

If $\mathfrak{D}$ contains more than the zero function, it is clear that (2.1) represents the zero operator on $\mathfrak{D}$ only if all the coefficients $\alpha_0, \cdots, \alpha_n$ are zero; and since the product of two operators of this form is another such operator, the reciprocal of (2.1) cannot be expressed in the same form unless it is a scalar multiple of I. Thus it is usual for $\mathfrak{A}(\mathfrak{D})$ to contain operators other than those of the form (2.1). In general it seems to be difficult to decide whether $\mathfrak{A}(\mathfrak{D})$ is commutative or not; but it is shown in § 4 that $\mathfrak{D}$ can be chosen, of moderate size, so that $\mathfrak{A}(\mathfrak{D})$ is not commutative.

The Laplace transformation is naturally associated with the idea of a V-operator, because it converts the shift operators to exponential factors. A locally integrable function x has an absolutely convergent Laplace integral if x is of exponential order at infinity, in the sense that $x(t) = O(e^{ct})$ as $t \to \infty$, for some real number c (depending on x). One can consider V-operators on spaces consisting of such functions, and for some of these spaces the following result is available.

THEOREM 3. *Let A be a V-operator on a space $\mathfrak{D}$ consisting of all*

[2] A property at infinity might be regarded as 'local', but this interpretation is to be excluded here.

the functions in $\mathfrak{X}$ which satisfy some (possibly empty) set of local conditions and are of exponential order at infinity. Then there are positive numbers b, c, and τ such that $|Ax(t)| \leqq be^{ct}$ whenever $t \geqq \tau$ and $|x(t)| \leqq 1$ for all t, with x in $\mathfrak{D}$.

Proof. Assuming the theorem to be false, we shall construct inductively a sequence $\{x_n\}$ in $\mathfrak{D}$, and a sequence $\{t_n\}$ of positive numbers, such that, for each positive integer n,

(i) $|x_n(t)| \leqq 2^{-n}$ for all values of t,

(ii) $t_n \geqq n$,

(iii) $x_n(t) = 0$ if $0 \leqq t \leqq t_{n-1}$, where $t_0 = 0$,

(iv) $|\sum_{j=1}^{n} Ax_j(t_n)| \geqq e^{nt_n}$.

In the first place, if the theorem is false, we can choose x_1 so that $|x_1(t)| \leqq \frac{1}{2}$ for all values of t and $|Ax_1(t)| \geqq e^t$ for some value of t, say t_1, greater than 1. Suppose, then, that the first $m-1$ terms of each sequence have been chosen, where $m > 1$, so that (i)-(iv) hold when $n \leqq m-1$. Let

$$y_m = \sum_{j=1}^{m-1} Ax_j .$$

Since y_m belongs to $\mathfrak{D}$, there is a real number c_m such that $|y_m(t)| \leqq e^{c_m t}$ when t is sufficiently large. We can choose x_m so that $|x_m(t)| \leqq 2^{-m}$ for all t, $x_m(t) = 0$ if $0 \leqq t \leqq t_{m-1}$, and

$$|Ax_m(t_m)| \geqq 2e^{(c_m+m)t_m} ,$$

where t_m is chosen so that $t_m \geqq m$ and $|y_m(t_m)| \leqq e^{c_m t_m}$. Then

$$\left|\sum_{j=1}^{m} Ax_j(t_m)\right| \geqq |Ax_m(t_m)| - |y_m(t_m)| \geqq e^{(c_m+m)t_m} \geqq e^{mt_m} .$$

Thus (i)–(iv) hold when $n = m$.

Now let $x_0 = \sum_{n=1}^{\infty} x_n$. Then $|x_0(t)| \leqq 1$ for all t, by virtue of (i); and x_0 belongs to $\mathfrak{D}$ since, by (iii), it has the appropriate local properties. Hence there is a real number c_0 such that $Ax(t) = O(e^{c_0 t})$ as $t \to \infty$; so that, by (ii), $Ax(t_n) = O(e^{c_0 t_n})$ as $n \to \infty$. But, by (iii) and (iv), and Theorem 1, $|Ax(t_n)| \geqq e^{nt_n}$ for each n. This contradiction proves the theorem.

3. Strong continuity. If the field of complex numbers is given either the discrete topology or the usual topology, the space $\mathfrak{X}$ can be given the corresponding topology of uniform convergence on finite closed intervals. The first of these topologies for $\mathfrak{X}$ has the property that every V-operator is continuous with respect to it, as Theorem 1 shows; but it does not make $\mathfrak{X}$ a topological vector space (it has the defect that $n^{-1}x \to 0$ as $n \to \infty$ only if x is the zero function). The second topology for $\mathfrak{X}$

is more interesting, and will be referred to as the *strong* topology. In fact we shall consider this only in relation to the closed subspace, $\mathfrak{C}_0$, consisting of all the continuous functions x for which $x(0) = 0$. For each x in $\mathfrak{C}_0$, and each non-negative number t, we define $\| x \|_t$ to be the least upper bound of $| x(u) |$ with $0 \leqq u \leqq t$. We can then give $\mathfrak{C}_0$ a metric, which determines the strong topology, by taking the distance between functions x and y to be

$$\sum_{n=1}^{\infty} 2^{-n} \| x - y \|_n / (1 + \| x - y \|_n) .$$

In this way $\mathfrak{C}_0$ becomes a Fréchet space.

In the case of $\mathfrak{C}_0$, which is an example of a space $\mathfrak{D}$ satisfying (1.1) and (1.2), a large class of V-operators, including those of the form (2.1), can be defined in terms of Riemann-Stieltjes convolution integrals. If ν is a function which belongs to $\mathfrak{X}$ and has bounded variation in every finite interval $[0, t]$, then the formula

$$Ax(t) = \int_0^t x(t - u) d\nu(u) \tag{3.1}$$

where x is any function in $\mathfrak{C}_0$, defines a V-operator A on $\mathfrak{C}_0$ (cf. [5], Theorem 3). Moreover, if $0 \leqq v \leqq t$ then

$$| Ax(v) | \leqq \int_0^v | x(v - u) | | d\nu(u) | \leqq \int_0^t \| x \|_t | d\nu(u) | , \qquad (t \geqq 0) ,$$

so that

$$\| Ax \|_t \leqq \| x \|_t \int_0^t | d\nu(u) | ;$$

whence it follows that A is strongly continuous (continuous with respect to the strong topology). The theorem we are about to prove shows that every strongly continuous V-operator on a sufficiently large space $\mathfrak{D}$ of continuous functions can be represented in this way (and can therefore be extended from $\mathfrak{D}$ to the whole of $\mathfrak{C}_0$).

If A is a linear operator on a subspace $\mathfrak{D}$ of $\mathfrak{C}_0$, and if $t \geqq 0$, we denote by '$\| A \|_t$' the least upper bound of $\| Ax \|_t$ with x in $\mathfrak{D}$ and $\| x \|_t \leqq 1$. It is clear that A is strongly continuous if and only if $\| A \|_t$ is finite for all values of t (or, equivalently, for all sufficiently large values of t).

THEOREM 4. *Let A be a strongly continuous V-operator on a strongly dense subspace $\mathfrak{D}$ of $\mathfrak{C}_0$, and let t be any positive number. Then there is a function ν in $\mathfrak{X}$, with $\nu(0) = 0$ and $\nu(u-) = \nu(u)$ whenever $0 < u \leqq t$, such that $Ax(t)$ is given by* (3.1) *for every x in $\mathfrak{D}$. This function ν is uniquely determined by A, and is independent of t; its total variation* *in the interval* $[0, t]$ *is* $\| A \|_t$.

Proof. For each function x in $\mathfrak{D}$, and for each positive number t, let x_t be the restriction of x to the closed interval $[0, t]$. Then, for a fixed value of t, the mapping $x \to x_t$ is a linear transformation of $\mathfrak{D}$ on to a subspace $\mathfrak{D}_t$ of the complex Banach space $C[0, t]$, consisting of all continuous functions on the interval $[0, t]$; moreover, $\| x_t \| = \| x \|_t$. If $x_t = 0$ then $Ax(t) = 0$, by Theorem 1; we can therefore define a linear functional φ on $\mathfrak{D}_t$ by the formula

$$\varphi(x_t) = Ax(t) .$$

This functional is continuous, with $\| \varphi \| = \| A \|_t$.

An integral representation of φ can be found by adapting a construction used by Banach ([1], 59-60). By a well-known theorem[3], φ can be extended without change of norm to the complex Banach space $M[0, t]$, which contains the characteristic functions of all the subintervals of $[0, t]$. A function ν_t can then be defined on $[0, t]$ so that $\nu_t(0) = 0$ and

$$\text{(i)} \qquad \int_0^t | d\nu_t(u) | \leqq \| \varphi \| ,$$

$$\text{(ii)} \qquad \varphi(f) = \int_0^t f(t-u) d\nu_t(u)$$

for every function f in $C[0, t]$.

Without affecting the validity of (i) or (ii), we can adjust ν_t so that it is continuous on the left at each interior point of the interval $[0, t]$. Moreover, if f is a continuous function such that $f(0) = 0$, then the jump of ν_t at the point t makes no contribution to the integral in (ii); therefore, as far as such functions f are concerned, we may suppose ν_t chosen so that $\nu_t(t-) = \nu_t(t)$, giving left-hand continuity throughout the interval $(0, t]$, and retaining (i). Under these conditions, ν_t is uniquely determined by A. For, if $0 < v \leqq t$ and $0 < \delta < v$, there is a function f_δ in $C[0, t]$ such that $\| f_\delta \| = 1$ and

$$f_\delta(u) = \begin{cases} 0 & (0 \leqq u \leqq t - v) \\ 1 & (t - v + \delta \leqq u \leqq t) . \end{cases}$$

Thus

$$\varphi(f_\delta) = \int_0^{v-\delta} d\nu_t(u) + \int_{v-\delta}^{v} f_\delta(t-u) d\nu_t(u) ,$$

and therefore

$$| \varphi(f_\delta) - \nu_t(v - \delta) | \leqq \int_{v-\delta}^{v} | d\nu_t(u) | ,$$

[3] The Hahn-Banach-Bohnenblust-Sobczyk extension theorem: see, for example, [8], 113.

so that $\varphi(f_\delta) \to \nu_t(v)$ as $\delta \to 0$.[4] But since $\mathfrak{D}$ is strongly dense in $\mathfrak{C}_0$, f_δ belongs to the closure of $\mathfrak{D}_t$, in $C[0, t]$; so that, φ being continuous, $\varphi(f_\delta)$ is uniquely determined by A, for each value of δ. This establishes the uniqueness of ν_t.

Now suppose that $t' > t$. By what has been proved, we have, for any x in $\mathfrak{D}$,

$$Ax(t) = \int_0^t x(t-u)d\nu_t(u) .$$

But $Ax(t) = I_{t'-t}Ax(t')$, and $I_{t'-t}A = AI_{t'-t}$; hence

$$Ax(t) = \int_0^{t'} I_{t'-t}x(t'-u)d\nu_{t'}(u) = \int_0^t x(t-u)d\nu_{t'}(u) .$$

It follows that $\nu_t(u) = \nu_{t'}(u)$ whenever $0 \leqq u \leqq t$; in particular, $\nu_t(t) = \nu_{t'}(t)$. Hence if we define the function ν by

$$\nu(t) = \nu_t(t) \qquad (t \geqq 0) ,$$

we obtain the required representation of A.

Finally, (i) shows that

$$\int_0^t | d\nu(u) | \leqq \| A \|_t ,$$

and we have previously noted that, for any x in $\mathfrak{D}$,

$$\| Ax \|_t \leqq \| x \|_t \int_0^t | d\nu(u) | .$$

Thus $\int_0^t | d\nu(u) | = \| A \|_t$, and the proof is complete.[5]

As a corollary, we have

THEOREM 5. *Suppose that the formula*

$$Ax(t) = \int_0^t K(t, u)x(u)du \qquad (t \geqq 0)$$

defines a V-operator A on $\mathfrak{C}_0$, the kernel K being such that $\int_0^t | K(t,u) | du$ exists as a Lebesgue integral which is locally bounded with respect to t. Then there is a function k in $\mathfrak{X}$ such that, for each t, $K(t, u) = k(t-u)$ for almost all values of u.

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[4] Here we use the fact that if a function of bounded variation is continuous on the left, then so is its total variation.

[5] In this proof we have not fully used the fact that A maps $\mathfrak{D}$ into itself: it is enough that A maps $\mathfrak{D}$ into C_0.

Proof. For each t, let $\| K \|_t$ be the least upper bound of $\int_0^v |K(v, u)| du$ with $0 \leqq v \leqq t$; this is finite, by hypothesis. Then, for each x in $\mathfrak{C}_0$,

$$\| Ax \|_t \leqq \| K \|_t \| x \|_t ,$$

so that A is strongly continuous. But

$$Ax(t) = \int_0^t K(t, t - u) x(t - u) du ,$$

so that if

$$L_t(u) = \int_0^u K(t, t - v)\, dv$$

then

$$Ax(t) = \int_0^t x(t - u) dL_t(u) .$$

Hence, by Theorem 4, $L_t = \nu$, a function which is independent of t. Since ν has bounded variation, there is a function k such that

$$k(u) = \frac{d}{du} \nu(u)$$

except when u is in a set E whose Lebesgue measure is 0. However, for each value of t,

$$\frac{d}{du} \nu(u) = \frac{d}{du} L_t(u) = K(t, t - u)$$

except when u is in a set E_t of measure 0. Thus

$$K(t, u) = k(t - u)$$

except when u is in the set $t - (E_t \cup E)$, which has measure 0.

The functions in $\mathfrak{C}_0$ which are of exponential order at infinity form a subspace $\mathfrak{E}_0$. The perfect functions form a smaller subspace, $\mathfrak{D}_0$ (in fact $\mathfrak{D}_0$ is the largest subspace of $\mathfrak{E}_0$ which is invariant under the differential operator, D).

THEOREM 6. *$\mathfrak{D}_0$ is strongly dense in $\mathfrak{C}_0$.*

Proof. It is easily seen that $\mathfrak{E}_0$ is strongly dense in $\mathfrak{C}_0$: in fact, if x is in $\mathfrak{C}_0$ and x_n is defined by

$$x_n(t) = \begin{cases} x(t) & (0 \leqq t \leqq n) \\ x(n) & (t \geqq n) , \end{cases}$$

then x_n belongs to $\mathfrak{C}_0$, for each n, and $x_n \to x$ strongly as $n \to \infty$. To show that $\mathfrak{D}_0$ is dense in $\mathfrak{C}_0$, let x be any function in $\mathfrak{C}_0$ and, for each positive number δ, let $g_{(\delta)}$ be a positive perfect function such that if $t \geqq \delta$ then $g_{(\delta)}(t) = 0$ and $\int_0^t g_{(\delta)}(u)du = 1$ (for example, we could take $g_{(\delta)}$ to be $Dh_{(\delta)}$, where $h_{(\delta)}$ is given by Lemma 1 of [5]). Let $x_{(\delta)} = x*g_{(\delta)}$. Then $x_{(\delta)}$ belongs to $\mathfrak{D}_0$ ('$x*$' is a perfect operator), and, if $v \geqq \delta$,

$$x_{(\delta)}(v) - x(v) = \int_0^v x(v-u)g_{(\delta)}(u)du - x(v)$$

$$= \int_0^\delta \{x(v-u) - x(v)\}g_{(\delta)}(u)du \ .$$

Now let t and ε be any positive numbers. Since x is uniformly continuous in the interval $[0, t]$, with $x(0) = 0$, we can choose δ so that

$$| x(v-u) - x(v) | < \varepsilon$$

whenever $\delta \leqq v \leqq t$, and $| x(v) | < \frac{1}{2}\varepsilon$ whenever $0 \leqq v \leqq \delta$; then

$$| x_{(\delta)}(v) - x(v) | < \varepsilon \int_0^\delta g_{(\delta)}(u)du = \varepsilon$$

if $\delta \leqq v \leqq t$, and if $0 \leqq v \leqq \delta$,

$$| x_{(\delta)}(v) - x(v) | \leqq \int_0^\delta | x(v-u) | g_{(\delta)}(u)du + | x(v) |$$

$$\leqq \tfrac{1}{2}\varepsilon \int_0^\delta g_{(\delta)}(u)du + \tfrac{1}{2}\varepsilon = \varepsilon \ .$$

Thus $\| x_{(\delta)} - x \|_t < \varepsilon$. It follows that $\mathfrak{D}_0$ is strongly dense in $\mathfrak{C}_0$.

In [5] it is shown that any positive perfect operator has the representation (3.1), with ν a non-decreasing function (in fact this holds for any positive V-operator on a space $\mathfrak{D}$ such that $\mathfrak{D}_0 \subseteqq \mathfrak{D} \subseteqq \mathfrak{C}_0$). It follows that the linear combinations of positive perfect operators, which form a linear algebra $\mathfrak{M}(\mathfrak{D}_0)$[6], are strongly continuous. On the other hand, there are strongly continuous perfect operators which do not belong to $\mathfrak{M}(\mathfrak{D}_0)$: for example, if $\nu(t) = \sin(e^{t^2} - 1)$, and A is defined on $\mathfrak{D}_0$ according to (3.1), then, as is shown in [5], A is a perfect operator which is not in $\mathfrak{M}(\mathfrak{D}_0)$; but of course A is strongly continuous. However, it is possible to characterize $\mathfrak{M}(\mathfrak{D}_0)$ in terms of seminorms, as follows.

THEOREM 7. *A V-operator A on $\mathfrak{D}_0$ is an element of $\mathfrak{M}(\mathfrak{D}_0)$ if and only if there is a real number c such that $\| A \|_t = O(e^{ct})$ as $t \to \infty$.*

Proof. By Theorem 1 of [5], an operator A on $\mathfrak{D}_0$ is in $\mathfrak{M}(\mathfrak{D}_0)$ if

[6] $\mathfrak{M}(\mathfrak{D}_0)$ is denoted in [5] by '$\mathfrak{M}$'.

and only if it admits the representation (3.1) with ν a linear combination of positive non-decreasing functions which are of exponential order at infinity. This condition on ν is equivalent to the existence of a real number c such that $\int_0^t |d\nu(u)| = O(e^{ct})$ as $t \to \infty$. Therefore, by Theorems 4 and 6 above, A is in $\mathfrak{M}(\mathfrak{D}_0)$ if and only if $\| A \|_t = O(e^{ct})$ as $t \to \infty$.

Each function y in $\mathfrak{C}_0$ determines a strongly continuous V-operator A on $\mathfrak{C}_0$ according to the formula $Ax = x*y$; for, integration by parts shows that this formula is equivalent to (3.1), with

$$\nu(t) = D^{-1}y(t) = \int_0^t y(u)du \qquad (t \geqq 0) .$$

An important property of convolution in $\mathfrak{C}_0$ is the fact that it obeys the associative law (as well as the commutative law); more generally, we have

THEOREM 8. *Let A and B be strongly continuous V-operators, on $\mathfrak{C}_0$ and on a subspace $\mathfrak{D}$ of $\mathfrak{C}_0$ respectively. If x is any function in $\mathfrak{D}$ then Ax belongs to the strong closure of $\mathfrak{D}$; if Ax is in $\mathfrak{D}$ itself, then $ABx = BAx$. In particular, if y is a function in $\mathfrak{C}_0$ such that $x*y$ is in $\mathfrak{D}$, then $B(x*y) = (Bx)*y$.*

Proof. Let A be represented by a function ν in accordance with Theorem 4. Then for any x in $\mathfrak{D}$, each value $Ax(t)$ can be arbitrarily approximated by sums of the form

$$\sum_{j=1}^{n}\{\nu(u_j) - \nu(u_{j-1})\}x(t - u_j) ,$$

where $0 \leqq u_1 \leqq \cdots \leqq u_n \leqq t$; and this approximation is locally uniform with respect to t. Now the above sum is the value at t of the function

$$\text{(i)} \qquad \sum_{j=1}^{n}\alpha_j I_{u_j} x ,$$

where $\alpha_j = \nu(u_j) - \nu(u_{j-1})$. This function belongs to $\mathfrak{D}$, since $\mathfrak{D}$ satifies (1.2). Thus Ax belongs to the strong closure of $\mathfrak{D}$. Further, the points u_j can be chosen in such a way that, while Ax is strongly approximated by (i), ABx is simultaneously approximated, in the same sense, by

$$\text{(ii)} \qquad \sum_{j=1}^{n}\alpha_j I_{u_j} Bx .$$

But, since B is a V-operator, (ii) is the same as

$$B\sum_{j=1}^{n}\alpha_j I_{u_j} x .$$

Since B is strongly continuous, it follows that $ABx = BAx$ if Ax is an operand of B.

We can now prove a partial converse of Theorem 1, namely.

THEOREM 9. *Let A be a non-zero strongly continuous V-operator on $\mathfrak{C}_0$. Then there is a non-negative number τ such that* (i) *for any function x in C_0, $Ax(t) = 0$ whenever $0 \leqq t \leqq \tau$, and* (ii) *if $Ax(t) = 0$ whenever $0 \leqq t \leqq t_0$, where x belongs to $\mathfrak{C}_0$ and $t_0 \geqq \tau$, then $x(t) = 0$ whenever $0 \leqq t \leqq t_0 - \tau$. In particular, $x = 0$ if $Ax = 0$.*

Proof. Let ν be the function representing A according to Theorem 4, and let τ be the greatest lower bound of the numbers t for which $\nu(t) \neq 0$. Obviously, τ has the property (i) required by the theorem. Suppose that x is a function in $\mathfrak{C}_0$ such that $Ax(t) = 0$ whenever $0 \leqq t \leqq t_0$, where $t_0 \geqq \tau$. Let $g_{(\delta)}$ be defined as in the proof of Theorem 6, and let $x_{(\delta)} = x * g_{(\delta)}$. Then, for each value of δ, $x_{(\delta)}$ has a derivative $x'_{(\delta)}$ in $\mathfrak{C}_0$; in fact $x'_{(\delta)} = x * g'_{(\delta)}$. Also, if $0 \leqq t \leqq t_0$,

$$\int_0^t x'_{(\delta)}(t-u)\nu(u)du = Ax_{(\delta)}(t) = (Ax) * g_{(\delta)}(t)$$

$$= \int_0^t Ax(t-u)g_{(\delta)}(u)du = 0 .$$

Therefore, by a theorem of Titchmarsh [4, 327], $x'_{(\delta)}(t) = 0$ whenever $0 \leqq t \leqq t_0 - \tau$ (we cannot have $\nu(t) = 0$ for almost all t in a neighbourhood of τ, since ν is continuous on the left). Hence $x_{(\delta)}(t) = 0$ whenever $0 \leqq t \leqq t_0 - \tau$. Since $x_{(\delta)}(t) \to x(t)$ as $\delta \to 0$, the theorem follows.

It is a consequence of Theorem 8 that every strongly continuous V-operator on $\mathfrak{D}_0$ is a perfect operator (the converse is false; in fact it is easy to see that the differential operator D is not strongly continuous). Thus an operator A represented by (3.1) is a perfect operator if and only if it maps $\mathfrak{D}_0$ into itself. An equivalent condition is given by

THEOREM 10. *The formula* (3.1), *with x in $\mathfrak{D}_0$, represents a perfect operator A if and only if there is a positive integer n such that $D^{-n}\nu$ belongs to $\mathfrak{C}_0$, where*

$$D^{-n}\nu(t) = \int_0^t \cdots \int_0^{u_2} \nu(u_1)du_1 \cdots du_n \qquad (t = u_{n+1} \geqq 0) .$$

Proof. For any perfect function x and any positive integer n, we have from (3.1), after integration by parts,

$$Ax(t) = \int_0^t x^{(n+1)}(t-u)D^{-n}\nu(u)du \qquad (t \geqq 0) .$$

Thus if $D^{-n}\nu$ belongs to $\mathfrak{C}_0$ for some value of n, then A is a perfect operator. On the other hand, suppose that A, given by (3.1), is a perfect operator (when restricted to $\mathfrak{D}_0$). By a general representation theorem for perfect operators [6], there is a function y in $\mathfrak{C}_0$ such that, for some positive integer n, and every perfect function x,

$$Ax(t) = \int_0^t x^{(n+1)}(t-u)y(u)du \qquad (t \geqq 0) .$$

Hence $x^{(n+1)}*(y - D^{-n}\nu) = 0$, so that, by Theorem 9, $y = D^{-n}\nu$.

If $\nu(t) = e^{e^t}$, the V-operator A given by (3.1) does not map $\mathfrak{D}_0$ into itself, since ν does not satisfy the condition of Theorem 10.

Every perfect operator A has a Laplace transform, $\bar{A}$: if A is given by (3.1), $\bar{A}$ may or may not be given by

$$\bar{A}(z) = \int_0^\infty e^{-zt}d\nu(t) , \tag{3.2}$$

the integral being convergent when $\Re z$ is sufficiently large. This representation of $\bar{A}$ is certainly valid if A belongs to $\mathfrak{M}(\mathfrak{D}_0)$ (cf. [5], Theorem 4); and also if $\nu(t) = \sin(e^{t^2}-1)$, for example. But if $D^{-1}\nu(t) = \sin(e^{t^2}-1)$ the integral in (3.2) does not converge for any value of z (as can be seen on integrating twice by parts). However, (3.2) holds whenever the integral is convergent, as the following result shows.

THEOREM 11. *Let A be any strongly continuous perfect operator, and let ν be a function such that A is represented by* (3.1). *Then the Laplace transform $\bar{A}$ is represented by* (3.2), *with $\Re z$ sufficiently large, if the infinite integral is interpreted in the sense of summability* (C, n), *where n is any non-negative integer such that $D^{-n}\nu$ belongs to $\mathfrak{C}_0$.*

Proof. Let B be the perfect operator obtained on replacing ν by $D^{-1}\nu$ in (3.1). Then, if x is any perfect function, and $t \geqq 0$,

$$DBx(t) = Bx'(t) = \int_0^t x'(t-u)\nu(u)du = \nu(0)x(t) + \int_0^t x(t-u)d\nu(u) .$$

Thus $DB = \nu(0)I + A$. If ν belongs to $\mathfrak{C}_0$ then, since B is determined by the function ν in the sense that $Bx = x*\nu$, B has the same Laplace transform as ν; that is to say, when $\Re z$ is sufficiently large,

$$\bar{B}(z) = \int_0^\infty e^{-zt}\nu(t)dt .$$

Therefore, in this case,

$$\bar{A}(z) = z\bar{B}(z) - \nu(0) = \int_0^\infty ze^{-zt}\{\nu(t) - \nu(0)\}dt = \int_0^\infty e^{-zt}d\nu(t) ,$$

so that (3.2) holds, the integral being convergent.

We now proceed by induction. Suppose that, for some non-negative integer n, (3.2) holds in the sense of summability (C, n) provided that $D^{-n}\nu$ belongs to $\mathfrak{C}_0$ and $\Re z$ is sufficiently large. If $D^{-n-1}\nu$ belongs to $\mathfrak{C}_0$, and $t > 0$, then

$$\int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu}d\nu(u) = -\nu(0) + z\int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu}dD^{-1}\nu(u)$$
$$+ \frac{n+1}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{n} e^{-zu}dD^{-1}\nu(u) .$$

But, by the induction hypothesis (with $D^{-1}\nu$ in place of ν),

$$\bar{B}(z) = \lim_{t\to\infty}\int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu}dD^{-1}\nu(u) = \lim_{t\to\infty}\int_0^t \left(1 - \frac{u}{t}\right)^{n} e^{-zu}dD^{-1}\nu(u)$$

when $\Re z$ is sufficiently large; so that

$$\lim_{t\to\infty}\int_0^t \left(1 - \frac{u}{t}\right)^{n+1} e^{-zu}d\nu(u) = -\nu(0) + z\bar{B}(z) = \bar{A}(z) .$$

Thus

$$\bar{A}(z) = \int_0^\infty e^{-zt}d\nu(t) \quad (C, n+1) ,$$

and the theorem follows.

If $\mathfrak{D}$ is any subspace of $\mathfrak{C}_0$ satisfying (1.1) and (1.2), the strongly continuous V-operators on $\mathfrak{D}$ form a subalgebra of $\mathfrak{A}(\mathfrak{D})$, say $\mathfrak{N}(\mathfrak{D})$. If $\mathfrak{D}$ is strongly dense in $\mathfrak{C}_0$, it follows from Theorem 4 that $\mathfrak{N}(\mathfrak{D})$ effectively consists of those operators in $\mathfrak{N}(\mathfrak{C}_0)$ which leave $\mathfrak{D}$ invariant. In this case, Theorems 8 and 9 show that $\mathfrak{N}(\mathfrak{D})$ is an integral domain (it is commutative, and has no divisors of zero). The full algebra $\mathfrak{N}(\mathfrak{C}_0)$[7] has the further property that any operator which is inverse to an operator in $\mathfrak{N}(\mathfrak{C}_0)$ is itself in $\mathfrak{N}(\mathfrak{C}_0)$: this is special case of

THEOREM 12. *Let A and B be strongly continuous V-operators on a strongly closed subspace $\mathfrak{D}$ of $\mathfrak{C}_0$, and suppose that there is an operator C on $\mathfrak{D}$ such that $A = BC$. Suppose also that $Bx = 0$ only if $x = 0$. Then C is a strongly continuous V-operator.*

[7] $\mathfrak{N}(\mathfrak{C}_0) = \mathfrak{M}(\mathfrak{C}_0)$, consisting of the linear combinations of positive V-operators on $\mathfrak{C}_0$.

Proof. If $u > 0$ and x is any function in $\mathfrak{D}$ then, since A and B are V-operators,

$$B(I_u Cx - CI_u x) = I_u Ax - AI_u x = 0;$$

so that, by the hypothesis concerning B, $I_u Cx = CI_u x$. In a similar way it can be verified that C is linear, and is therefore a V-operator. To show that C is strongly continuous, let $\{x_n\}$ be a strongly convergent sequence in $\mathfrak{D}$ such that the sequence $\{Cx_n\}$ is also strongly convergent. Since A and B are strongly continuous,

$$B(\lim_{n\to\infty} Cx_n - C\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Ax_n - A\lim_{n\to\infty} x_n = 0 ,$$

so that $\lim_{n\to\infty} Cx_n = C\lim_{n\to\infty} x_n$; thus the graph of C is closed. Now $\mathfrak{D}$, being strongly closed, is a Fréchet space relative to the strong topology; hence, by Banach's closed-graph theorem [**1**, 41], C is strongly continuous.

4. Operators that commute with convolution. It is a consequence of Theorem 8 that a subspace $\mathfrak{D}$ of $\mathfrak{C}_0$, satisfying (1.1) and (1.2), is closed under convolution if it is strongly closed. On the other hand, $\mathfrak{D}_0$ is closed under convolution though it is not strongly closed. If $\mathfrak{D}$ is any subspace of $\mathfrak{C}_0$ which is closed under convolution (so forming an integral domain with no unit element), an operator A on $\mathfrak{D}$ will be said to *commute with convolution* if

$$A(x*y) = (Ax)*y$$

for all x and y in $\mathfrak{D}$. Such operators are necessarily linear (cf. [**5**], § 4), and, for a given choice of $\mathfrak{D}$, they form an integral domain $\mathfrak{D}^\#$ in which $\mathfrak{D}$ is isomorphically embedded (by the correspondence $x \to x*$).

A shift operator belongs to $\mathfrak{D}^\#$ if it maps $\mathfrak{D}$ into itself. Hence if $\mathfrak{D}$ satisfies (1.1) and (1.2), in addition to being closed under convolution, then all the operators in $\mathfrak{D}^\#$ are V-operators; in fact $\mathfrak{D}^\#$ is then a maximal commutative subalgebra of $\mathfrak{A}(\mathfrak{D})$. In this case, Theorem 8 shows that every strongly continuous V-operator commutes with convolution; so that

$$\mathfrak{N}(\mathfrak{D}) \subseteqq \mathfrak{D}^\# \subseteqq \mathfrak{A}(\mathfrak{D}) .$$

If, further, $\mathfrak{D}$ is strongly closed, then $\mathfrak{N}(\mathfrak{D}) = \mathfrak{D}^\#$: for, if B is defined by $Bx = x*y$, with y in $\mathfrak{D}$, and $A = BC$, where C is any operator in $\mathfrak{D}^\#$, then, for any x in $\mathfrak{D}$,

$$Ax = (Cx)*y = C(x*y) = C(y*x) = (Cy)*x;$$

thus the conditions of Theorem 12 are satisfied, so that C belongs to $\mathfrak{N}(\mathfrak{D})$. In particular, the operators on $\mathfrak{C}_0$ that commute with convolution are precisely the strongly continuous V-operators on $\mathfrak{C}_0$ (and can therefore be represented according to Theorem 4).

An operator A on $\mathfrak{F}_0$ which commutes with convolution can be extended to the whole of $\mathfrak{C}_0$ so as to preserve this property. For, if x is any function in $\mathfrak{C}_0$, let x_n be defined, for each positive integer n, as in the proof of Theorem 6: then x_n belongs to $\mathfrak{F}_0$, and Theorem 1 shows that $Ax_n(t)$ is independent of n provided that $n \geqq t$; therefore, if $t \geqq 0$, we can define $Ax(t)$ to be $Ax_n(t)$, where $n \geqq t$, without ambiguity. Since convolution is defined locally this extension of A is an operator on $\mathfrak{C}_0$ which commutes with convolution. It follows that A is strongly continuous, and that its extension to $\mathfrak{C}_0$ is unique (since $\mathfrak{F}_0$ is strongly dense in $\mathfrak{C}_0$).

The integration operator, D^{-1}, is an example of an operator on $\mathfrak{C}_0$ which commutes with convolution. Since $\mathfrak{D}_0$ can be expressed as $\bigcap_{n=1}^{\infty} D^{-n}\mathfrak{F}_0$, any operator on $\mathfrak{C}_0$ which commutes with convolution and leaves $\mathfrak{F}_0$ invariant must leave $\mathfrak{D}_0$ invariant. The converse of this is false: for, if A is defined by (3.1), ν being such that $D^{-2}\nu$ belongs to $\mathfrak{F}_0$ but $D^{-1}\nu$ does not, and $\nu(0) = 0$, then A maps $\mathfrak{D}_0$ into itself, by Theorem 10; however, if $x(t) = t$ then

$$Ax(t) = \int_0^t (t-u)d\nu(u) = D^{-1}\nu(t) ,$$

so that x is in $\mathfrak{F}_0$ but Ax is not.

The operators on $\mathfrak{D}_0$ that commute with convolution are the perfect operators. These can be characterized as those V-operators on $\mathfrak{D}_0$ which are continuous in a sense defined in terms of Laplace transforms [7][8]. The strongly continuous perfect operators are the strongly continuous V-operators on $\mathfrak{D}_0$, constituting the algebra $\mathfrak{N}(\mathfrak{D}_0)$; this algebra, and also its subalgebra $\mathfrak{M}(\mathfrak{D}_0)$, can be characterized in terms of convolution, as follows.

THEOREM 13. *A perfect operator belongs to $\mathfrak{N}(\mathfrak{D}_0)$ if and only if it can be extended to the whole of $\mathfrak{C}_0$ so as to commute with convolution; it belongs to $\mathfrak{M}(\mathfrak{D}_0)$ if and only if this extension (necessary unique) leaves $\mathfrak{F}_0$ invariant.*

Proof. If an operator A on $\mathfrak{D}_0$ can be extended to $\mathfrak{C}_0$ so as to commute with convolution, then its extension belongs to $\mathfrak{N}(\mathfrak{C}_0)$, so that A itself belongs to $\mathfrak{N}(\mathfrak{D}_0)$. On the other hand, any operator A in $\mathfrak{N}(\mathfrak{D}_0)$ admits the representation (3.1), which provides an extension of A to $\mathfrak{C}_0$: this extension, being strongly continuous, commutes with convolution;

[8] It is not at present known whether there are any V-operators on $\mathfrak{D}_0$ which are not perfect; that is to say, it is not known whether $\mathfrak{A}(\mathfrak{D}_0)$ is commutative or not (but there are linear operators on $\mathfrak{D}_0$ which commute with D and are not perfect [**6**]).

it is also unique, since $\mathfrak{D}_0$ is strongly dense in $\mathfrak{C}_0$.

If a perfect operator A has a strongly continuous extension to $\mathfrak{C}_0$ which leaves $\mathfrak{C}_0$ invariant, we can regard A as a V-operator on $\mathfrak{C}_0$; then, by Theorem 3, there is a real number c such that $\| A \|_t = O(e^{ct})$ as $t \to \infty$, and this implies, by Theorem 7, that A belongs to $\mathfrak{M}(\mathfrak{D}_0)$. On the other hand, if A belongs to $\mathfrak{M}(\mathfrak{D}_0)$ then the extension of A to $\mathfrak{C}_0$ given by (3.1) leaves $\mathfrak{C}_0$ invariant, by Theorem 3 of [5].

Finally, we give an example of a V-operator, on a strongly dense subspace of $\mathfrak{C}_0$, which does not commute with convolution. Let h be the Heaviside unit function ($h(t) = 1$ if $t \geqq 0$), and let $\mathfrak{D}_1$ be the class of all functions x given by

$$x = D^{-1}(y + Bh)\ , \tag{4.1}$$

where y belongs to $\mathfrak{C}_0$ and B is an operator of the type (2.1). Then $\mathfrak{D}_0 \subseteqq \mathfrak{D}_1 \subseteqq \mathfrak{C}_0$, and $\mathfrak{D}_1$ satiyfies (1.1) and (1.2); moreover, $\mathfrak{D}_1$ is closed under convolution. It is clear that y and B in (4.1) are uniquely determined by x, and that the mapping $x \to y$ is a V-operator, say A, on $\mathfrak{D}_1$. The operator D^{-1} maps $\mathfrak{D}_1$ into itself and commutes with convolution. However, $AD^{-1}x = x$ and $D^{-1}Ax = y$, so that $AD^{-1} \neq D^{-1}A$. Hence A does not commute with convolution. It follows that the algebra $\mathfrak{A}(\mathfrak{D}_1)$, of all V-operators on $\mathfrak{D}_1$, is not commutative.

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