# THE STRUCTURE OF THREADS

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A *thread*, as defined by A. H. Clifford, is a connected topological semigroup in which the topology is the interval topology induced by a total order. A résumé of papers on the subject can be found in the introduction of [1] or in section three of [3].

Briefly, the main classes of threads which have been described are: that of compact threads with an identity and a zero for which the underlying space is a real interval [4]; that of threads defined on the real interval $[0, \infty)$ in which "zero" and "one" play their usual roles [6]; and the class of compact threads with idempotent endpoints, [1] and [2]. Since the separability of the real numbers is not needed for the proofs involved, we will interpret the results of [4] and [6] as applying also to threads in which the underlying space is not real.

The object of this paper is to investigate the structure of more general threads. In the second, third and forth sections we study maximal subgroups, subthreads and the minimal ideal respectively of an arbitrary thread. Theorem 5.5 generalizes the result in [6] by describing all threads $S$ with a zero as an endpoint for which $S^2=S$. In the final section, we are able to describe at least half of any thread satisfying $S^2=S$. More explicitly, if such a thread has no minimal ideal, or if it is itself the minimal ideal, then the entire structure of the thread is determined; while, if there is a proper minimal ideal, then the set of elements larger or the set of elements smaller than the minimal ideal forms a subthread which, satisfying the hypotheses of Theorem 5.5, can be completely described.

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**1. Preliminaries.** As defined in [1], a *standard thread* is a compact thread in which the minimal element is a zero and the maximal element an identity. The primary examples are the real interval $[0, 1]$ under the natural order and multiplication and the Rees quotient of $[0, 1]$ by the ideal $[0, \frac{1}{2}]$. The structure of any standard thread can be given as follows [7, Theorem B]: The set of idempotents is closed and thus its complement is a union of disjoint open intervals. If $(e, f)$ is one of these intervals, then $[e, f]$ is a subthread isomorphic with one of the two examples just given. Finally, if $e$ is an idempotent and if $x \leqq e \leqq y$,

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then $xy = yx = x$.

We say that a thread with a zero and an identity is a *positive thread* if the zero is a least element and if there is no greatest element. The result in [6] is that, in a positive thread, there exists a largest idempotent $e$ less than the identity, $[0, e]$ is a standard thread, $\{t \mid e < t\}$ is isomorphic with the group of positive real numbers, and $xy = yx = x$ whenever $x \leqq e \leqq y$.

Given a thread $S$ which has a zero as a least element, we construct a new thread which we denote by $\mathscr{R}(S)$. Let $S'$ be a copy of $S\backslash\{0\}$, and let $x'$ be the element of $S'$ corresponding to the element $x$ of $S\backslash\{0\}$; put $0' = 0$. Let $\mathscr{R}(S) = S' \cup S$, and extend the order on $S$ to $\mathscr{R}(S)$ by reversing the order in $S'$ and declaring each element of $S'$ to be less than every element of $S$. Now extend the multiplication in $S$ to $\mathscr{R}(S)$ by defining $x'y = yx' = (xy)'$ and $x'y' = xy$. It is easy to verify that $\mathscr{R}(S)$ is a thread.

We state Lemma 1 of [1] which will be repeatedly used without reference. *If $a$, $b$ and $c$ are elements in a thread, then* $[ac, bc] \subset [a, b]c$ *and* $[ca, cb] \subset c[a, b]$. *The same holds for open and for half-open intervals.* The proof is a simple application of the fact that a continuous image of a connected set is connected.

If there is a homeomorphism between threads $S$ and $T$ which is also an algebraic homomorphism, $S$ and $T$ are *iseomorphic* and we write $S \approx T$. If the iseomorphism is also order preserving (it must either preserve or reverse the order), then $S$ and $T$ are *isomorphic* and we write $S \cong T$. A *subthread* is, of course, a connected subsemigroup. The *order dual* of a thread is the thread obtained by reversing the order while leaving the multiplication unchanged. As in [8], $H(e)$ is the maximal subgroup containing the idempotent $e$, $\Gamma(x)$ is the topological closure of the set of powers of $x$, and $J(x)$ is the ideal generated by $x$.

The groups of positive and non-zero real numbers will be denoted by $\mathscr{P}$ and $\mathscr{X}$ respectively. Throughout the paper, $S$ will always be a thread.

**2. Maximal subgroups.** Let $e$ be an idempotent in an arbitrary thread $S$. We wish to investigate the maximal subgroup $H(e)$ of $S$ having $e$ as its identity. We recall that

$$H(e) = eSe \cap \{x \mid e \in xS \cap Sx\} .$$

Since $H(e)$ is an algebraic group and a topological semigroup, it is homogeneous. Thus, if $H(e)$ contains any open interval of $S$, it contains an open interval about $e$. Denoting the component of $H(e)$ containing $e$ by $G$, either $G = e$ or $e$ is a cut point of $G$. But $G$ is clearly a cancellative thread, and by a theorem of Acél and Tamari (as stated on page

81 of [1]), every such thread is isomorphic with a subthread of $\mathscr{P}$. Since the only subthread of $\mathscr{P}$ of which the identity is a cut point is $\mathscr{P}$ itself, we see that $G = e$ or $G \cong \mathscr{P}$.

Again, observe that translations of $eSe$, the set on which $e$ acts as an identity, by elements of $H(e)$ are homeomorphisms. Thus, if any element of $H(e)$ is a cut point of $eSe$, then $e$ is a cut point. Consequently, if $H(e)$ contains more than two elements, then $e$ is a cut point of $eSe$.

2.1 LEMMA. *If $e$ is an idempotent in $S$, then either $e = eSe$, or $e$ is an endpoint of $eS \cup Se$, or $e$ cuts $eSe$. In the first two cases, $H(e)$ contains at most two elements; while in the last, the identity component of $H(e)$ is isomorphic with $\mathscr{P}$.*

*Proof.* It will suffice for the proof to show that the following are equivalent: the identity component of $H(e)$ is isomorphic with $\mathscr{P}$; $H(e)$ contains more than two element; $e$ cuts $eSe$; $e \neq eSe$ and $e$ cuts $eS \cup Se$. Moreover, the first of these obviously implies the second; we have already seen that the second implies the third; and the third clearly implies the forth.

Suppose then that $e \neq eSe$ and that $e$ is a cut point of $eS \cup Se$. Since $eS \cap Se = eSe$, this means that $e$ cuts one of $eS$ and $Se$, and that $Se \neq e \neq eS$. The two cases being similar, assume that $e$ cuts $eS$, and choose $a$ and $b$ in $eS$ such that $a < e < b$. Using the continuity of multiplication, there exists an open interval $W$ about $e$ such that $W \subset (a, b)$ and $Wa < e < Wb$. Thus, if $x$ is in $W$, $e \in (xa, xb) \subset x(a, b)$. Repeating the argument, using $W$ in place of $(a, b)$, we obtain an open interval $V$ about $e$ such that $e \in zW$ for each $z$ in $V$. Now if $z \in V \cap eSe$, then there exist $x$ in $W$ and $s$ in $(a, b)$ such that $e = zx = xs$. Since $z \in Se$ while $s \in eS$,

$$z = ze = z(xs) = (zx)s = es = s .$$

Hence $V \cap eSe \subset H(e)$. Observing that $V \cap eSe$ is a non-degenerate interval containing $e$, it follows from the argument of the first paragraph in this section that the identity component of $H(e)$ is isomorphic with $\mathscr{P}$.

2.2 THEOREM. *If $e$ is an idempotent in a thread $S$ and if $e$ cuts $eSe$, then $H(e) \cong \mathscr{P}$ or $H(e) \approx \mathscr{Z}$. Moreover, if the identity component $G$ is not all of $S$, then the boundary of $G$ in $S$ contains exactly one point $f$, either $G = (f, \infty)$ or $G = (-\infty, f)$, and $f$ acts as a zero for $G$.*

*Proof.* Assuming that $e$ cuts $eSe$, $G \cong \mathscr{P}$ by 2.1. Certainly, $H(e)$ is a topological group of which $G$ is a normal subgroup. Since the remainder of the theorem is evident otherwise, we assume $G \neq S$.

We claim now that if $M$ and $N$ are cosets of $G$ in $H(e)$ and if $t \in M^*\backslash M$, then $tN^*$ and $N^*t$ contain but one point each. For, since each coset is homeomorphic with $G$, each coset is open and connected, and thus has at most two boundary points. Since $t$ does not belong to $H(e)$, $Nt$ misses $H(e)$. Thus $Nt \subset (NM)^*\backslash NM$. But $Nt$ is connected and $(NM)^*\backslash NM$, the boundary of some coset, contains at most two points. Hence $Nt$ consists of a single element, and by continuity, the same must be true of $N^*t$. Likewise, $tN^*$ contains only one element.

Now take $f$ in $G^*\backslash G$, and let $C$ be any coset. If $t \in C^*\backslash C$, then, using the result of the preceding paragraph, $tG^* = te = t$ and $G^*t = et = t$. In particular, $f$ acts as an identity on $C^*\backslash C$. But applying the result again, $fC^*$ and $C^*f$ contain one point each. Thus the coset $C$ has exactly one boundary point. Taking $C = G$, we see that $G$ has only one boundary point $f$ and thus $G = (f, \infty)$ or $G = (-\infty, f)$. Moreover, $fG^* = G^*f = f$ implies that $G$ is iseomorphic (we do not know whether $f$ is the least or the greatest element of $G^*$) with the thread of non-negative real numbers. If $H(e) = G$, the proof is complete.

Assuming $H(e) \neq G$, it follows from the fact that each coset has only one boundary point in $S$ that there can be only one other coset besides $G$. Take $b \in H(e)\backslash G$ and observe that the function on $G^*$ which takes $g$ into $b^{-1}gb$ is a continuous automorphism which (since $b^2 \in G$ and $G$ is commutative) is its own inverse. But the only such automorphism of the non-negative real numbers is the identity, and thus $b^{-1}gb = g$ for each $g$ in $G$. It follows that $H(e)$ is commutative, and from this it is easy to verify that $H(e)$ is iseomorphic with $\mathscr{X}$.

2.3 THEOREM. *Let $e$ be an idempotent in a thread $S$.*

(1) *If $H(e) = \{d, e\}$ with $d < e$, then $Se = eS = [d, e]$ and there exists a zero for $S$ in $(d, e)$. Denoting the zero by $z$, $[z, e]$ is a standard thread and $[d, e] \cong \mathscr{R}([z, e])$.*

(2) *If $H(e) \approx \mathscr{X}$ then the complete structure of $S$ is determined. Namely, there exists a positive thread $T$ such that $S \approx \mathscr{R}(T)$.*

*Proof.* Let $H(e) = \{d, e\}$ with $d < e$, and observe that $[d, e] \subset eSe$. Then $eS \cup Se \leqq e$, for otherwise $e$ cuts $eS$ or $Se$ and 2.1 yields a contradiction. Now since $d$ is in $H(e)$ and $d^2 = e$, left multiplication by $d$ is a strictly decreasing function from $eS$ onto itself. Hence

$$d = de \leqq d(eS) = eS ,$$

so that $[d, e] = eS$. Moreover, there exists a unique element $q$ in $eS$ such that $dq = q$. However, if $s$ is any element of $S$, then $qs \in eS$ and

$d(qs) = qs$. Since $q$ is unique, $q$ is a left zero for $S$. Similarly $[d, e] = Se$ and there exists a right zero for $S$ in $Se$. Evidently these two one sided zeros are equal, and putting $z = q$, $z$ is a zero for $S$. Now, $[d, e]$ is a subthread with an identity $e$ and a zero $z$ in which $d^2 = e$. Applying part one of Theorem 6.2 in [4], we conclude that $[z, e]$ is a standard thread and that $[d, e] \cong \mathscr{R}([z, e])$.

Turning to the proof of (2), let $H(e) \approx \mathscr{X}$. Since $S$ is iseomorphic with its order dual, we may assume that $e$ is larger than the element $u$ which corresponds to $-1$. Each coset in $H(e)$ has exactly one boundary point in $S$ and thus $H(e) = (-\infty, h) \cup (f, \infty)$ where $h \leqq f$. Since we have assumed that $u < e$, $(f, \infty) \cong \mathscr{P}$.

One sees easily that $f^2 = h^2 = f$ and that $fh = hf = h$, i.e., $H(f) = \{h, f\}$. If $h = f$ then $S$ is iseomorphic with the multiplicative thread of all real numbers which is certainly $\mathscr{R}(T)$ where $T$ is the thread of non-negative reals. Assuming $h < f$, we may apply the conclusion of (1). Thus $S$ has a zero between $h$ and $f$, $[z, f]$ is a standard thread, $[h, f]$ is commutative, and $Sf = fS = [h, f]$.

Since $f$ is an identity for $[z, f]$ and a zero for $G$, each element of $G$ acts as an identity on $[z, f]$. Consequently, $[z, \infty)$ is a positive thread.

If $y \in [z, f]$, then $uy = u(fy) = (uf)y$ and $yu = y(fu)$. Now, $f$ commutes with $u$, and since $uf \in [h, f]$, $uf$ commutes with $y$. Thus $u$ commutes with each element of $[z, f]$ as well as with each element of $(f, \infty)$. Armed with these facts, it is a straightforward exercise to show that the function $g$ defined on $\mathscr{R}([z, \infty))$ by $g(t) = t$ and $g(t') = ut$ is an iseomorphism onto $S$.

2.4 COROLLARY. *If $x^k < x < x^p$ for some $x$ in a thread $S$ and for some positive integers $k$ and $p$, then $S \approx \mathscr{R}(T)$ for some positive thread $T$. Moreover, if $e$ is the identity of $S$, then $x \in H(e)$ and $e$ separates $x$ and $x^2$.*

*Proof*. Since $x$ is evidently not an idempotent, we assume that $x < x^2$. The case where $x^2 < x$ is entirely similar. Taking $j$ to be the least positive integer such that $x^{j+1} < x$, we have $2 \leqq j$ and $x < x^j$. Now $x \in (x^{j+1}, x^2)$ and $(x^{j+1}, x^2) \subset x(x, x^j) \cap (x, x^j)x$, so $x = xs = tx$ for some $s$ and $t$ in $(x, x^j)$. It follows that $s$ is a right identity on $Sx$ and that $t$ is a left identity on $xS$. But $(x, x^j) \subset (x^{j+1}, x^j) \subset xS \cap Sx$, hence $s = ts = t$. Putting $e = s$, $e \in (x, x^j)$ and $(x, x^j) \subset xS \cap Sx = exS \cap Sxe \subset eSe$, so that $e$ is a cut point of $eSe$. By 2.2, $H(e) \cong \mathscr{P}$ or $H(e) \approx \mathscr{X}$. But $e \in xS \cap Sx$ and $x \in eSe$ imply that $x \in H(e)$, and in view of the hypothesis on the powers of $x$, $H(e) \cong \mathscr{P}$ is impossible. The result now follows from 2.3.

The following facts concerning the sets $eS$ and $Se$ will be useful later.

2.5 LEMMA. *Let $e$ be an idempotent in a thread $S$.*

(1) *If $e = eSe$, then either $eS = e$ or $Se = e$; and in either case, $SeS$ is the minimal ideal of $S$. It is a closed connected set of one sided zeros.*

(2) *Either $eS \subset Se$ or $Se \subset eS$, and thus $SeS = eS \cup Se$.*

*Proof.* Let $e = eSe$, and recall that $eSe = eS \cap Se$. By way of contradiction suppose that $eS \neq e$ and that $Se \neq e$. Then either $eS \leqq e \leqq Se$ or $Se \leqq e \leqq eS$; and in either case, $e$ is in the interior of $eS \cup Se$. Thus there exists an open interval $V$ about $e$ such that $V^2 \subset eS \cup Se$. Choosing $x$ and $y$ in $V$ such that $x \in eS$, $x \neq e$, $y \in Se$, and $y \neq e$, we have $yx \in eS \cup Se$. But if $yx \in eS$, then

$$e = e(yx)e = (yx)e = (ye)xe = y(exe) = ye = y ,$$

contrary to the choice of $y$; and if $yx \in Se$, then similarly, $e = x$, contrary to the choice of $x$.

Now if $eS = e$, $SeS = Se$. Since $Se$ is the image of the connected set $S$ under right translation by $e$, it is connected; and since it is the set on which right translation by $e$ agrees with the identity mapping, it is closed. Moreover, for each $k$ in $SeS$,

$$kS = (ke)S = k(eS) = ke = k .$$

Thus, $SeS$ is a closed connected set of left zeros and is clearly the minimal ideal of $S$. If $Se = e$, then $SeS$ consists of right zeros.

In order to prove (2), consider the three cases of 2.1. If $e = eSe$, then one of $eS$ and $Se$ is just $\{e\}$ and is clearly contained in the other. If $e$ is an endpoint of $eS \cup Se$, then since $eS$ and $Se$ are connected sets extending from $e$ in the same direction, one evidently contains the other. Finally, if $e$ cuts $eSe$, then the identity component of $H(e)$ extends to one end of the thread. Since $H(e) \subset eS \cap Se$, the result again follows from the connectedness of $eS$ and $Se$.

## 3. Subthreads.

3.1 LEMMA. *Let $A$ be a subset of $S$ which contains, with $x$, all elements larger than $x$. If $A$ contains no idempotents and if $a < a^2$ for some $a$ in $A$, then $A$ is a subthread in which $\max\{x, y\} < xy$ for each pair of elements $x$ and $y$ in $A$.*

*Proof.* Let $a$ be an element in $A$ such that $a < a^2$, and let $x$ be any element of $A$. If $x^2 < x$, then, since $A$ is evidently connected and since the function mapping each element onto its square is continuous, there is an idempotent between $x$ and $a$ contrary to the assumption that $A$ contains no idempotents. Hence $x < x^2$. If $x^n < x$, for some positive integer $n$, then there is an idempotent between $x$ and $x^2$ by 2.4. And again, if $\Gamma(x)$ is bounded, it is a compact semigroup and thus contains an idempotent. Hence $x \in A$ implies that $x \leqq \Gamma(x)$ and that $\Gamma(x)$ is unbounded.

Now suppose that $yz = y$ with $y$ and $z$ in $A$. For each positive integer $n$, $yz^n = y$, thus $z^n$ is a right identity for $Sy$. But both $\Gamma(y)$ and $\Gamma(z)$ are unbounded, so for some $n$ and $m$, $y^2 < z^n < y^m$. Thus $z^n$ is in $Sy$ and $z^n z^n = z^n$. Since $A$ contains no idempotents, $yz = y$ is impossible.

Finally, if $yz < y$, then, by the continuity of right multiplication by $z$ and the fact that $z < zz$, there exists a $t$ between $y$ and $z$ for which $tz = t$, a contradiction. Hence $y < yz$, and dually $z < yz$.

3.2 LEMMA. *If $e$ is an idempotent, if $eS \cup Se \leqq e$, if $C$ is a connected set containing $e$ as a least element, and if $[e, x) \subset xC \cap Cx$ whenever $x \in C$; then $e \leqq C^2$.*

*Proof*. Appealing to 2.5 we will lose no generality by assuming that $eS \subset Se$. Thus $t \in eC$ implies $et = te = t$. Moreover, if $t = ex$ with $x$ in $C$, then $e = sx$ for some $s$ in $C$, and thus

$$(es)t = (es)(ex) = [(es)e]x = (es)x = e(sx) = e\ .$$

It follows that $eC$ is a subgroup of $H(e)$. But $eC$ is connected and contains $e$ while, by 2.1, $H(e)$ contains at most two elements. Hence $eC = e$.

Now suppose that $xy < e$ for some $x$ and $y$ in $C$. Clearly $e < x$ and therefore $e < xt$ for some $t$ in $C$. Now $xy < e < xt$ implies that $e = xw$ for some $w$ between $y$ and $t$. But if $y < w$, then $xy \in xwC = eC = e$; and if $t < w$, then $xt \in xwC = eC = e$. Since this contradicts $xy < e < xt$, we have $e \leqq xy$. Hence, $e \leqq C^2$.

The following result, which is a generalization of Faucett's Lemma 4 in [5], will be extremely useful in the remainder of the paper.

3.3 THEOREM. *If $e$ and $f$ are idempotents in a thread $S$ and if $eS \cup Se \leqq e < f$, then $[e, f]$ is a standard thread. If, in addition, $f$ cuts $fSf$, then $[e, \infty)$ is a positive thread.*

*Proof*. Since $ef \in eS$ and $fe \in Se$, neither $ef$ nor $fe$ is larger than $e$. But $ef \in Sf$ and $fe \in fS$, and these sets are connected. Thus $e \in Sf \cap fS$, and $f$ acts as an identity on $[e, f]$. Then, for each $x$ in $[e, f]$,

$[e, x] \subset [ex, fx] \cap [xe, xf] \subset [e, f]x \cap x[e, f]$. Consequently, by 3.2, $e \leqq [e, f]^2$, and in particular, $e$ acts as a zero for $[e, f]$.

Now if $fS \cup Sf \leqq f$; then $[e, f] \subset fS$ implies $[e, f]^2 \subset fS$ and the theorem is established. If on the other hand, $fS \cup Sf \not\leqq f$; then, by 2.1, $f$ cuts $fSf$.

Finally, if $f$ cuts $fSf$; then, since $e$ cannot be in $H(f)$, it follows from 2.2 that there exists an idempotent $h$ in $[e, f)$ such that $(h, \infty)$ is isomorphic with $\mathscr{P}$. Since $hS \cup Sh \leqq h$, the preceding paragraphs show that $[e, h]$ is a standard thread (it may of course be simply one point if $e = h$). Evidently then, $[e, \infty)$ is a positive thread of which $[e, f]$ is a standard subthread.

3.4 LEMMA. *If $[a, b]$ and $[b, c]$ are subthreads, then so is $[a, c]$.*

*Proof.* Let $x \in [a, b]$, let $y \in [b, c]$, and suppose that $c < xy$. Then, since $xb \in [a, b]$, $[b, c] \subset [xb, xy] \subset x[b, c]$. Now $\Gamma(x)$ and $[b, c]$ are both compact, and by Wallace's Theorem 1 in [11], we conclude that $[b, c] = x[b, c]$ contrary to $c < xy$. Thus $xy \leqq c$; and similarly, one proves that $a \leqq xy$ and that $a \leqq yx \leqq c$.

3.5 THEOREM. *If $e$ and $f$ are any two idempotents in a thread, then the closed interval between them is a subthread.*

The proof of this result will be postponed until the end of section four. The proof will be much easier then, and we promise not to apply the result in the meanwhile.

## 4. The minimal ideal.

4.1 THEOREM. *If $S$ has no minimal ideal, then a zero may be adjoined as an endpoint and the resulting semigroup is again a thread.*

*Proof.* We show first that $S$ has no bounded ideals. Indeed, if $M$ is a bounded ideal, then $M^*$ is a compact ideal. In particular, $M^*$ is a compact topological semigroup, and as such (see Theorem 3 in [10]), there is an idempotent $e$ in $M^*$ such that $eM^*e$ is a group. But $M^*$ is an ideal and thus $eSe = eM^*e$, thus $eSe$ is a compact connected group. It follows from 2.1 and 2.5 that $eSe = e$ and that $SeS$ is the minimal ideal of $S$. Hence, $S$ has no bounded ideals.

Next observe that every ideal contains a connected ideal. For if $x$ is any element of an ideal $J$, then $SxS$ is a connected ideal contained in $J$.

Now fix $y$ in $S$ and let $J$ be an ideal contained in $S\backslash y$. Such an ideal does exist, for if not, then $y$ is in each ideal of $S$, the intersection of all ideals is not empty, and $S$ has a minimal ideal. Since we may take $J$ to be connected, we lose no generality if we assume that $J < y$.

If $x < y$ then again there is a connected ideal $M$ contained in $S \backslash x$. In fact, $M < x$, for otherwise $M \cap J$ is a bounded ideal. Thus $M^*$ is a connected, closed, unbounded ideal whose elements are all less than or equal to $x$. Hence, for each $x$ less than $y$, there exists a $c$ not greater than $x$ such that $(-\infty, c]$ is an ideal. Evidently a zero can be adjoined as a least element.

4.2 THEOREM. *If $S$ has a minimal ideal $K$, then either $S = K$ and $S \cong \mathscr{P}$, or there exists an idempotent $e$ such that $e = eSe$. In the second case, it follows from* 2.5 *that $K = SeS$ and is a closed connected set of one sided zeros.*

*Proof.* Let $x \in K$ and consider the subthread $xK$. We claim that $xK$ contains an idempotent. If not, we may assume without loss of generality that $a < a^2$ for some $a$ in $xK$. It follows from 3.1 that $a < (xK)a(xK)$. But $K(ax)K$ is an ideal contained in $K$ and must therefore be equal to $K$. Consequently $(xK)a(xK) = xK$ so that $a \in (xK)a(xK)$. Hence, $xK$ (and by an analogous proof, $Kx$ as well) contains an idempotent for each $x$ in $K$.

Let $e$ be an idempotent in $K$ and recall that one of $eS$ and $Se$ contains the other by 2.5. Assuming $eS \subset Se$, we have $eSe = eS = eK$. Notice that $eSe$ contains no idempotents other than $e$. For if $f \in eSe$, then $f = ef = fe$. But also, $f \in K$ so that $e \in SfS = Sf \cup fS$, hence $e = f$.

Now if $x \in eSe$, then $xK$ contains an idempotent. But

$$xK = (xe)K = x(eK) = x(eSe) \subset eSe ,$$

and $eSe$ contains only one idempotent. Hence $x \in eSe$ implies $e \in x(eSe)$, i.e. $eSe$ is a group.

Since $eSe$ is also connected, either $e = eSe$ or $eSe \cong \mathscr{P}$. In the latter case, $eSe$ is both open and closed and hence $eSe = S$. Thus $S \cong \mathscr{P}$ and $S = K$.

We are now in a position to give the overdue proof of Theorem 3.5. We are to show that the closed interval between two idempotents in a thread is a subthread.

*Proof of* 3.5. Since we can adjoin a zero if not, we assume that $S$ has a minimal $K$; and since the assertion is vacuously true otherwise, we assume that $K$ consists of one sided zeros. Observe that because of the trivial multiplication within $K$, any closed interval contained in $K$ is a subthread.

If $f$ is an idempotent larger than each element of $K$, and if $k = \sup K$, then $[k, f]$ is a standard thread by 3.3. Similarly, if $f < K$ and if

$l = \inf K$, then $\lfloor f, l \rfloor$ is the order dual of a standard thread. Moreover, the interval between any two idempotents in a standard thread is again a standard thread.

Finally, using these facts along with Lemma 3.4, which allows us to sew the subthreads together, the theorem follows easily.

## 5. Threads with a zero.

The principal result of this section is the characterization in 5.5 of all threads which have a zero as an endpoint and for which $S^2 = S$. However, the series of lemmas leading to this result will be used again in the following section; consequently they are more troublesome than is apparently necessary.

It will be convenient to introduce the following partial order whenever $S$ has a zero:

$$x \prec y \text{ if and only if } 0 \leqq x < y \text{ or } y < x \leqq 0 .$$

Obviously this does define a partial order on $S$.

5.1 LEMMA. *Let $S$ be a thread with a zero in which each idempotent $e$* is an endpoint *of $eSe$. Then $\Gamma(x)$ is compact for each $x$ in $S$, $J(x) \leqq x$ when $0 < x$, and $x \leqq J(x)$ when $x < 0$.*

*Proof.* We show first that $0 < x$ implies $\Gamma(x) \leqq x$. This is clear if $x \in [0, e]$ for some idempotent $e$, for $[0, e]$ is a standard thread by 3.3. Assume that $x$ is larger than each idempotent, and let $e$ be the largest idempotent. Now if $x < x^2$, then by 3.1, $\max\{y, x\} < xy$ for each $y$ larger than $e$. By continuity, $x \leqq xe$, and thus, $0 < e < x$ while $x \in Se$. But using 2.1, this implies that $e$ cuts $eSe$, contrary to hypothesis. Hence $x^2 < x$, and it follows from 2.4 and the assumption that each idempotent $e$ is an endpoint of $eSe$ that $\Gamma(x) \leqq x$. Repeating the argument with all inequalities reversed, $x \leqq \Gamma(x)$ when $x < 0$.

Next we prove that $\Gamma(\mathrm{x})$ is compact for each $x$ larger than 0. This is obvious if $\Gamma(x) \subset [0, x]$. If $\Gamma(x) \not\subset [0, x]$, let $x^j$ be the first power of $x$ which is less than 0. Since $x^j \leqq \Gamma(x^j)$, $x^{jn} \in [x^j, x]$ for each positive integer $n$. By the choice of $j$, $x^i \in [x^j, x]$ for each positive integer $i$ less than $j$ as well; therefore $\Gamma(x) \subset [x^j, x]^2 \cup [x^j, x]$, a compact set. Similarly, $\Gamma(x)$ is compact when $x$ is less than 0.

To establish the last statement of the lemma, let $0 < x$ and suppose that $x \leqq sxt$. Then $[0, x] \subset s[0, x]t$, while $[0, x]$, $\Gamma(s)$, and $\Gamma(t)$ are compact. By Corollary 2 in [11], $[0, x] = s[0, x]t$. Therefore $SxS \leqq x$, and using the one sided analogues of the result just used, it can be proved that $Sx \leqq x$ and that $xS \leqq x$. This gives $J(x) \leqq x$, and it follow similarly that $x \leqq J(x)$ when $x < 0$.

5.2 LEMMA. *Let $S$ be a thread with a zero. If $S^2 = S$, then, for* *each $x$ larger than 0, there exist an element $u$ and a compact set $A$ such that $x = uA$ and such that $x$ is in the interior of $uV$ for each open set $V$ which contains $A$.*

*Proof.* Given $x$ larger than 0, choose $y$ larger than $x$; or if $x$ is maximal, put $y = x$. Since $S^2 = S$, we can choose $u$ and $v$ in $S$ so that $y = uv$. Now if $0 < v$, let

$$p = \inf\{t \mid 0 \leqq t \leqq v \text{ and } x \leqq u[t, v]\} ,$$
$$q = \sup\{t \mid p \leqq t \leqq v \text{ and } x = u[p, t]\} ,$$

and let $A = [p, q]$. And if $v < 0$, define $p$, $q$, and $A$ analogously. The details are easy to verify in either case. Actually, this proof is just a slight generalization of the usual proof of the intermediate value theorem for continuous functions on the real line.

5.3 LEMMA. *Let $S$ have a zero, let $S^2 = S$, and let $J(x) \leqq x$ for $x > 0$. If $T$ is a connected set containing 0 such that $Tu$ is bounded for each $u$ in $S$, and if $h$ is defined on $\{x \mid 0 \leqq x\}$ by $h(x) = \sup Tx$, then $h$ is continuous.*

*Proof.* Since $Tx \subset J(x) \leqq x$, $0 \leqq h(x) \leqq x$ for each $x$ greater than 0, and consequently, $h$ is continuous at 0.

Now let $0 < x$ and let $a < h(x) < b$. Choose $c$ and $t$ so that $t \in T$, $a < tx$, and $h(x) < c < b$, and let $u$ and $A$ be as in 5.2. We have

$$(Tu)^*A \subset (TuA)^* = (Tx)^* \leqq h(x) < c ,$$

and since $Tu$ is bounded by hypothesis, $(Tu)^*$ and $A$ are both compact. Thus (Lemma 2 in [9]) there exists an open set $V$ such that $A \subset V$ and $TuV < c$. If $y \in uV$, then $h(y) = \sup Ty \leqq c < b$; and by 5.2, $uV$ contains an open set about $x$.

Since $a < tx$, there is another open set $W$ about $x$ such that $a < tW$. Thus, $y \in W$ implies

$$a < ty \leqq \sup Ty = h(y) .$$

Taking the intersection of $W$ and the interior of $uV$, we have produced a neighborhood of $x$ which is mapped into $(a, b)$. Thus $h$ is continuous.

5.4 LEMMA. *Let $S$ have a zero and let $A$ be a set such that $\Gamma(a)$ is compact for each $a$ in $A$. If $[0, x) \subset Ax$ for each $x$ greater than 0, then $rt \leqq st$ whenever $0 \leqq r < s$.*

*Proof.* If 0 lies strictly between $rt$ and $st$, then there exists $c$ in $(r, s)$ for which $ct = 0$. But then $r \in [0, c)$ so that $rt \in (Ac)t = A(ct) = 0$

which contradicts the assumption that zero lies strictly between $rt$ and $st$. Hence $rt$ and $st$ are at least comparable with respect to $\prec$.

Since $r \in [0, s)$, we can choose an $a$ in $A$ such that $r = as$. Now if $st \leqq rt$, then

$$\{x \mid 0 \leqq x \leqq st\} \subset \{x \mid 0 \leqq x \leqq ast\} \subset a\{x \mid 0 \leqq x \leqq st\} ;$$

and since both $\Gamma(a)$ and $\{x \mid 0 \leqq x \leqq st\}$ are compact, we have $\{x \mid 0 \leqq x \leqq st\} = a\{x \mid 0 \leqq x \leqq st\}$ (Theorem 1, [11]). Thus $rt = ast \leqq st$.

5.5 THEOREM. *If $S$ is a thread with a zero as a least element and if $S^2 = S$, then $S$ is a standard thread, or $S$ is a standard thread with its identity removed, or $S$ is a positive thread.*

*Proof.* If there exists an idempotent $f$ in $S$ which cuts $fSf$, then $S$ is a positive thread by 3.3. Hence, assume that no idempotent $e$ cuts $eSe$. By 5.1, $\Gamma(x)$ is compact and $J(x) \subset [0, x]$ for each $x$ in $S$.

If we put $h(x) = \sup Sx$, then $h$ is continuous by 5.3. We claim moreover that $h$ is the identity. For suppose $h(a) \neq a$. Then $a \neq 0$ and $h(a) < a$. Using the continuity of $h$ we choose an element $t$ and an open interval $V$, containing $a$, such that $h(V) < t < V$. Since $S^2 = S$, we can write $a = yx$ and thus $h(0) < a \leqq h(x)$. Again using continuity, choose, $b$ so that $a = h(b)$. Now take any $c$ in $V$ such that $c < a$, and observe that $c \in Sb = S(Sb)$. Thus $c \in Sp$ for some $p$ in $Sb$. But then $c \leqq p \leqq a$ so that $p \in V$, and hence $h(p) < t < c$ contrary to $c \in Sp$.

Since $h$ is the identity, $[0, x) \subset Sx$ for each $x$; and an analogous argument gives $[0, x) \subset xS$. Thus we conclude from 5.4 and its left-right dual that the multiplication in $S$ is monotone.

If $S$ is compact with $w$ as its largest element, then $w$ is an idempotent and $S$ is a standard thread. Indeed, we can write $w = xy$, and it then follows from $J(x) \leqq x$ and $J(y) \leqq y$ that $w = x = y$.

If $S$ is not compact, then let $T$ be the semigroup obtained by adjoining an identity to $S$, and extend the order of $S$ to $T$ by declaring that the identity is larger than each element of $S$. Since $S$ is not compact, $T$ is evidently connected. Finally, the continuity of multiplication in $T$ follows immediately from the continuity and monotonicity is $S$ along with the relation $[0, x) \subset xS \cap Sx$. Thus, $T$ is a thread, and in fact, a standard thread.

5.6 COROLLARY. *If $S$ is a thread with no idempotents, and if $S^2 = S$, then $S$ is iseomorphic with the real interval* $(0, 1)$ *under the natural multiplication.*

*Proof.* Since $S$ has no idempotents, it follows from 4.2 that $S$ has no minimal ideal; and by 4.1, a zero may be adjoined as an endpoint to $S$. Then either the extended thread or its order dual satisfies the hypotheses of 5.5. Thus, $S$ must be the result of removing both the zero and the identity from a standard thread which has no other idempotents and which has no nilpotent elements. But Faucett proved in Theorem 2 of [5] that any such standard thread is iseomorphic with $[0, 1]$.

**6. Threads in which $S^2 = S$.** Let $S$ be a thread satisfying $S^2 = S$. If $S$ has no minimal ideal, then a zero may be adjoined as an endpoint. After taking the order dual, if necessary, the extended thread can then be described by 5.5. Consequently, the structure of $S$ is determined. If S does have a minimal ideal, and if $K = S$, then the structure of $S$ is given by 4.2.

Thus, we have left only the case where $S$ has a proper minimal ideal which consists either of left zeros or of right zeros. We include, of course, the special case in which $S$ has a zero. Throughout this section, when we say that $S$ has a minimal ideal $K$, it will be tacitly assumed that $K$ is proper and thus consists of zeros.

The following notation will be used when there exists a minimal ideal $K$:

$$R = \{t \mid k \leqq t \text{ for each } k \text{ in } K\} ,$$
$$L = \{t \mid t \leqq k \text{ for each } k \text{ in } K\} ,$$

If $S$ has a zero, we have, $R = \{t \mid 0 \leqq t\}$ and $L = \{t \mid t \leqq 0\}$.

6.1 Lemma. *If $S$ has a minimal ideal $K$, if $S^2 = S$, and if there exists a connected proper ideal of $S$ containing $L$, then $R^2 = R$.*

*Proof.* Let $J$ be a connected proper ideal containing $L$, and let $c = \sup J$. If $J^* = S$, then $S \backslash J = c$; and since $S^2 = S$, $c$ is an idempotent. Thus by 3.3, $R$ is a standard thread, and certainly $R^2 = R$.

Now assume that $J^*$ is a proper ideal, and let $B = \{t \mid c \leqq t\}$. Since $J^*$ is closed and connected, $T = S/J^*$ is a non-degenerate thread with a zero as a least element and with $T^2 = T$. By 5.5, $T$ is a positive thread or $T$ is a standard thread with or without its identity. In any case, $[0, t) \subset tT \cap Tt$ for each $t$ larger than zero in $T$. Since the natural homomorphism of $S$ onto $T$ is strictly increasing on $B$ and takes $J^*$ onto 0, we conclude that $[c, b) \subset bB \cap Bb$ for each $b$ larger than $c$ in $S$.

Taking $k = \sup K$, $k$ is the least element of $R$ and $kS \cup Sk \leqq k$. Since $bR$ and $Rb$ are connected sets, $[k, b) \subset bR \cap Rb$ for each $b$ larger than $c$. Now fix $b$ larger than $c$ and let $r$ be any element of $R$ such that $r \leqq c$. Then there exist $s$ and $t$ in $R$ such that $r = sb = bt$. Thus,

$$[k, r) \subset [sk, sb) \subset s[k, b) \subset sbR = rR ,$$

and similarly, $[k, r) \subset Rr$. Hence, for each $r$ in $R$, $[k, r) \subset rR \cap Rr$. Applying 3.2, with $C = R$ and $e = k$, we have $R^2 \subset R$. On the other hand, $R^2 \supset R$ follows immediately from the facts that $J$ is a proper ideal containing $L$ and that $S^2 = S$.

6.2 LEMMA. *Let $S$ have a zero, and let $S^2 = S$. If $R \subset LS \cup SL$ and if there exists a set $A$ such that $(d, 0] \subset dA \cap Ad$ for each $d$ less than 0 and such that $\Gamma(a)$ is compact for each $a$ in $A$, then the multiplication in $S$ is monotone with respect to $\prec$ and 0 is an endpoint of $L^2$, $R^2$, $LR$, and $RL$.*

*Proof.* First, notice that the second conclusion follows from the first. Indeed, it suffices to show that if $x$ and $y$ are $\prec$-comparable and if $u$ and $v$ are $\prec$-comparable, then so are $xu$ and $yv$. But if $x \prec y$ and $u \prec v$; then, assuming that the multiplication is monotone, $xu \leqq yu$ and $yu \leqq yv$, so that $xu \leqq yv$.

To prove monotonicity, observe that (using both order and left-right duality) 5.4 gives $dt \leqq pt$ and $td \leqq tp$ whenever $p < d \leqq 0$. Since $R \subset LS \cup SL$, while each of $LS$ and $SL$ is a connected set containing 0, either $R \subset LS$ or $R \subset SL$; and without loss of generality we assume that $R \subset LS$.

Now if $x > 0$, choose $d$ in $L$ and $q$ in $S$ such that $x = dq$. Then

$$[0, x) = [0q, dq) \subset (d, 0]q \subset Adq = Ax .$$

Thus, again by 5.4, $rt \leqq st$ whenever $0 \leqq r < s$.

The only case left to demonstrate is $tr \leqq ts$ for $0 \leqq r < s$. Again choose $d$ and $q$ with $d$ in $L$ so that $dq = s$. Then $r \in [0q, dq)$ so that $r = pq$ for some $p$ in $(d, 0]$. Since $d < p \leqq 0$, we have $tp \leqq td$, i.e., either $0 \leqq tp \leqq td$ or $td \leqq tp \leqq 0$. In either case we can multiply on the right by $q$ to obtain

$$tr = tpq \leqq tdq = ts .$$

6.3 LEMMA. *If $S$ has a zero, if $S^2 = S$, and if either $L^2 = L$ or $R^2 = R$; then the conclusions of* 6.2 *hold.*

*Proof.* The other case being quite similar, let us assume that $L^2 = L$. By 5.5, the order dual of $L$ is a positive thread or a standard thread with or without its identity. In the first case, $L$ has an identity $e$, $\Gamma(x)$ is compact for each $x$ in $[e, 0]$, and $(d, 0] \subset d[e, 0] \cap [e, 0]d$ for each $d$ less than 0. In the second case, $\Gamma(x)$ is compact for each $x$ in $L$ and $(d, 0] \subset dL \cap Ld$ when $d < 0$.

Hence, if $R \subset LS \cup SL$ as well, then monotonicity follows from 6.2. However, even if $R \not\subset LS \cup SL$, we may still apply 5.4 to conclude that

$dt \leqq pt$ and $td \leqq tp$ for $p < d \leqq 0$. Thus, if we show that $R^2 = R$, then monotonicity follows by dualizing the foregoing argument.

Now assume that $R \not\subset LS \cup SL$; we must show that $R^2 = R$. If $R$ contains an idempotent $e$ which cuts $eSe$, this is an immediate consequence of 3.3. Assume that each idempotent $e$ in $R$ is an endpoint of $eSe$.

If each idempotent $f$ in $L$ is also an endpoint of $fSf$, then by 5.1, $J(x) \leqq x$ whenever $0 < x$. From this it follows that $L \cup SL \cup LS$ is an ideal, and thus a connected proper ideal containing $L$. If some idempotent $f$ in $L$ cuts $fSf$, then by 3.3 and 2.5, $fS \cup Sf$ is a connected proper ideal containing $L$. Thus, in either case, 6.1 yields $R^2 = R$.

6.4 LEMMA. *If $S$ has a zero, if $S^2 = S$, if $J(x) \leqq x$ for $x > 0$, and if $x \leqq J(x)$ for $x < 0$; then either $L \subset L^2$ or $R \subset R^2$.*

*Proof.* Suppose by way of contradiction that neither $L \subset L^2$ nor $R \subset R^2$. Since $L \subset S^2 = L^2 \cup SR \cup RS$, while each of the three sets on the right is connected and contains 0, $L$ must be contained in one of the three. Consequently $L \subset SR$ or $L \subset RS$.

If $L \subset SR$, then

$$R \subset S^2 = SL \cup SR \subset S(SR) \cup SR = SR = R^2 \cup LR ,$$

and thus $R \subset LR$. Now

$$R \subset LR \subset L(LR) = (L^2 \cap L)R \cup (L^2 \cap R)R \subset (L^2 \cap L)R \cup R^2 .$$

Again, $R \subset (L^2 \cap L)R$; and hence $L \subset SR \subset S(L^2 \cap L)R$.

If $L \subset RS$, we obtain similarly, $L \subset R(L^2 \cap L)S$. But then, in either case, $L \subset S(L^2 \cap L)S$; and choosing $d$ less than $L^2$, $d \in SpS$ for some $p$ in $L^2 \cap L$ contrary to $p \leqq J(p)$.

6.5 LEMMA. *Let $S$ have a zero, let $S^2 = S$, let $Sx$ be bounded for each $x$, let $J(x) \leqq x$ for $0 < x$, let $x \leqq J(x)$ for $x < 0$, and define a function $f$ on $S$ by:*

$$f(x) = \begin{cases} \sup Sx , & \text{if } 0 \leqq x , \\ \inf Sx , & \text{if } x \leqq 0 . \end{cases}$$

*Then $f$ is continuous. Moreover, if $f$ is the identity on a set $B$, then $f$ also acts as the identity on $BS$.*

*Proof.* The continuity of $f$ is immediate from 5.3 and its order dual.

Since $Sx$ is connected and contains 0, $f(x) = x$ if and only if $\{y \mid y \leqq x\} \subset Sx$. Now if $f(b) = b$, and if $t = bs$, then

$$\{y \mid y \leqq t\} = \{y \mid y \leqq bs\} \subset \{y \mid y \leqq b\}s \subset Sbs = St.$$

Thus $f(b) = b$ implies $f(bs) = bs$.

6.6 LEMMA. *Let $S$ be a thread with a zero in which $J(x) \leqq x$ for $0 < x$ and $x \leqq J(x)$ for $x < 0$. Let $\Gamma(x)$ be compact for each $x$ in $S$, and let $R^2 = S$. Then $R = S$.*

*Proof.* Since $J(x) \leqq x$ for $0 < x$, $L \cup LS \cup SL$ is an ideal. If $R \not\subset LS \cup SL$, then $L \cup LS \cup SL$ is a proper connected ideal containing $L$, and by 6.1, $S = R^2 = R$. Hence we assume that $R \subset LS \cup SL$.

If $J(x)$ is unbounded for some $x$ larger than 0, then $L$ is unbounded and $L \subset J(x)$. But then, $R$ is unbounded, and at the same time $R \subset LS \cup SL \subset J(x) \leqq x$. Hence $x \geqq 0$ implies that $J(x)$ is bounded. If $J(x)$ is unbounded for some $x < 0$, then $R$ is unbounded and $R \subset J(x)$. Since $R^2 = S$, $x \in J(r)$ for some $r$ in $R$. Hence $R \subset J(r) \leqq r$, a contradiction. Thus, $Sx$ and $xS$ are bounded for each $x$ in $S$.

In the remainder of the proof we will prove that $(d, 0] \subset Sd \cap dS$ for each $d$ less than 0. Actually we only prove that $(d, 0] \subset Sd$; the other case depends on an analogous argument. Then we will be able to apply 6.2 and conclude that 0 is an endpoint of $R^2$, and thus $S = R$.

Let $a \in S$ and choose $h$ in $R$ such that $a \in Sh$. From $S^2 = S$ it follows that $a \in Sa_1$ for some $a_1$ in $Sh$. Continuing inductively, we construct an infinite sequence $\{a_n\}$ such that $a_n \in Sa_{n+1}$ and $a_{n+1} \in Sh$. Replacing $\{a_n\}$ by an infinite subsequence if necessary, we may assume that either $\{a_n\} \subset L$ or $\{a_n\} \subset R$. In either case, it follows from the hypotheses that $a_n \leqq a_{n+1}$.

Since each $a_n \in Sh$ while $Sh$ is bounded, the least upper bound of $\{a_n\}$ with respect to $\prec$ exists. Let $b$ be this least upper bound. Let $f$ be the function defined in Lemma 6.5. Then $a_n \leqq f(a_{n+1}) \leqq a_{n+1}$, and since $f$ is continuous, $f(b) = b$. This means that $\{x \mid 0 \leqq x \prec b\} \subset Sb$. Now if $a_1 = b$ then $a \in Sa_1 = Sb$, and if $a_1 \prec b$ then $a \in Sa_1 \subset S(Sb) = Sb$. We have shown that for each $a$ in $S$ there exists $b$ such that $a \in Sb$ and $f(b) = b$.

Let $B = \{x \mid f(x) = x\}$ and let $A = BS$. We have just proved that $SB = S$ and thus $SA = S$. Moreover, $f$ is the identity on $A$ by 6.5; and since we can write $A = \cup \{bS \mid b \in B\}$, $A$ is a connected right ideal.

Suppose that neither $L \subset A$ nor $R \subset A$. Then choose $d$ in $L$ and $r$ in $R$ such that $d < A < r$. Since $SA = S$, there exist $s$ and $t$ in $A$ such that $d \in Ss$ and $r \in St$. It follows from $d < A$ and $A < r$ that $t < 0 < s$. But then $s \in [0, r]$ and $[0, r] \subset St$, so that $d \in Ss \subset St$ contrary to $t \leqq J(t)$. Hence, either $L \subset A$ or $R \subset A$. Moreover, if $R \subset A$ then $L \subset R^2 \subset AR \subset A$, and thus $L \subset A$ in any case. Finally, $f$ acts as the identity on $L$ and thus $(d, 0] \subset Sd$ for each $d$ less than 0.

6.7 THEOREM. *Let $S$ be a thread with a proper minimal ideal $K$, and let $S^2 = S$. Then, passing to the order dual if necessary, $R^2 = R$ and is thus completely described by* 5.5. *Moreover, if $L \subset L^2$ then $L = L^2$ as well. Finally, $K$ does not separate $R^2$, $L^2$, $LR$, or $RL$, and the multiplication in $S/K$ is monotone with respect to $\prec$.*

*Proof.* Since $K$ is connected and closed, $T = S/K$ is a thread which obviously has a zero and satisfies $T^2 = T$. We show first that, after passing to the order dual if necessary, $R^2 = R$ in $T$.

If some idempotent $f$ cuts $fSf$ in $T$ then by 3.3 either $L$ or $R$ is a positive thread, and clearly either $L^2 = L$ or $R^2 = R$. Otherwise, each idempotent $e$ in $T$ is an endpoint of $eSe$ and 5.1 can be applied. Thus $J(x) \leqq x$ for $x > 0$, $J(x) \geqq x$ for $x < 0$, and $\Gamma(x)$ is compact for each $x$. Now by 6.4, either $L \subset L^2$ or $R \subset R^2$, and passing to the order dual if necessary, we assume that $R \subset R^2$. Since $d \leqq J(d)$ for each $d$ less than 0, $R^2$ is itself a thread. Moreover, it satisfies the hypotheses of 6.6, and thus $R^2 = R$.

Next, applying 6.3 to $T$, we see that the multiplication in $T$ is monotone with respect to $\prec$, and that 0 is an endpoint of $L^2$, $R^2$, $LR$, and $RL$. This evidently gives the last assertion of the theorem.

Finally, going back to $S$ itself, we clearly have $R \subset R^2$. Since $K$ does not separate $R^2$, $K \cup R$ is a thread satisfying 6.1 and thus $R = R^2$. Likewise, if $L \subset L^2$ in $S$, then $L = L^2$.

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