TWO EXTREMAL PROBLEMS

MARVIN ROSENBLUM AND HAROLD WIDOM

1. Introduction. Let $\mathscr{P}_0$ be the class of all complex trigonometric polynomials P of the form $P_0 + P_1 e^{i\phi} + P_2 e^{2i\phi} + \cdots$. Let σ and μ be, respectively normalized Lebesgue measure and any finite non-negative Borel measure on the real interval $(-\pi, \pi]$. Suppose $\mu = \mu_A + \mu_S$, with $d\mu_A(\phi) = f(\phi)d\sigma(\phi)$, is the Lebesgue decomposition of μ into absolutely continuous and singular measures. In this note we shall be concerned with two generalizations of the problem Q_0: Find

$$I_0(\mu) = \inf_{P \in \mathscr{P}_0} \left[\int | 1 + e^{i\phi} P(e^{i\phi}) |^2 \, d\mu(\phi) \right]^{\frac{1}{2}} .$$

Q_0 was solved by Szegö for the case $\mu = \mu_A$ and in general by M. G. Krein and Kolmogorov. They showed that $I_0(\mu) = \exp \frac{1}{2} \int \log f \, d\sigma$ if $\log f$ is integrable and $I_0(\mu) = 0$ otherwise. (See [3], pp. 44, 231.)

We shall consider:

Problem Q_1: Suppose $\int | g |^2 d\mu < \infty$. Find

$$I_1(g, \mu) = \inf_{P \in \mathscr{P}_0} \left[\int | g + P |^2 \, d\mu \right]^{\frac{1}{2}} ,$$

and

Problem Q_2: Suppose $\int | h | \, d\sigma < \infty$. Find

$$I_2(h, \mu) = \sup_{P \in \mathscr{P}_0} \left\{ \left| \int Ph \, d\sigma \right| \Big/ \left[\int | P |^2 \, d\mu \right]^{\frac{1}{2}} \right\} .$$

Clearly $I_1(e^{-i\phi}, \mu) = I_0(\mu)$. Also

$$[I_2(1, \mu)]^{-1} = \inf_{P \in \mathscr{P}_0} \left\{ \left[\int | P |^2 \, d\mu \right]^{\frac{1}{2}} \Big/ \int | P d\sigma | \right\} = I_0(\mu) ,$$

so Q_0 is a particularization of both Q_1 and Q_2. There are other special cases of Q_1 and Q_2 that can be found in the work of Szegö [5] and Grenander and Szegö [3]. Of particular interest are the following:

(i) Let $g(\phi) = e^{-i(k+1)\phi}$, where k is a positive integer. Then Q_1 is the problem of linear prediction k units ahead of time ([3], p. 184).

(ii) Let $h(\phi) = 1/(1 - \alpha e^{-i\phi})$, $| \alpha | < 1$. Then

$$I_2(h, \mu) = \sup_{P \in \mathscr{P}_0} \left\{ | P(\alpha) | \Big/ \left[\int | P |^2 \, d\mu \right]^{\frac{1}{2}} \right\} .$$

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See [3], p. 48.

Throughout we shall indulge in the following notational conveniences: We shall write $I_1(g, f)$ and $I_2(h, f)$ for $I_1(g, \mu_A)$ and $I_2(h, \mu_A)$ respectively, and, in certain contexts, consider two functions identical that are equal everywhere except for a set of Lebesgue measure zero.

We have divided this note into six sections. First we indicate an interesting duality between $I_1(e^{-i\phi}g(\phi), f)$ and $I_2(g, 1/f)$ that relates the problems Q_1 and Q_2 under certain restrictive hypotheses. In section three we fashion the theory that will handle Q_1 and Q_2. This is the solution of a Riemann-Hilbert problem (which we call problem Q_3), which is applied in §§ 4, 5 and 6 to Q_1 and Q_2.

2. Duality of I_1 and I_2. This will fall out of the following Banach space lemma:

Let $\mathscr{P}_0$ be a subspace of a Banach space $\mathscr{L}$ and let $\mathscr{P}_0^\perp$ be the annihilator of $\mathscr{P}_0$ in the dual space $\mathscr{L}^*$. If $g \in \mathscr{L}$, then

$$\inf \{\| g + P \| : P \in \mathscr{P}_0\} = \sup \{| l(g)| : l \in \mathscr{P}_0^\perp, \| l \| \leqq 1\} .$$

For a proof see Bonsall [2].

THEOREM 1. *Suppose f and $1/f$ are in $L^1(-\pi, \pi)$ and $\int | g |^2 f \, d\sigma < \infty$. Then*

$$I_1(e^{-i\phi}g(\phi), f) = I_2(g, 1/f) .$$

Sketch of proof. By the above lemma

$$I_1(e^{-i\phi}g(\phi), f) = \sup \left\{ \left| \int e^{-i\phi}g(\phi)h(\phi)f(\phi)d\sigma \right| \Big/ \left[\int | h |^2 f d\sigma \right]^{\frac{1}{2}} \right\} ,$$

where the supremum is taken over all h such that $\int e^{in\phi}h(\phi)f(\phi)d\sigma = 0$ for $n = 0, 1, 2, \cdots$. Through the substitution $e^{-i\phi}hf = P$ if follows that

$$I_1(e^{-i\phi}g(\phi), f) = \sup \left\{ \left| \int Pf \, d\sigma \right| \Big/ \left[\int | P |^2 \frac{1}{f} d\sigma \right]^{\frac{1}{2}} \right\} ,$$

where now the supremum is taken over all P such that $\int e^{in\phi}P(\phi)d\sigma = 0$ for $n = 1, 2, \cdots$. It can be shown that it is sufficient merely to consider suprema for $P \in \mathscr{P}_0$, which proves the theorem.

The restrictive condition $1/f \in L^1(-\pi, \pi)$ seems essential to the formulation of the preceding duality relation, but at least this relation indicates that there exist close tie-ins between Q_1 and Q_2. We shall solve a Riemann-Hilbert problem for the unit circle that, when applied to Q_1 and Q_2, solves both.

3. The Riemann-Hilbert problem Q_3. Let f be a non-negative function in $L^1 = L^1(-\pi, \pi)$, and suppose that $\mathscr{P}$ is the closure of $\mathscr{P}_0$ in the Hilbert space $L^2(f)$ of functions square integrable with respect to the measure $fd\sigma$. Thus, for example, $\mathscr{P}$ in $L^2(1) = L^2$ can be identified with the Hardy space H^2. The problem Q_3 is:

Given $k \in L^1$, find functions $P \in \mathscr{P}$ and q satisfying

$$Pf = k + q \;, \quad \text{and} \tag{1}$$

$$\int q e^{-in\phi}\, d\sigma = 0 \;, \qquad n = 0, 1, \cdots \;. \tag{2}$$

$\left(\text{Note that since } \int |P|^2 f\, d\sigma < \infty, \text{ we have } Pf \in L^1 \text{ and so } q = Pf - k \in L^1.\right)$

We first list some prefactory material. We associate with any non-negative $f \in L^1$ such that $\log f \in L^1$ the analytic functions

$$\begin{aligned} F^+(z) &= \exp \frac{1}{2} \int \frac{e^{i\phi} + z}{e^{i\phi} - z} \log f(\phi)\, d\sigma(\phi), \; |z| < 1 \;, \\ F^-(z) &= \exp \frac{1}{2} \int \frac{z + e^{i\phi}}{z - e^{i\phi}} \log f(\phi)\, d\sigma(\phi), \; |z| > 1 \;. \end{aligned} \tag{3}$$

F^+ and F^- belong to H^2 and K^2 respectively, and $\overline{F^-(z)} = F^+(1/\bar{z})$ if $|z| > 1$. (A function $F(z)$ is said to belong to K^p if $F(1/z)$ belongs to H^p.) Since the boundary functions in H^2 and K^2 exist in mean square, we can define

$$\begin{aligned} f^+(\phi) &= \lim_{r \to 1-} F^+(re^{i\phi}) \;, \\ f^-(\phi) &= \lim_{r \to 1+} F^-(re^{i\phi}) \;. \end{aligned} \tag{4}$$

These functions satisfy

$$f(\phi) = f^-(\phi) f^+(\phi) = |f^+(\phi)|^2 = |f^-(\phi)|^2 \;. \tag{5}$$

For any non-negative $f \in L^1$ and $\varepsilon > 0$ we define $F_\varepsilon^\pm(z)$, $f_\varepsilon^\pm(\phi)$ by (3) and (4) with f replaced by $f_\varepsilon = f + \varepsilon$. Here we need not assume that $\log f \in L^1$. Note that since $f + \varepsilon \geqq \varepsilon > 0$, we have $1/F_\varepsilon^+ \in H^\infty$ and $1/F_\varepsilon^- \in K^\infty$. Moreover $|f_\varepsilon^+(\phi)|^2 = f(\phi) + \varepsilon$, so $|f_\varepsilon^-(\phi)| = |f_\varepsilon^+(\phi)| \geqq [f(\phi)]^{1/2}$.

Next we define an operator $(\;\;)_+$ as follows. Its domain D consists of all L^1 functions k with Fourier series $\sum_{-\infty}^{\infty} c_n e^{in\phi}$ such that $\sum_0^\infty |c_n|^2 < \infty$, and k_+ is the function with Fourier series $\sum_0^\infty c_n e^{in\phi}$. We define the operator $(\;\;)_-$ by $k_- = k - k_+$. Notice that $k_+ \in H^2$ and $k_- \in K^1$ with $\int k_- d\sigma = 0$.

Our discussion of Q_3 proceeds in the following order. First we prove uniqueness. Then we solve Q_3 in certain special cases (these being sufficient, it will turn out, to handle Q_1), and finally find the solution in the general case.

We are indebted to the referee for the proof of the next lemma.

LEMMA 2. *Q_3 has at most one solution.*

Proof. Suppose $Pf = q$ where $P \in \mathscr{P}$ and q satisfies (2). Then P is orthogonal, in $L^2(f)$, to all exponentials $e^{in\phi}$ $(n \geqq 0)$. Since P belongs to the closed manifold $\mathscr{P}$ spanned by these exponentials we conclude $P = 0$.

One can formally solve Q_3 by means of the usual factorization methods (see [4], for example). Write $f = f^+f^-$, so $Pf = k + q$ implies

$$Pf^+ = \frac{k}{f^-} + \frac{q}{f^-} .$$

Applying $(\ )_+$ to both sides we obtain $Pf^+ = (k/f^-)_+$, $P = (1/f^+)(k/f^-)_+$. The following theorem justifies this procedure in certain cases.

THEOREM 3. (i) *Suppose* $\log f \in L^1$ *and* $k/f^- \in D$. *Then* Q_3 *has the solution*

$$P = \frac{1}{f^+}\left(\frac{k}{f^-}\right)_+ \qquad q = -f^-\left(\frac{k}{f^-}\right)_- . \tag{6}$$

(ii) *Suppose* $\log f \notin L^1$ *and* $k^2/f \in L^1$. *Then* Q_3 *has the solution*

$$P = \frac{k}{f} \qquad q = 0 .$$

Proof. (i) Let $\varepsilon > 0$. Since the function f^+ is outer, it follows from a theorem of Beurling [1] that there exists a $P_0 \in \mathscr{P}_0$ such that

$$\int \left| \left(\frac{k}{f^-}\right)_+ - P_0 f^+ \right|^2 d\sigma < \varepsilon .$$

Therefore by (5)

$$\int \left| \frac{1}{f^+}\left(\frac{k}{f^-}\right)_+ - P_0 \right|^2 f d\sigma < \varepsilon ,$$

so P as defined in (6) belongs to $\mathscr{P}$. Furthermore, with q as defined in (6),

$$Pf - q = f^-\left[\left(\frac{k}{f_-}\right)_+ + \left(\frac{k}{f^-}\right)_-\right] = k .$$

It remains to show that $q \in K^1$. Certainly q belongs to $K^{1/2}$ since it is the product of the two K^1 functions $-f^-$ and $(k/f^-)_-$. But since also

$q = Pf - k$, it belongs to L^1. Therefore ([6], p. 163) $q \in K^1$.

(ii) If $\log f \notin L^1$, the space $\mathscr{P}$ is identical with $L^2(f)$ ([3], § 33) and so $k/f \in \mathscr{P}$.

We now give the complete solution of Q_3.

THEOREM 4. (i) *The limit*

$$\lim_{\varepsilon\to 0+} \int \left| (k/f_\varepsilon^-)_+ \right|^2 d\sigma$$

exists either finitely or infinitely.

(ii) *A necessary and sufficient condition that Q_3 have a solution P, q is that the limit be finite.*

(iii) *If the limit is finite then*

$$P = \lim\, (1/f_\varepsilon^+)(k/f_\varepsilon^-)_+$$

in the space $L^2(f)$, and

$$\int |P|^2 f\, d\sigma = \lim_{\varepsilon\to 0+} \int |(k/f_\varepsilon^-)_+|^2\, d\sigma\ .$$

Proof. Assume first that Q_3 has a solution P, q and divide both sides of (1) by f_ε^-. Since $q/f_\varepsilon^- \in K^1$ and $\int q/f_\varepsilon^- d\sigma = 0$ we have $q/f_\varepsilon^- \in D$ and $(q/f_\varepsilon^-)_+ = 0$; also $Pf/f_\varepsilon^- \in L^2 \subset D$. Therefore we can apply $(\ )_+$ to both sides, obtaining

$$(Pf/f_\varepsilon^-)_+ = (k/f_\varepsilon^-)_+\ .$$

Consequently

$$\int |(k/f_\varepsilon^-)_+|^2\, d\sigma \leqq \int |Pf/f_\varepsilon^-|^2\, d\sigma \leqq \int |P|^2 f d\sigma\ , \tag{7}$$

and so

$$\limsup_{\varepsilon\to 0+} \int |(k/f_\varepsilon^-)_+|^2\, d\sigma < \infty\ . \tag{8}$$

Conversely suppose that $\{\varepsilon_n\}$ is a sequence of ε's such that $\varepsilon_n \to 0+$ and

$$\int |(k/f_\varepsilon^-)_+|^2\, d\sigma = O(1) \text{ for } \varepsilon = \varepsilon_n\ . \tag{9}$$

By Theorem 3(i) there corresponds to each $\varepsilon = \varepsilon_n$ a solution P_ε, q_ε of $(f + \varepsilon)P_\varepsilon = k + q_\varepsilon$. We have

$$\int |P_\varepsilon|^2 f d\sigma \leqq \int |P_\varepsilon|^2 f_\varepsilon d\sigma = \int |(k/f_\varepsilon^-)_+|^2\, d\sigma = O(1)\ . \tag{10}$$

Thus there exists a subsequence of $\{\varepsilon_n\}$ such that $\{P_\varepsilon\}$ converges weakly in $L^2(f)$ to an element $P \in \mathscr{P}$. It will follow that P, $Pf - k$ satisfies Q_3 if the L^1 function $q = Pf - k$ satisfies (2). We have for $n = 0, 1, 2, \cdots$

$$\begin{aligned}\int q(\phi)e^{-in\phi}d\sigma &= \int \{P_\varepsilon(\phi)[f(\phi) + \varepsilon] - k(\phi)\}e^{-in\phi}d\sigma \\ &\quad + \int [P(\phi) - P_\varepsilon(\phi)]f(\phi)e^{-in\phi}d\sigma - \varepsilon \int P_\varepsilon(\phi)e^{-in\phi}d\sigma \\ &= J_1 + J_2 + J_3 .\end{aligned}$$

Theorem 3(i) implies that $J_1 = 0$. By the weak convergence of the P_ε we can make J_2 as small as desired by taking ε_n sufficiently small. Finally (10) implies that $\int | \varepsilon^{1/2}P_\varepsilon |^2 d\sigma = O(1)$, so by the Schwarz inequality $| J_3 | \leqq \varepsilon^{1/2} \int | \varepsilon^{1/2}P_\varepsilon | d\sigma = O(\varepsilon^{1/2})$ as $\varepsilon_n \to 0$. Thus P, q satisfy Q_3, so (8), holds and (9) is true for *any* sequence $\{\varepsilon_m\}$ of ε's that converge to $0+$. By what we have shown there corresponds to any such sequence $\{\varepsilon_m\}$ a subsequence such that P_ε converges weakly to the unique (Lemma 2) element P. Thus we can consider ε to be a real variable and conclude that P_ε converges weakly in $L^2(f)$ to $P \in \mathscr{P}$ as $\varepsilon \to 0+$ provided that

$$\liminf_{\varepsilon \to 0+} \int | k/f_\varepsilon^-)_+ |^2 d\sigma < \infty .$$

We next prove that in fact P_ε converges strongly to P in $L^2(f)$. It suffices to show that $\int | P_\varepsilon |^2 f d\sigma \to \int | P |^2 f d\sigma$. Weak convergence gives

$$\liminf_{\varepsilon \to 0+} \int | P_\varepsilon |^2 f d\sigma \geqq \int | P |^2 f d\sigma .$$

On the other hand, as in (7),

$$\int | P_\varepsilon |^2 f d\sigma \leqq \int | P_\varepsilon |^2 f_\varepsilon d\sigma = \int |(k/f_\varepsilon^-)_+ |^2 d\sigma \leqq \int | P |^2 f d\sigma .$$

so

$$\limsup_{\varepsilon \to 0+} \int | P_\varepsilon |^2 f d\sigma \leqq \int | P |^2 f d\sigma .$$

Thus

$$\lim_{\varepsilon \to 0+} \int | P_\varepsilon |^2 f d\sigma$$

exists, and equals

$$\lim_{\varepsilon \to 0+} \int |(k/f_\varepsilon^-)_+ |^2 d\sigma = \int | P |^2 f d\sigma .$$

Thus the proof is complete.

4. Solution of Q_1. In Q_1 we wish to find

$$I_1(g, \mu) = \inf_{P \in \mathscr{P}_0} \left[\int | g + P |^2 d\mu \right]^{\frac{1}{2}} ,$$

where g is a given function in $L^2(\mu)$. Since $I_1(g, \mu)$ represents the distance from g to the manifold $\mathscr{P}_0$ in $L^2(\mu)$, there exists a (unique) function P belonging to the closure $\mathscr{P}'$ of $\mathscr{P}_0$ in $L^2(\mu)$ such that

$$I_1(g, \mu) = \left[\int | g + P |^2 d\mu \right]^{\frac{1}{2}} .$$

This function P is such that $g + P$ is orthogonal to $\mathscr{P}_0$, so

$$\int [g(\phi) + P(\phi)] e^{-in\phi} d\mu(\phi) = 0 \qquad n = 0, 1, 2, \cdots .$$

It follows from a theorem of the brothers Riesz ([6], p. 158) that the measure ν given by

$$\nu(E) = \int_E [g(\phi) + P(\phi)] d\mu(\phi)$$

is absolutely continuous with respect to Lebesgue measure. Let F be a Borel set of Lebesgue measure zero such that $\mu_S((-\pi, \pi] - F) = 0$. Then $g + P$ vanishes on F almost everywhere with respect to μ_S, so

$$\int_F | g + P |^2 d\mu_S = 0$$

and

$$\int | g + P |^2 d\mu = \int_{\mathscr{C}} | g + P |^2 d\mu_A = \int | g + P |^2 f d\sigma .$$

Since $\mu \geqq \mu_A$ it follows that $I_1(g, \mu) = I_1(g, f)$, and this common value is attained by the same extremizing function $P \in \mathscr{P}' \subset \mathscr{P}$.

Now,

$$\int [g(\phi) + P(\phi)] e^{-in\phi} f(\phi) d\sigma = 0 \qquad n = 0, 1, \cdots ,$$

so if we set $q = (g + P)f$ we have $Pf = - gf + q$, where $P \in \mathscr{P}$ and q satisfies (2). Since $(gf)^2/f = g^2 f \in L^1$, we can apply Theorem 3 to this situation. The extremizing function

$$P = \begin{cases} -(1/f_+)(gf_+)_+ & \text{if} \quad \log f \in L^1 \\ -g & \text{if} \quad \log f \notin L^1 , \end{cases}$$

and since

$$I_1(g, f) = \left[\int | g + P |^2 f d\sigma\right]^{\frac{1}{2}} = \left[\int | q |^2 / f d\sigma\right]^{\frac{1}{2}}$$

we have

$$I_1(g, \mu) = I_1(g, f) = \begin{cases} \left[\int | (gf^+)_- |^2 \, d\sigma\right]^{\frac{1}{2}} & \text{if } \log f \in L^1 \\ 0 & \text{if } \log f \notin L^1 . \end{cases}$$

5. Solution of Q_2. Given $h \in L^1$, we will evaluate

$$I_2(h, \mu) = \sup_{P \in \mathscr{P}_0} \left\{ \left| \int Ph \, d\sigma \right| \Big/ \left[\int | P |^2 \, d\mu\right]^{\frac{1}{2}} \right\} .$$

Since $\mu \geqq \mu_A$ it is clear that if $I_2(h, f)$ is finite so is $I_2(h, \mu)$. We shall show that, conversely, if $I_2(h, \mu)$ is finite then so is $I_2(h, f)$ and in fact $I_2(h, f) = I_2(h, \mu)$. So now suppose $I_2(h, \mu) < \infty$. Then the linear functional L on $\mathscr{P}_0$ given by

$$L(P) = \int Ph \, d\sigma$$

is bounded on $L^2(\mu)$. Therefore if $\mathscr{P}'$ denotes the closure of $\mathscr{P}_0$ in $L^2(\mu)$, there is a uniquely determined $Q \in \mathscr{P}'$ such that $L(P) = \int P\bar{Q} \, d\mu$. Then we have

$$\int e^{-in\phi}[Q(\phi)d\mu(\phi) - \bar{h}(\phi)d\sigma(\phi)] = 0 \qquad n = 0, 1, \cdots .$$

We again apply the F. and M. Riesz theorem, and deduce that the measure ν given by

$$\nu(E) = \int_E Qd\mu - \int_E h \, d\sigma$$

is absolutely continuous with respect to Lebesgue measure. Letting F be a Borel set of Lebesgue measure zero such that $\mu_S((-\pi, \pi] - F) = 0$, we see that Q vanishes on F almost everywhere with respect to μ_S. Consequently

$$\int e^{-in\phi}[Q(\phi)f(\phi) - \bar{h}(\phi)]d\sigma(\phi) = 0 \qquad n = 0, 1, \cdots ,$$

so $Qf = \bar{h} + q$, where $Q \in \mathscr{P}' \subset \mathscr{P}$ and q satisfies (2). Thus the linear functional

$$L(P) = \int Ph \, d\sigma = \int P\bar{Q}f \, d\sigma ,$$

$P \in \mathscr{P}_0$, is bounded on $L^2(f)$, so $I_2(h, f)$ is finite and in fact equals $I_2(h, \mu)$. We deduce from Theorem 4 that

$$I_2(h, \mu) = I_2(h, f) = \lim_{\varepsilon \to 0+} \left[\int | (\bar{h}/f_\varepsilon^-)_+ |^2 d\sigma \right]^{\frac{1}{2}} ,$$

and Q may be exhibited as an $L^2(f)$ limit in the mean.

6. Some formulae for $I_2(h, \mu)$. We can obtain a simpler formula for $I_2(h, \mu)$ if we assume that $h^2/f \in L^1$ and apply Theorem 3. Then

$$I_2(h, \mu) = \begin{cases} \left[\int | (\bar{h}/f^-)_+ |^2 d\sigma \right]^{\frac{1}{2}} = \left[\int |(e^{-i\phi}h(\phi)/f^+(\phi))_- |^2 d\sigma(\phi) \right]^{\frac{1}{2}} & \\ & \text{if } \log f \in L^1 , \\ \left[\int | h |^2/f d\sigma \right]^{\frac{1}{2}} & \text{if } \log f \notin L^1 . \end{cases}$$

This, in conjunction with our solution of Q_1, gives the duality discussed in Theorem 1. Note that the hypothesis $1/f \in L^1$ of Theorem 1 implies that $\log f \in L^1$.

Another simple formula for $I_2(h, \mu)$ is available if we know that the Fourier series $\sum_{-\infty}^{\infty} h_n e^{in\phi}$ of h is such that $h_{-n} = O(R_0^{-n})$ as $n \to +\infty$ for some $R_0 > 1$. Then the function $H(z) = \sum_0^\infty h_{-n} z^{-n}$ is analytic in $| z | > 1/R_0$. We have

$$\int | (\bar{h}/f_\varepsilon^-)_+ |^2 d\sigma = \int | (e^{-i\phi}h(\phi)/f_\varepsilon^+(\phi))_- |^2 d\sigma ,$$

which by the Parseval relation equals

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int e^{in\phi} h(\phi) f_\varepsilon^+(\phi) d\sigma \right|^2 &= \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=1} z^{n+1} H(z) \Big/ F_\varepsilon^+(z) dz \right|^2 \\ &= \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=R} z^{n+1} H(z) \Big/ F_\varepsilon^+(z) dz \right|^2 , \end{aligned}$$

where $1/R_0 < R < 1$. Let us also assume that $\log f \in L^1$, so F^+ is well-defined and

$$H(Re^{i\phi})/F_\varepsilon^+(Re^{i\phi}) \longrightarrow H(Re^{i\phi})/F^+(Re^{i\phi})$$

in L^2 as $\varepsilon \to 0+$. It follows that

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \frac{1}{2\pi} \int_{|z|=R} z^{n+1} H(z) \Big/ F^+(z) dz \right|^2 .$$

Now, if we write

$$\frac{1}{F^+(z)} = \sum_{n=0}^{\infty} f_n z^n ,$$

then

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} h_{-n-m} f_m \right|^2 .$$

Thus if H is the Hankel matrix $[h_{-n-m}]_{n,m=0}^{\infty}$, and Φ the column vector with components $f_0, f_1, \cdots$, then

$$I_2(h, \mu) = \| H\Phi \| ,$$

where the norm is that of l^2.

For example, let α be such that $|\alpha| < 1$ and consider

$$\sup_{P \in \mathscr{P}} \left\{ | P(\alpha) | \Big/ \left(\int | P |^2 d\mu \right)^{\frac{1}{2}} \right\} .$$

Thus we wish to evaluate $I_2(1/(1 - \alpha e^{-i\phi}), \mu)$. Here $h_{-n} = \alpha^n$, $n = 0, 1, \cdots$, so

$$I_2(h, \mu)^2 = \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\infty} \alpha^{n+m} f_m \right|^2 = 1/[(1 - |\alpha|^2) | F^+(\alpha)|^2] ,$$

as in [2], p. 48.

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