CONCERNING BOUNDARY VALUE PROBLEMS[1]

J. W. NEUBERGER

1. Introduction. This paper follows work on integral equations by H. S. Wall [4], [5], J. S. MacNerney [1], [2] and the present author [3]. Some results of these papers are used here to investigate certain boundary value problems.

In § 2, results of Wall and MacNerney are used to study a linear boundary value problem which includes problems of the following kind: Suppose that each of a_{ij}, $i, j = 1, \cdots, n$ is a continuous function, a and b are numbers and each of b_{ij}, c_{ij} and d_i, $i, j = 1, \cdots, n$ is a number. Is there a unique function n-tuple $f_1, \cdots, f_n$ such that

$$f_i' = \sum_{j=1}^{n} a_{ij} f_j \quad \text{and} \quad \sum_{j=1}^{n} [b_{ij} f_j(a) + c_{ij} f_j(b)] = d_i \, , \qquad i = 1, \cdots, n?$$

Section 3 contains some observations concerning a nonlinear boundary value problem which includes the problem of solving a certain system of nonlinear first order differential equations together with a nonlinear boundary condition. An example is given in the final section.

S denotes a normed, complete, abelian group (norms are denoted by $\|\cdot\|$). B denotes the normed, complete, abelian group of all bounded endomorphisms from S to S (the norm of an element T of B is the g.l.b. of the set of all M such that $\| Tx \| \leqq M \| x \|$ for all x in S). B^* denotes the set to which T belongs only if T is a continuous function from S to S. If $[a, b]$ denotes a number interval, then $C_{[a,b]}$ denotes the set to which f belongs only if f is a continuous function from $[a, b]$ to S. The identity function on the numbers is denoted by j.

The reader is referred to [1] for a definition of the integral of a function from a number interval $[a, b]$ to B with respect to a function from $[a, b]$ to B and to [3] for a definition of the integral of a function from $[a, b]$ to S with respect to a function from $[a, b]$ to B^*. [1] and [3] contain existence theorems for these integrals and a discussion of some of their properties.

2. A linear boundary value problem. Suppose that $[a, b]$ is a number interval and F is a continuous function from $[a, b]$ to B which is of bounded variation on $[a, b]$. The following are theorems:

(i) There is a unique continuous function M from $[a, b] \times [a, b]$ to B such that $M(t, u) = I + \int_u^t dF \cdot M(j, u)$ for each of t and u in $[a, b]$. (I denotes the identity element in B)

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(ii) $M(t, u)M(u, v) = M(t, v)$ if each of t, u and v is in $[a, b]$.

(iii) If h is a continuous function from $[a, b]$ to S and c is in $[a, b]$, then the only element X of $C_{[a,b]}$ such that $X(t) = h(t) + \int_c^t dF \cdot X$ for each t in $[a, b]$ is given by $X(t) = M(t, c)h(c) + \int_c^t M(t, j)dh$ for each t in $[a, b]$.[2]

THEOREM A. *Suppose that H is a function from $[a, b]$ to B which is of bounded variation on $[a, b]$. A necessary and sufficient condition that there be a unique element Y of $C_{[a,b]}$ such that*

(*) $Y(t) = Y(u) + g(t) - g(u) + \int_a^t dF \cdot Y$ *and* $\int_a^b dH \cdot Y = C$ *for each C in S and each g in $C_{[a,b]}$ is that $\int_a^b dH \cdot M(j, a)$ have an inverse which is from S onto S.*

Proof. Consider first the following lemma. If Y is in $C_{[a,b]}$ and satisfies (*) for each of u and t in $[a, b]$, then

$$\left[\int_a^b dH \cdot M(j, a)\right]Y(a) = C - \int_a^b\left[\int_j^b dH(s) \cdot M(s, j)\right]dg .$$

Suppose Y is in $C_{[a,b]}$ and satisfies (*) for each of u and t in $[a, b]$. By (iii), $Y(t) = M(t, a)Y(a) + \int_a^t M(t, j)dg$ for each t in $[a, b]$ and thus

$$C = \int_a^b dH \cdot Y = \left[\int_a^b dH \cdot M(j, a)\right]Y(a) + \int_a^b dH(s) \cdot \left[\int_a^s M(s, j)dg\right]$$

$$= \left[\int_a^b dH \cdot M(j, a)\right]Y(a) + \int_a^b\left[\int_j^b dH(s) \cdot M(s, j)\right]dg .^{3}$$

Hence,

$$\left[\int_a^b dH \cdot M(j, a)\right]Y(a) = C - \int_a^b\left[\int_j^b dH(s) \cdot M(s, j)\right]dg .$$

Denote $\int_a^b dH \cdot M(j, a)$ by Q. Suppose that (*) has a unique solution for each g in $C_{[a,b]}$ and each C in S.

Denote by W a point of S, by g an element of $C_{[a,b]}$,

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[2] Certain essential ideas for Theorems (i) and (ii) were given by Wall in [**4**]. In [**5**,] Wall gave these theorems for S an n-dimensional Euclidean space or suitable infinite dimensional space. In [**1**], MacNerney extended Wall's theory in proving these theorems for any normed, linear and complete space. Modifications of MacNerney's proofs to the case of S a normed, complete, abelian group are so slight that the proofs are omitted. Discussion concerning the properties and computation of M can be found in each paper listed as reference to this paper.

[3] A proof that $\int_a^b dH(s) \cdot \left[\int_a^s M(s, j)dg\right] = \int_a^b\left[\int_j^b dH(s) \cdot M(s, j)\right]dg$ which follows closely a similar argument for ordinary integrals, is ommitted.

$$W + \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg$$

by C and by X the unique element of $C_{[a,b]}$ satisfying (*) for this g and C. By the above lemma, $QX(a) = C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg = W$. Thus each point of S is the image of some point of S under Q, that is, Q takes S onto S.

Suppose that Q is not reversible and denote by each of W, U and V a point in S such that $QU = W$, $QV = W$ and $U \neq V$. Denote by Y and Z two elements of $C_{[a,b]}$ such that $Y(t) = U + g(t) - g(a) + \int_a^t dF \cdot Y$ and $Z(t) = V + g(t) - g(a) + \int_a^t dF \cdot Z$ for each t in $[a, b]$. Thus, $Y(t) = Y(u) + g(t) - g(u) + \int_u^t dF \cdot Y$ and $Z(t) = Z(u) + g(t) - g(u) + \int_u^t dF \cdot Z$, for each of u and t in $[a, b]$. Since $Y(a) = U$ and $Z(a) = V$, it follows that $Y \neq Z$. As in the proof of the lemma,

$$\int_a^b dH \cdot Y = QU + \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg$$

and

$$\int_a^b dH \cdot Z = QV + \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg$$

and so

$$\int_a^b dH \cdot Y = \int_a^b dH \cdot Z ,$$

which means that there is a boundary value problem of the type (*) which has two solutions, which contradicts the above assumption. Thus if (*) has a unique solution for each g in $C_{[a,b]}$ and each C in S, Q takes S onto S reversibly.

Suppose that Q takes S onto S reversibly. Denote by g an element of $C_{[a,b]}$ and by C a point in S. Denote

$$\left[\int_a^b dH \cdot M(j, a)\right]^{-1} \left\{C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg\right\}$$

by U and denote by X the element of $C_{[a,b]}$ such that $X(t) = U + g(t) - g(a) + \int_a^t dH \cdot X$ for each t in $[a, b]$. Noting that $X(t) = X(u) + g(t) - g(u) + \int_u^t dH \cdot X$ and that $X(t) = M(t, a)U + \int_a^t M(t, j) dg$ for each of u and t in $[a, b]$ and substituting for X in $\int_a^b dH \cdot X$, it is seen that $\int_a^b dH \cdot X = C$. Thus X satisfies (*) for this g and C. Suppose Y is in $C_{[a,b]}$ and satisfies (*). Then, by the above lemma,

$$QY(a) = C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg$$

and so $Y(a) = U$ which means that $Y(t) = U + g(t) - g(u) + \int_a^t dF \cdot Y$ and hence by (iii), $X = Y$. Thus if Q takes S onto S reversibly, there is a unique solution to (*) for each g in $C_{[a,b]}$ and C in S.

THEOREM B. *If $\int_a^b dH \cdot M(j, a)$ has a bounded inverse which takes S onto S, that is, if $\left[\int_a^b dH \cdot M(j, a)\right]^{-1}$ is in B, then there is a function R from $[a, b]$ to B and a function K from $[a, b] \times [a, b]$ to B such that if g is in $C_{[a,b]}$ and C is in S, then the only element Y of $C_{[a,b]}$ satisfying (*) for each of t and u in $[a, b]$ is given by $Y(t) = R(t)C + \int_a^b K(t, j)dg$ for each t in $[a, b]$. Moreover, such a pair of functions R and K is given by $R(t) = \left[\int_a^b dH \cdot M(j, t)\right]^{-1}$ and*

$$K(t, u) = \begin{cases} -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_u^b dH \cdot M(j, u) + M(t, u) & \text{if } a \leqq u \leqq t \\ -\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_u^b dH \cdot M(j, u) & \text{if } t \leqq u \leqq b. \end{cases}$$

Proof. Suppose that g is in $C_{[a,b]}$ and C is in S. From Theorem A, (*) has a unique solution Y for this C and g, and from the lemma in the proof of Theorem A,

$$\left[\int_a^b dH \cdot M(j, a)\right] X(a) = C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg$$

and so

$$X(a) = \left[\int_a^b dH \cdot M(j, a)\right]^{-1} \left\{C - \int_a^b \left[\int_j^b dH(s) \cdot M(s, j)\right] dg\right\}.$$

Using (iii) and the fact that

$$M(t, a)\left[\int_a^b dH \cdot M(j, a)\right]^{-1} = \left[\int_a^b dH \cdot M(j, t)\right]^{-1},$$

$$\begin{aligned} X(t) &= \left[\int_a^b dH \cdot M(j, t)\right]^{-1} C - \int_a^b \left\{\left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_j^b dH(s) \cdot M(s, j)\right\} dg \\ &\quad + \int_a^t M(t, j) dg \\ &= R(t)C + \int_a^b K(t, j) dg \end{aligned}$$

where R and K are defined as in the statement of the theorem.

3. A nonlinear boundary value problem. Here a problem is considered which includes the one in the preceding section. Essentially, the requirements of § 2 that each of $F(t)$ and $H(t)$ be an element of B for every t in $[a, b]$ and that F and H be of bounded variation are replaced by considerably weaker conditions. Theorem D gives a necessary and sufficient condition for the nonlinear problem considered to have a unique solution. First a fundamental theorem for a certain type of integral equation is given.

THEOREM C. *Suppose that* $[a, b]$ *is a number interval and* F *is a function from* $[a, b]$ *to* B^* *such that if* A *is in* S *and* $r > 0$, *there is a variation function* U *on* $[a, b]$ *and a variation function* V *on* $[a, b]$ *such that*

$$\|[F(p) - F(q)]x\| \leqq U(p, q)$$

and

$$\|[F(p) - F(q)]x - [F(p) - F(q)]y\| \leqq V(p, q)\|x - y\|$$

if each of p and q is in $[a, b]$, $\|A - x\| \leqq r$ and $\|A - y\| \leqq r$. Then, if c is in $[a, b]$, there is a segment Q' containing c such that if Q is the common part of Q' and $[a, b]$, there is only one continuous function Y from Q to S such that $Y(t) = A + \int_c^t dF \cdot Y$ if t is in Q.

This follows from Theorem F of [3].

DEFINITION. Suppose F is a function from $[a, b]$ to B^* and c is in $[a, b]$. If there is a point A in S and an element Y of $C_{[a,b]}$ such that $Y(t) = A + \int_c^t dF \cdot Y$ for each t in $[a, b]$, then the set which contains only each such point A is denoted by $F_{c;[a,b]}$.

LEMMA 4.1. *Suppose that* F *satisfies the hypothesis of Theorem C and for some number* c *in* $[a, b]$ *and that there is a segment* Q' *as in the theorem which has* $[a, b]$ *as subset. Then, for each number* u *in* $[a, b]$, *there is a set* $F_{u;[a,b]}$.

Proof. Given such a number c and segment Q', then $Q = [a, b]$ and there is a point A in S and an element Y of $C_{[a,b]}$ such that $Y(t) = A + \int_c^t dF \cdot Y$ for each t in $[a, b]$. Thus if u is in $[a, b]$, $Y(u) = A + \int_c^u dF \cdot Y$ and $Y(t) = Y(u) + \int_u^t dF \cdot Y$ for each t in $[a, b]$. Thus there is a set $F_{u;[a,b]}$.

DEFINITION. Suppose the hypothesis of Lemma 4.1 holds. M denotes a function from $[a, b] \times [a, b]$ such that if each of t and u is in $[a, b]$, $M(t, u)$ is the function from $F_{u;[a,b]}$ to $F_{t;[a,b]}$ such that if A is in $F_{u;[a,b]}$, $M(t, u)A$ is $Y(t)$ where Y is the element of $C_{[a,b]}$ satisfying $Y(s) = A + \int_u^s dF \cdot Y$ for each s in $[a, b]$.

LEMMA 4.2. *Under the hypothesis of Lemma* 4.1, $M(s, t)M(t, u) = M(s, u)$ *for each of* s, t *and* u *in* $[a, b]$.

Proof. Suppose that each of s, t and u is in $[a, b]$ and A is in $F_{u;[a,b]}$. Then, $Y(s) = A + \int_u^s dF \cdot Y$ and $Y(t) = A + \int_u^t dF \cdot Y$ so that $Y(s) = Y(t) + \int_t^s dF \cdot Y$, $Y(t) = M(t, u)A$ and $Y(s) = M(s, u)A$. Therefore, $Y(s) = M(t, u)A + \int_t^s dF \cdot Y$ and $Y(s) = M(s, t)[M(t, u)A] = [M(s, t)M(t, u)]A$. Thus, $M(s, u) = M(s, t)M(t, u)$.

THEOREM D. *Suppose that in addition to the hypothesis of Theorem C, it is true that for some* c *in* $[a, b]$, *there is a set* $F_{c;[a,b]}$. *Suppose furthermore that* T *is a function from* $C_{[a,b]}$ *to* S *and that* C *is in* S. *The following two statements are equivalent*:

(i) *There is only one element* Y *of* $C_{[v,b]}$ *such that*

(**) $TY = C$ *and* $Y(t) = Y(u) + \int_u^t dF \cdot Y$ *for each of* t *and* u *in* $[a, b]$.

(ii) *For some* u *in* $[a, b]$, *the function* R *from* $F_{u;[a,b]}$, *defined by* $RA = T[M(j, u)A]$ *for each* A *in* $F_{u;[a,b]}$ *takes only one element of* $F_{u;[a,b]}$ *into* C.

Proof. Suppose that for some u in $[a, b]$, the function R as defined in Theorem D takes only the point U of $F_{u;[a,b]}$ into C. Denote by Y the element of $C_{[a,b]}$ such that $Y(t) = U + \int_u^t dF \cdot Y$ for each t in $[a, b]$. Thus, $Y(t) = Y(s) + \int_s^t dF \cdot Y$ and $Y(t) = M(t, u)U$ for each of t and s in $[a, b]$ and $TY = T[M(j, u)Y(u)] = C$. Suppose X is in $C_{[a,b]}$ and satisfies (**). Then, $X(t) = M(t, s)X(s)$ for each of t and s in $[a, b]$ and so $TX = T[M(j, u)X(u)]$ which means that $R[X(u)] = C$ which in turn implies that $X(u) = U$ and so $X(t) = U + \int_u^t dF \cdot X$ for each t in $[a, b]$. By Theorem B, $X = Y$. Thus the existence of such a u in $[a, b]$ and such a function R implies that (**) has a unique solution.

Suppose that (**) has a unique solution Y which is in $C_{[a,b]}$. Denote by u a number in $[a, b]$. Thus $Y(t) = Y(u) + \int_u^t dF \cdot Y$ and $Y(t) = M(t, u)Y(u)$ for each t in $[a, b]$ and so $TY = T[M(j, u)Y(u)]$. Denote by R the function from $F_{u;[a,b]}$ to S so that $RA = T[M(j, u)A]$ for each A in $F_{u;[a,b]}$. Thus $R[Y(u)] = C$. Suppose that $V \neq Y(u)$ and $RV = C$. Denote by X the element of $C_{[a,b]}$ so that $X(t) = V + \int_u^t dF \cdot X$ for each t in $[a, b]$. $X \neq Y$ as $X(u) \neq Y(u)$. But $X(t) = X(s) + \int_s^t dF \cdot X$ for each of t and s in $[a, b]$ and $TX = [M(j, u)X(u)] = T[M(j, u)V] = RV = C$, a contradiction. Thus there is not such a point V in $F_{u;[a,b]}$ and so the existence of a unique element of $C_{[a,b]}$ satisfying (**) implies the existence of the required function R.

4. An example. Suppose that $[a, b]$ is a number interval, S the number plane, each of p and q a continuous function from $[a, b]$ to a number set such that $p(t) > 0$ for each t in $[a, b]$ and each of a_{ij}, b_{ij} and c_i, $i, j = 1, 2$, a number. The problem of solving

$$(\Delta) \qquad \begin{aligned} &(py')'\, qy = G \\ &a_{11}y(a) + a_{12}p(a)y'(a) + b_{11}y(b) + b_{12}p(b)y'(b) = c_1 \\ &a_{12}y(a) + a_{22}p(a)y'(a) + b_{21}y(b) + b_{22}p(b)y'(b) = c_2 \end{aligned}$$

for each continuous function G from $[a, b]$ to a number set and each ordered number pair (c_1, c_2) is equivalent to the problem of finding a function pair f_1, f_2 each of which is from $[a, b]$ to a number set such that

$$\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} 0 & 1/q \\ q & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f_1(a) \\ f_2(a) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f_1(b) \\ f_2(b) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

i.e., the problem of finding a continuous function f from $[a, b]$ to S such that

$$(\delta) \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

$$f(t) = f(u) + g(t) - g(u) + \int_u^t dF \cdot f$$

and

$$\int_a^b dH \cdot f = A_1 f(a) + A_2 f(b) = C$$

for each of u and t in $[a, b]$ where $g(t) = \begin{bmatrix} 0 \\ G(t) \end{bmatrix}$, $F(t)$ is the linear transformation from S to S associated with

$$\begin{bmatrix} 0 & \int_a^t (1/p)dj \\ \int_a^t q\,dj & 0 \end{bmatrix}$$

for each t in $[a, b]$, each of A_1 and A_2 is a linear transformation from S to S with A_1 associated with $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and A_2 associated with $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and H is defined in the following way: $H(a) = N_b$, the transformation which takes each point of S into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H(u) = A_1$ if $a < u < b$ and $H(b) = A_1 + A_2$. Suppose that M satisfies $M(t, u) = I + \int_u^t dF \cdot M(j, u)$ for each of t and u in $[a, b]$. From § 2, for (δ) to have a unique continuous solution for each g and each C it is necessary and sufficient that $\int_a^b dH \cdot M(j, a) = A_1 + M(b, a)A_2$ have an inverse which is from S onto S_r. Here is $\int_a^b dH \cdot M(j, a)$ has an inverse, it is from S to S and is bounded

Suppose that $\int_a^b dH \cdot M(j, a)$ has an inverse, G is a continuous function from $[a, b]$ to a number set, C is in S and $g = \begin{bmatrix} 0 \\ G \end{bmatrix}$. By Theorem B, there is a function K from $[a, b] \times [a, b]$ to B and a function R from $[a, b]$ to B such that $f(t) = R(t)C + \int_a^b K(t, j)dg$ for each t in $[a, b]$. Denote by each of $R_{ij}, K_{ij}, i, j = 1, 2$ a function from $[a, b]$ to a number set such that if each of t and u is in $[a, b]$, $R(t)$ is associated with

$$\begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}$$

and $K(t, u)$ is associated with

$$\begin{bmatrix} K_{11}(t, u) & K_{12}(t, u) \\ K_{21}(t, u) & K_{22}(t, u) \end{bmatrix}.$$

Thus, $f_1(t) = R_{11}(t)c_1 + R_{12}(t)c_2 + \int_a^b K_{12}(t, j)dG$ for each t in $[a, b]$ and f_1 is the unique solution to (Δ).

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THE ILLINOIS INSTITUTE OF TECHNOLOGY
AND THE UNIVERSITY OF TENNESSEE