MULTIPLICATION ON CLASSES OF PSUEDO-ANALYTIC FUNCTIONS

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Lipman Bers [1, 2] has formulated a theory of solutions of linear elliptic partial differential equations in terms of classes of psuedo-analytic functions on a plane domain D. The theory for each class of psuedo-analytic functions is based on the notion of a generating pair of Holder continuous complex valued functions F and G defined on D and satisfying $\Im\mathrm{m}\,[\overline{F(z)}G(z)] > 0$ in D.

If w is any function defined on D, then there exist two real valued functions ϕ and ψ such that w can be written uniquely as

$$w(z) = \phi(z)F(z) + \psi(z)G(z) \; . \tag{1}$$

A function w defined on D is said to be (F, G)-psuedo-analytic (of the first kind) if a certain generalized derivative exists or equivalently if the equations

$$\begin{aligned} \phi_x F_1 - \phi_y F_2 + \psi_x G_1 - \psi_y G_2 &= 0 \\ \phi_y F_1 + \phi_x F_2 + \psi_y G_1 + \psi_x G_2 &= 0 \end{aligned} \tag{2}$$

are satisfied in D, where the subscripts x and y refer to partial derivatives with respect to x and y and the subscripts 1 and 2 refer to the real and imaginary parts of the functions F and G. If $F = 1$ and $G = i$, these equations reduce to the Cauchy-Riemann equations.

Given a generating pair (F, G) let B denote the class of all functions which are (F, G)-psuedo-analytic. If $F = 1$ and $G = i$, then B is the class of analytic functions on D, which will be referred to in this paper as A.

Any B has many of the properties of the ring of analytic functions. In particular very close analogues of the identity theorem, the Cauchy theorem, the Cauchy integral formula, the standard convergence theorems, and power series expansions have been proved.

With each class B is associated a class B' of psuedo-analytic functions of the second kind. This association is made by a mapping η of B into B' defined by

$$\eta(\phi F + \psi G) = \phi + i\psi \; .$$

On the class A of analytic functions this mapping is clearly the identity.

Each class B is a vector space with the usual definition of addition

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of functions and multiplication by scalars and η is a vector space isomorphism of B onto B'. The class A is a ring under the usual pointwise multiplication of functions. Since the classes of psuedo analytic functions each bear such marked resemblances to the class A of analytic functions, the question arises as to whether there exist for other classes appropriate generalizations of the ordinary multiplications of function. We shall prove that if such a multiplication bears a certain slight resemblance to the point wise multiplication, then B is multiplicatively isomorphic to A under the mapping η and conversely.

We denote the ordinary multiplication of functions by juxtaposition. Let m denote any mapping from $B \times B$ to the set of all functions from D to the plane. In particular let m_p be the mapping defined as follows: if $w = \phi F + \psi G$ and $w' = \phi' F + \psi' G$, let

$$m_p(w, w') = (\phi\phi' - \psi\psi')F + (\phi\psi' + \psi'\phi)G \ .$$

THEOREM. *Let B be a system of psuedo-analytic functions on the plane domain D and let m be a multiplication on B* (any mapping from $B \times B$ to the set of all functions from D to the plane). *Let m be associative and bilinear with respect to addition in B. Then a necessary and sufficient condition for the mapping η to be a multiplicative isomorphism of B onto the ring A of all analytic functions on D is that there exists a non-constant w in B such that $m(w, G) = m_p(w, G)$ and $m_p(w, G) \in B$.*

The proof of this theorem will be preceded by a lemma.

LEMMA. *Suppose that for all w and w' in B, $m(w, w') = m_p(w, w')$. Then the mapping η defined above is an isomorphism of B onto the ring A of analytic functions on D if and only if $F_1 = G_2$ and $F_2 = -G_1$.*

Proof of Lemma. A simple calculation shows that η is an isomorphism of B onto B' if and only if $m = m_p$. So the condition concerning isomorphism in the lemma is that $B' = A$.

By adding and subtracting terms involving ψ the system (2) is seen to be equivalent to

$$\begin{aligned} &F_1(\phi_x - \phi_y) - F_2(\phi_y + \psi_x) + \psi_y(F_1 - G_2) + \psi_x(F_2 + G_1) = 0 \\ &F_1(\phi_y + \psi_x) + F_2(\phi_x - \psi_y) + \psi_y(F_2 + G_1) - \psi_x(F_1 - G_2) = 0 \ . \end{aligned} \tag{3}$$

First suppose $F_1 = G_2$ and $F_2 = -G_1$. Then this system becomes

$$\begin{aligned} &F_1(\phi_x - \psi_y) - F_2(\phi_y + \psi_x) = 0 \\ &F_1(\phi_y + \psi_x) + F_2(\phi_x - \psi_y) = 0 \ . \end{aligned} \tag{4}$$

It is clear that if $w^* = \phi + i\psi$ is analytic, then $\eta^{-1}(w^*)$ satisfies the system (4). Therefore $A \subset B'$

Suppose then that B' contains a function $w = \phi + i\psi$ which at some point z of D does not satisfy the Cauchy-Riemann equations. For this point the system (4) is a system of homogeneous algebraic equations with a non zero determinant whose value is

$$[\phi_x(z) - \psi_y(z)]^2 + [\phi_y(z) + \psi_y(z)]^2 .$$

Hence the only solution at z is the zero solution and thus

$$\mathfrak{Im}\,[\overline{F(z)}G(z)] = F_1^2(z) + F_2^2(z) = 0$$

which contradicts the definition of generating pair. Thus $B' \subset A$ and we have proved that $A = B'$.

Conversely suppose that η is an isomorphism onto A so that $A = B'$. Let $w^* = \phi + i\psi$ be a non constant analytic function in B'. Then the system (3) becomes for this w^*

$$\begin{aligned} \psi_y(F_1 - G_2) + \psi_x(F_2 + G_1) &= 0 \\ \psi_y(F_2 + G_1) - \psi_x(F_1 - G_2) &= 0 . \end{aligned} \tag{5}$$

If for some z the equations $F_1(z) = G_2(z)$ and $F_2(z) = -G_1(z)$ do not both hold, then the determinant of this system is non-zero at z and hence by continuity of F and G the determinant is non-zero in some neighborhood of z and hence $\psi_x = \psi_y = 0$ on this neighborhood. By the identity theorem for harmonic functions ψ_x and ψ_y must then be zero everywhere so that ψ is constant. A similar argument demonstrates the constancy of ϕ so that w^* is constant contrary to assumption. This completes the proof of the lemma.

Proof of Theorem. Suppose first that η is a multiplicative isomorphism of B onto the ring A of analytic functions in D. Then as before m is identically equal to m_p so that for $w = \phi F + \psi G \in B$ we have $m(w, G) = -\psi F + \phi G$. Substituting this function for $\phi F + \psi G$ in the system (1) yields that $m(w, G)$ is in B if and only if

$$\begin{aligned} -\psi_x F_1 + \psi_y F_2 + \phi_x G_1 - \phi_y G_2 &= 0 \\ -\psi_y F_1 - \psi_x F_2 + \phi_y G_1 + \phi_x G_2 &= 0 . \end{aligned} \tag{6}$$

By the lemma $F_1 = G_2$ and $F_2 = -G_1$. Using this to substitute for the G's in the system (6) we obtain the system (4) and this system must be satisfied because $\phi + i\psi$ is analytic. Thus if w is in B then so is $m(w, G)$ and the condition of the theorem is necessary.

Conversely suppose that there exists a non-constant w in B such that $m(w, G) = m_p(w, G) = -\psi F + \phi G$ and this function is in B. Then ϕ and ψ satisfy both (1) and (6) and since w is non-constant there must exist a z such that this system of four equations has a non-zero solution, i.e., the determinant of this system must be zero.

The determinant of this system is

$$[(F_1 - G_1)^2 + (F_2 + G_1)^2][(F_1 + G_2)^2 + (F_2 - G_1)^2] \, . \tag{7}$$

Now $\mathfrak{Im}(\bar{F}G) = F_1G_2 - F_2G_1$ which must be everywhere positive since F and G form a generating pair. If the second factor of (7) is zero, then it follows that

$$\mathfrak{Im}(\bar{F}G) = -F_1^2 - F_2^2 < 0 \, .$$

Hence the first factor must be zero and the lemma implies that η is an isomorphism of B onto A.

Bibliography

1. Lipman Bers, *Theory of Psuedo-analytic functions,* Mimeographed Lecture Notes, New York University, 1953.
2. ———, *Local Theory of Psuedo-analytic functions,* Lectures on Functions of a Complex Variable, Ann Arbor, Michigan, 1955, pp. 213–244.

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