THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS ON A CLOSED 2-CELL

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1. Introduction. If X is a metric space with metric ρ and $T(X)=X$ is a self-homeomorphism of X, then T is said to be expansive[1] provided there exists a $\delta > 0$ depending only upon X and T such that corresponding to each distinct pair $x, y \in X$ there exists an integer $n(x, y)$ for which $\rho(T^n(x), T^n(y)) > \delta$. W. H. Gottschalk [2] has asked if the n-cell can carry an expansive homeomorphism. B. F. Bryant [1] obtained a partial answer to this question when he essentially showed that there are no expansive self-homeomorphisms of a closed 1-cell, that is, of an arc. In this paper we show that there are no expansive self-homeomorphisms of a closed 2-cell and, in the final section, point out an error in a paper of R. F. Williams. The authors wish to acknowledge the referee's assistance in condensing the paper.

Throughout the paper, X will denote a metric space with metric ρ and $T(X) = X$ will denote a self-homeomorphism of X. The set $0(x)= \bigcup_{n\in I}\{T^n(x)\}$, where I denotes the integers, is called the orbit of x under T. A set $M \subset X$ is said to be minimal under T if, and only if, M is non-vacuous and M is the closure of the orbit of each of its points. If $x, y \in X$, then $0(x)$ and $0(y)$ are said to be positively (negatively) asymptotic if corresponding to $\varepsilon > 0$, there exists an integer N such that

$$\rho(T^n(x), T^n(y)) < \varepsilon \text{ for all } n > N (n < N).$$

If $0(x)$ and $0(y)$ are both positively and negatively asymptotic, then the orbits are said to be doubly asymptotic.

2. Self-homeomorphisms of the 2-cell. In this section we show with the aid of results of van Kampen that there is no expansive self-homeomorphism of a circle, and from this obtain the same result for a simple closed curve and a closed 2-cell.

THEOREM. *If T is a homeomorphism of a closed 2-cell onto itself, then T is not expansive.*

Proof. If there is an expansive homeomorphism, T, of a closed 2-cell onto itself then, since the boundary of the 2-cell is invariant under

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[1] In most of the literature cited, the term "unstable" is used in place of "expansive".

T, T must be expansive on the simple closed curve forming the boundary of the 2-cell. Since T is an expansive self-homeomorphism of a simple closed curve, there must be an expansive self-homeomorphism of a circle since it is known [1] that if T is an expansive self-homeomorphism of a metric space X and $g(X) = Y$ is a homeomorphism onto the metric space Y such that g^{-1} is uniformly continuous, then gTg^{-1} is an expansive self-homeomorphism of Y.

Hereafter we assume that T is an expansive self-homeomorphism of a circle, C. We first show that T cannot have a periodic point. If T has at least two distinct periodic points on C, then for some integer m, $T^m = \Phi$ has at least two fixed points on C and it is easy to see that either Φ or Φ^2 leaves an arc invariant. Powers of an expansive homeomorphism are expansive [3] and hence either Φ or Φ^2 is an expansive self-homeomorphism of an arc in violation of the cited result of Bryant.

If T has exactly one periodic point on C, then the point must be fixed under T and the orbit of every other point is doubly asymptotic to the fixed point. There are uncountably many such orbits contrary to the fact that when X is compact and T is an expansive self-homeomorphism of X, then the number of distinct orbits doubly asymptotic to any fixed point is at most countably infinite.

Since we have shown that T has no periodic point on C, C is either a minimal set under T, or [4] there is a minimal set which is a Cantor set and which consists of the common cluster points of orbits. In the first instance T is topologically equivalent to a rotation and is therefore not expansive. In the second instance, a component, A, of the complement of the minimal set is chosen. Now, $T^n(A)$ is an open arc and its diameter goes to zero with increasing or decreasing n. Taking two distinct points of A which are sufficiently close, they remain close for all n by virtue of the continuity of T. This contradicts the hypothesis that T is expansive and the theorem is proved.

3. An example of Williams. R. F. Williams [5] has given two examples of non-degenerate continua and self-homeomorphisms of them which are said to be expansive. One example, where the continuum is the inverse limit space of the unit circle in the complex plane under the bonding map $g(z) = z^2$ and with the shift homeomorphism, is expansive. The other example contains an error which we now explain.

Using the notation of Williams' example, let

$$a = \frac{10^n - 1}{10^n}, \qquad b = \frac{10^n + 1}{10^n}$$

and consider the points

$$x = (a, a/2, a/2^2, a/2^3, \cdots),$$
$$y = (a, b/2, b/2^2, b/2^3, \cdots)$$

for an arbitrary but fixed positive integer n. It is not difficult to see that the maximum value of $\rho(f^j(x), f^j(y))$ occurs for $j = -1$. Since

$$\rho(f^{-1}(x), f^{-1}(y)) = 1/10^n(1 + 1/2^2 + 1/2^4 + \cdots)$$

this maximum can be made arbitrarily small by taking n sufficiently large. Thus the homeomorphism f is not expansive.

The failure of this example suggests seeking another continuous function on $[0, 1]$ such that the shift homeomorphism of the inverse limit space onto itself is expansive. However, such an example is impossible. The authors can prove that the shift homeomorphism on the inverse limit space of any continuous transformation of an arc onto itself cannot be expansive. The proof of the theorem is long and will not be given here.

References

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