SINGULARITIES OF THREE-DIMENSIONAL HARMONIC FUNCTIONS

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Introduction. Recently G. Szegö [9] and Z. Nehari [8] have obtained some interesting results connecting the singularities of axially symmetric harmonic functions with those of analytic functions. In this paper we shall show that a similar connection also exists between the singularities of a three-dimensional harmonic function and a function of two complex variables. We may do this by considering the Whittaker-Bergman operator [10] [1] $B_3(f, \mathscr{L}, X_0)$ which transforms functions of two complex variables $f(t, u)$, into harmonic functions of three variables.

$$H(X) = B_3(f, \mathscr{L}, X_0),\ B_3(f, \mathscr{L}, X_0) = \frac{1}{2\pi i}\int_{\mathscr{L}} f(t, u)\frac{du}{u}$$

$$t = \left[-(x-iy)\frac{u}{2} + z + (x+iy)\frac{u^{-1}}{2}\right],$$

$$|X - X_0| < \varepsilon, \qquad X \equiv (x, y, z),\ X_0 \equiv (x_0, y_0, z_0)\ ,$$

where $\mathscr{L}$ is a closed differentiable arc[1] in the u-plane, and $\varepsilon > 0$ is sufficiently small. We may see how this operator maps the functions $f(t, u)$ into harmonic functions by considering the homogeneous polynomials of degree n in x, y, z, which are defined by

$$t^n = \left\{-(x-iy)\frac{u}{2} + z + (x+iy)\frac{u^{-1}}{2}\right\}^n = \sum_{m=-n}^{+n} h_{n,m}(x, y, z)u^{-m}\ .$$

The $h_{n,m}(x, y, z)$ are linearly independent polynomials, which form a complete system [4]. Now, any harmonic function regular in a neighborhood of the origin $|X| < \varepsilon$, may be expanded into a series

$$H(X) = H(x, y, z) = \textstyle\sum_{n=0}^{\infty} \sum_{1=-n}^{+n} a_{n,l} h_{n,l}(x, y, z),$$

which converges inside the smallest sphere on whose surface there is a singularity of $H(X)$.

From the definition of the harmonic polynomials we see that

$$\frac{1}{2\pi i}\int_{\mathscr{L}} t^n u^m \frac{du}{u} = h_{n,m}(x, y, z)\ ,$$

where $\mathscr{L}$ is, say, the unit circle. In spherical coordinates this result may be recognized as one of Heine's [7] integral representations for the

Received January 19, 1960. This work has been submitted to Carnegie Institute of Technology in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

[1] We shall usually consider $\mathscr{L}$ to be closed; however there is nothing preventing us from considering open arcs also.

associated Legendre functions.[2]

It follows then that if $H(X)$ is regular for $|X| < \varepsilon$ it may be generated by an integral operator

$$H(X) = \frac{1}{2\pi i}\int_{\mathscr{L}} f(t, u)\frac{du}{u} ,$$

where

$$f(t, u) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_{nm} t^n u^m .$$

The harmonic functions which are regular at infinity, $|X| > 1/\varepsilon$, are of the form

$$H^{\infty}(X) = \frac{1}{r} H\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) ,$$

and may also be generated by the Whittaker operator; however, in this case we use the functions

$$G(t, u) = \frac{1}{t}\sum_{n=0}^{\infty} \sum_{m=-n}^{+n} a_{nm} t^{-n} u^m .$$

How the functions $G(t, u)$ transform may be seen from Heine's other representation

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathscr{L}} t^{-n}u^m \frac{du}{tu} &= h^{\infty}_{n,m}(x, y, z) \\ &= \frac{(n-m)!\,(n+m)!}{(n!)^2 2^n}\,\frac{1}{r} h_{n,m}\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) \\ &\frac{(n-m)!}{n!}(-i)^m r^{-n-1} P_n^m(\cos\theta) e^{im\varphi} , \end{aligned}$$

where, as before, $\mathscr{L}$ is the unit circle.

Occasionally it is convenient to continue the arguments x, y, z to complex values in order to study the behavior of $H(X)$. For instance, if we introduce, as a particular continuation, the complex spherical coordinates

$$r = +\,(x^2 + y^2 + z^2)^{1/2} .$$

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[2] By introducing spherical coordinates

$$\begin{aligned} x &= r \sin\theta \cos\varphi , \\ y &= r \sin\theta \sin\varphi , \\ z &= r \cos\theta , \end{aligned}$$

the polynomials may be written in the form $h_{n,m}(x, y, z) = (n!/(n + m!))r^n P_n^m(\cos\theta)e^{im\varphi}$

Integrals of terms $t^n u^m$, where $|m| > |n| > 0$, vanish; consequently, we may restrict ourselves to just those functions where $|m| \leqq |n|$.

$$\zeta = +\left(\frac{x+iy}{x-iy}\right)^{1/2},$$

$$\xi = \frac{z}{r},$$

which reduce to $\zeta = e^{i\varphi}$, $\xi = \cos\theta$, for real x, y, z, we may obtain an inverse Whittaker operator.

LEMMA. *Let $V(r, \cos\theta, e^{i\varphi})$, be a harmonic function regular at infinity*; i.e.

$$V(r, \cos\theta, e^{i\phi}) \equiv H^{\infty}(X) = \frac{1}{2\pi i}\int_{\mathscr{L}} G(t, u)\frac{du}{u},$$

where

$$G(t, u) = \left(\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} a_{nm}t^{-n-1}u^{m}\right),$$

and $\mathscr{L}$ is the unit circle.

Then $G(s, u)$ may be generated by the integral transform

$$G(s, u) = \frac{1}{4\pi i}\int_{-1}^{+1}\left[\int_{\Gamma}\frac{r(s+t)}{(s-t)^2}V(\gamma, \xi, \zeta)\frac{d\zeta}{\zeta}\right]d\xi .$$

The integration path in the ξ-plane is the linear segment $-1 \leqq \xi \leqq 1$, the path in ζ-plane is the unit circle.

Proof. Let us define

$$\frac{1}{r}K\left(\frac{r}{t}, \xi, \frac{iu}{\zeta}\right) \equiv \sum_{n=0}^{\infty}\sum_{m=-n}^{+n}(2n+1)\frac{n!}{(n+m)!}\left(\frac{r}{t}\right)^{n}P_n^m(\xi)\left(\frac{iu}{\zeta}\right)^{m};$$

it follows then, directly from the orthogonality relation

$$\int_{-1}^{+1}P_n^m(\xi)P_s^m(\xi)d\xi = \delta_{ns}\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!},$$

that

$$s^{-n}u^{m} = \frac{1}{4\pi i}\int_{-1}^{+1}\left[\int_{\Gamma}K\left(\frac{r}{s}, \xi, \frac{iu}{\zeta}\right)\left(\frac{\zeta}{i}\right)^{m}\frac{d\zeta}{\zeta}\right]\frac{(n-m)!}{n!}r^{-n-1}P_n^m(\xi)d\xi$$

(where the integration paths are those mentioned in the hypothesis). Recalling the generating function for the spherical harmonics

$$\sum_{n=0}^{\infty}t^{n} = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n}\frac{n!}{(n+m)!}r^{n}P_n^m(\xi)\left(\frac{\zeta}{iu}\right)^{m}$$

we see that K may be formally summed to

$$K\left(\frac{r}{s}, \xi, \frac{iu}{\zeta}\right) = \left(1 - 2s\frac{\partial}{\partial s}\right)r\sum_{n=0}^{\infty}\left(\frac{t}{s}\right)^n = rs\frac{s+t}{(s-t)^2} ,$$

providing $(|t/s|) < 1$. In this case, K is an analytic function of t, and hence also analytic in r, ξ, and ζ. The harmonic functions $H^{\infty}(x, y, z)$, which are regular at infinity, have a Taylor series expansion of the form $\sum_{j,k,l=0}^{\infty} A_{jkl}x^{-j}y^{-k}z^{-l}$.

If this series converges for $x^2 + y^2 + z^2 > (1/\varepsilon^2)$, then the series

$$\sum_{j,k,l=0}^{\infty} A_{jkl}(x_1 + ix_2)^{-j}(y_1 + iy_2)^{-k}(z_1 + iz_2)^{-l} ,$$

if rewritten in the form

$$\sum_{\substack{a,b,c\\ r,s,t}} B_{abcrst}x_1^{-a}x_2^{-r}y_1^{-b}y_2^{-s}z_1^{-c}z_2^{-t} ,$$

will converge for $x_1^2 + y_1^2 + z_1^2 > (2/\varepsilon^2)$, and $x_2^2 + y_2^2 + z_2^2 > (2/\varepsilon^2)$. Hence, $H^{\infty}(x, y, z)$ is an analytic function of the complex variable x, y, z, in some neighborhood of infinity. The harmonic function $V(r, \xi, \zeta)$ obtained by replacing x, y, z in $H^{\infty}(x, y, z)$ by

$$x = \frac{r}{2}(\zeta + \zeta^{-1})\sqrt{1-\xi^2} ,$$

$$y = \frac{r}{2i}(\zeta - \zeta^{-1})\sqrt{1-\xi^2} ,$$

$$z = r\xi ,$$

consequently is an analytic function of r, ξ, ζ, except of course at $\xi = \pm 1$, and $\zeta = 0$.

It may be concluded, therefore, that the integrals involved in our representation for $G(s, u)$ are Cauchy-integrals, since the integrand is a single-valued analytic function of ξ and ζ.

II. Singularities of harmonic functions generated by the Whittaker-Bergman operator. Bergman [2] has considered a special class of harmonic functions generated by the Whittaker operator and has given a simple procedure for finding their singularities. He does this as follows:

Suppose that $(1/u)f(t, u)$ has the form $P(t, u)/Q(t, u)$, where P and Q are polynomials in t and u. In order to study the harmonic function

$$H(X) = B_3(f, \mathscr{L}, X_0) = \frac{1}{2\pi i}\int_{\mathscr{L}} f(t, u)\frac{du}{u} ,$$

we consider the singularity manifold of P/Q, i.e.

$$Z^3 = E\left\{Q\left[-(x-iy)\frac{u}{2} + z + (x+iy)(2u)^{-1}, u\right] = 0\right\}.$$

The manifold Z^3 may also be written in the form

$$Z^3 = E\{u = \phi_\nu(X), \nu = 1, 2, 3, \cdots n\},$$

where the $\phi_\nu(X)$ are algebraic functions of x, y, z, and the degree of u in Q is n. At every point (x, y, z), except those which satisfy the equation

$$\prod_{K \neq s} [\phi_K(X) - \phi_s(X)] = 0,$$

there are n distinct branches of $Z^3_\nu = E\{u = \phi_\nu(X), \nu = 1, 2, 3, \cdots, n\}$, of Z^3. We choose the contours $\mathscr{L}_\nu$, $v = 1, 2, 3, \cdots, n$, so that one and only one $u = \phi_\nu(X)$ lies inside $\mathscr{L}_\nu$. It follows from the residue theorem that

$$H_\nu(X) = \frac{1}{2\pi i}\int_{\mathscr{L}_\nu} \frac{P(t, u)}{Q(t, u)} du,$$

where $H_\nu(X)$ is the corresponding branch of

$$H(X) = \frac{P[-(x-iy)u/2 + z + (x-iy)(2u)^{-1}, u]}{\partial\{Q[-(x-iy)u/2 + z + (x-iy)(2u)^{-1}, u]\}/\partial u},$$

with

$$Q\left[-(x-iy)\frac{u}{2} + z + (x-iy)(2u)^{-1}, u\right] = 0.$$

We notice that $H(X)$ becomes singular for those values of (x, y, z) which satisfy the equations

$$Q\left[-(x-iy)\frac{u}{2} + z + (x-iy)(2u)^{-1}, u\right] = 0,$$

$$\partial\left\{Q\left[-(x-iy)\frac{u}{2} + z + (x-iy)(2u)^{-1}, u\right]\right\}/\partial u = 0.$$

We shall now show that Bergman's result does not depend on the fact that $(1/u)f(t, u)$ is an algebraic function, but holds under more general conditions. The only restriction we will impose is that the singularities of $(1/u)f(t, u)$ can be written in the implicit form $S(x, y, z, u) = 0$.

THEOREM 1. *If $Z^3 = E\{S(x, y, z, u) = 0\}$ is an implicit representation of the singularities of*

$$\frac{1}{u}f(t, u),\ \textit{then}\ H(X) = \frac{1}{2\pi i}\int_{\mathscr{L}} f(t, u)\frac{du}{u} ,$$

(where $\mathscr{L}$ is the unit circle) *is regular at* $X = (x, y, z)$, *providing this point does not lie simultaneously on the two surfaces*

$$S(x, y, z, u) = 0 ,$$

and

$$\frac{\partial}{\partial u}S(x, y, z, u) = 0 .$$

Proof. The proof of Theorem 1 will be based on a modified form of an idea employed by Hadamard in the proof of his theorem on the multiplication of singularities [8] [5]. The integral representation of $H(X)$ is valid for all points (x, y, z) which can be reached from an initial point by continuation along a curve $\Gamma(X)$ (in three dimensional real-space, R^3), provided no point of $\Gamma(X)$ corresponds to a singularity of $(1/u)f(t, u)$ on the integration path. This initial domain of definition of $H(X)$ can now be enlarged by continuously deforming the integration path provided, again, that in this process of deformation the integration path at no time crosses a singularity of $(1/u)f(t, u)$. Accordingly, we may now write $H(X)$ as

$$H(X) = \frac{1}{2\pi i}\int_{\mathscr{L}'} f(t, u)\frac{du}{u} ,$$

where $\mathscr{L}'$ is now a new integration path obtained by observing the above precautions.

Since t is dependent on $X = (x, y, z)$, the singularities of the integral move in the u-plane as we continue $H(X)$ along $\Gamma(X)$. Now, as long as we can avoid crossing such a singularity by deforming the contour $\mathscr{L}'$ we are still able to continue $H(X)$. Let us assume we have been able to continue $H(X)$ to the point $X_1 \equiv (x_1, y_1, z_1)$, and let us consider the singularities of the integral for $X = X_1$. The singularities of $(1/u)f(t, u)$ are those values of u satisfying $S(x_1, y_1, z_1, u) = 0$. From Taylor's theorem we may describe the local properties of S about some point $u = \alpha$, for which $S = 0$, by

$$S(x_1, y_1, z_1, u) = (u - \alpha)\frac{\partial}{\partial u}S(x_1, y_1, z_1, \alpha) + \frac{(u-\alpha)^2}{2!}\frac{\partial^2 S(x_1, y_1, z_1, \alpha)}{\partial u^2} \cdots .$$

Unless $\partial S/\partial u = 0$ at $u = \alpha$, in a neighborhood of $u = \alpha$ we may approximate S by

$$S(x_1, y_1, z_1, u) \cong (u - \alpha)\frac{\partial S(x_1, y_1, z_1, \alpha)}{\partial u} .$$

Therefore in some neighborhood of $u = \alpha$, say $|u - \alpha| < \varepsilon$, S does not vanish save at $u = \alpha$. Clearly, then, by deforming $\mathscr{L}'$ we can avoid crossing $u = \alpha$, or any other point $u = \beta$ for which $S(x_1, y_1, z_1, \beta) = 0$, if we follow an arc of the circle $|u - \alpha| = \varepsilon/2$ about $u = \alpha$. This completes our proof.

Using the language of real geometry we may say that unless we are in the neighborhood of the envelope $\mathscr{E}(x, y, z) = 0$ to $S(x, y, z, u) = 0$ (in which case there are an infinite number of such surfaces tangent to $\mathscr{E}(x, y, z) = 0$ we may avoid crossing these singularities by deforming $\mathscr{L}'$.

THEOREM 2. *Let $r = \Phi(\xi, \zeta)$ be a representation of the singularities of $V(r, \xi, \zeta) \equiv H^{\infty}(X)$, $X \in C^3$. The function of two complex variables*

$$G(s, u) = \frac{1}{4\pi i}\int_{-1}^{+1}\left[\oint_{\zeta}\frac{s + t}{(s - t)^2} r V(r, \xi, \zeta)\frac{d\zeta}{\zeta}\right]d\xi ,$$

is then regular at (s, u) providing (s, u) does not lie on the "envelope" of the two parameter family

$$\psi(s, u \,|\, \xi, \zeta) \equiv \Phi(\xi, \zeta)\left[\xi + \frac{i}{2}\sqrt{1 - \xi^2}\left(\frac{u}{\zeta} + \frac{\zeta}{u}\right)\right] - s = 0 .$$

Proof. The proof of this theorem closely parallels the one for Theorem 1. As before, we consider the analytic continuation of $G(s, u)$ along an arc $\Gamma^4(s^{-1}, u)$, beginning at $s^{-1} = 0, u = 1$. The integral representation of

$$G(s, u) = \frac{1}{4\pi i}\int_{-1}^{+1}\left[\oint_{\zeta}\frac{s + t}{(s - t)^2} r V(r, \xi, \zeta)\frac{d\zeta}{\zeta}\right]d\xi$$

will remain the same if either integration path (in ξ or ζ planes) is continuously deformed in such a manner so that at no time they cross a singularity of the integrand. Therefore, we may write $G(s, u)$ as

$$G(s, u) = \frac{1}{4\pi i}\int_{-1\mathscr{L}_{\xi}}^{+1}\left[\int_{\mathscr{L}_{\zeta}}\frac{s + t}{(s - t)^2} r V(r, \xi, \zeta)\frac{d\zeta}{\zeta}\right]d\xi$$

where $\mathscr{L}_{\xi}$ and $\mathscr{L}_{\zeta}$ are new integration paths obtained by observing the above precautions. Now, the kernel in our integral representation is singular whenever

$$t - s = r\left[\xi + \frac{i}{2}\sqrt{1 - \xi^2}\left(\frac{u}{\zeta} + \frac{\zeta}{u}\right)\right] - s = 0 ,$$

and the harmonic function is singular for $\Phi(\xi, \zeta) - r = 0$. We notice a significant difference in these two singularity manifolds; as $G(s, u)$ is continued along $\Gamma^4(s^{-1}, u)$ the singularities of the kernel move in the

ξ, ζ-planes, while those of the harmonic function remain fixed. By using the Hadamard idea we realize that we may always avoid an advancing singularity by deforming one of our contours with the possible exception occuring when the two manifolds coincide. Therefore, unless $r = \Phi(\xi, \zeta)$ as a function of ξ, and ζ also satisfies $t - s = 0, G(s, u)$ must be regular. This leads us to consider the two parameter family,

$$\psi(s, u \,|\, \xi, \zeta) \equiv \Phi(\xi, \zeta)\left[\xi + \frac{i}{2}\sqrt{1-\xi^2}\left(\frac{u}{\zeta} + \frac{\zeta}{u}\right)\right] - s = 0 \, ,$$

as the only possible singularities of $G(s, u)$.

Let us assume that we have been able to continue $G(s, u)$ to (s_0, u_0) and let us consider those values of ξ, ζ satisfying $\psi(s_0, u_0 \,|\, \xi, \zeta) = 0$. These values are singularities of the integrand which must be investigated to determine whether they are avoidable by deforming the paths of integration. Let $\xi = \alpha$, and $\zeta = \beta$ be singularities which may cross either $\mathscr{L}_\xi$ or $\mathscr{L}_\zeta$ respectively if $G(s, u)$ is continued further along $\Gamma^4(s^{-1}, u)$. In a bicylindrical neighborhood $|\xi - \alpha| < \varepsilon_1, |\zeta - \beta| < \varepsilon_2$, we may expand $\psi(s_0, u_0 \,|\, \xi, \zeta)$ in a double Taylor series as

$$\psi(s_0, u_0 \,|\, \xi, \zeta) = (\xi - \alpha)\frac{\partial}{\partial \xi}\psi(s_0, u_0 \,|\, \alpha, \beta) + (\zeta - \beta)\frac{\partial}{\partial \zeta}\psi(s_0, u_0 \,|\, \alpha, \beta)$$
$$+ \frac{1}{2}\left[(\xi - \alpha)^2\frac{\partial^2 \psi}{\partial \xi^2} + 2(\xi - \alpha)(\zeta - \beta)\frac{\partial^2 \psi}{\partial \xi \partial \zeta} + (\zeta - \beta)^2\frac{\partial^2 \psi}{\partial \zeta^2}\right] + \cdots \, .$$

Now, unless the first variation of $\psi(s_0, u_0 \,|\, \xi, \zeta)$ vanishes at (α, β), ψ may be approximated as

$$\psi(s_0, u_0 \,|\, \xi, \zeta) \cong (\xi - \alpha)\frac{\partial}{\partial \xi}\psi(s_0, u_0 \,|\, \alpha, \beta) + (\zeta - \beta)\frac{\partial}{\partial \zeta}\psi(s_0, u_0 \,|\, \alpha, \beta) \, .$$

In this case it is always possible to choose a secant to the circle $|\xi - \alpha| = \varepsilon_1/2$ not passing through $\xi = \alpha$, and a secant to the circle $|\zeta - \beta| = \varepsilon_2/2$ not passing through $\zeta = \beta$, such that $\psi(s_0, u_0 \,|\, \xi, \zeta) \neq 0$ on those portions of the secants inside the respective circles. It follows that, in this case, we may deform the paths $\mathscr{L}_\xi$, and $\mathscr{L}_\zeta$ so that they follow the secants about the singular point (α, β) and thereby continue $G(s, u)$ still further. The only possible singularities of $G(s, u)$ are therefore those values of s and u satisfying simultaneously

$$\psi(s, u \,|\, \xi, \zeta) = 0$$

and

$$\frac{\partial \psi}{\partial \xi} + \frac{\partial \psi}{\partial \zeta}\pi'(\xi) = 0 \, ,$$

where $\zeta = \pi(\xi)$ is an arbitrary relationship between ξ and ζ. This completes our proof.

We notice here, that a particular class of singularities of $G(s, u)$ may occur for s and u satisfying simultaneously

$$\psi(s, u \,|\, \xi, \zeta) = 0 ,$$

$$\frac{\partial \psi}{\partial \xi} = 0 ,$$

and

$$\frac{\partial \psi}{\partial \zeta} = 0 .$$

We have reduced the problem of locating the singularities of $G(s,u)$ to obtaining the envelope of a three parameter family of complex surfaces

$$\psi(s, u \,|\, r, \xi, \zeta) = 0 ,$$

where the parameters r, ξ, ζ are subject to the condition

$$A(r, \xi, \zeta) = 0 .$$

It was most natural, because of the Cauchy integrals involved, to consider ξ and ζ as independent parameters, and r the dependent parameter. However, unless we are in the neighborhood of a "singular point" of $A = 0$, it is no longer necessary to make this distinction.

For a point (s, u) to lie on the envelope $E(s, u) = 0$, the first variation,

$$\delta\psi = \frac{\partial \psi}{\partial r}\delta r + \frac{\partial \psi}{\partial \xi}\delta\xi + \frac{\partial \psi}{\partial \zeta}\delta\zeta ,$$

must vanish. If we proceed as before, and consider r dependent, we obtain

$$\left(\frac{\partial A}{\partial r}\frac{\partial \psi}{\partial \xi} - \frac{\partial A}{\partial \xi}\frac{\partial \psi}{\partial r}\right)\delta\xi + \left(\frac{\partial A}{\partial r}\frac{\partial \psi}{\partial \zeta} - \frac{\partial A}{\partial r}\frac{\partial \psi}{\partial r}\right)\delta\zeta = 0 ,$$

which implies that an arbitrary functional relationship exists between ξ and ζ, or more generally a relationship $B(r, \xi, \zeta) = 0$, such that

$$\left(\frac{\partial B}{\partial r}\frac{\partial \psi}{\partial \xi} - \frac{\partial B}{\partial \xi}\frac{\partial \psi}{\partial r}\right)\delta\xi + \left(\frac{\partial B}{\partial r}\frac{\partial \psi}{\partial \zeta} - \frac{\partial B}{\partial \zeta}\frac{\partial \psi}{\partial r}\right)\delta\zeta = 0 ,$$

where

$$\begin{vmatrix} \dfrac{\partial A}{\partial r}\dfrac{\partial \psi}{\partial \xi} - \dfrac{\partial A}{\partial \xi}\dfrac{\partial \psi}{\partial r} & \dfrac{\partial A}{\partial r}\dfrac{\partial \psi}{\partial \zeta} - \dfrac{\partial A}{\partial \zeta}\dfrac{\partial \psi}{\partial r} \\ \dfrac{\partial B}{\partial r}\dfrac{\partial \psi}{\partial \xi} - \dfrac{\partial B}{\partial \xi}\dfrac{\partial \psi}{\partial r} & \dfrac{\partial B}{\partial r}\dfrac{\partial \psi}{\partial \zeta} - \dfrac{\partial B}{\partial \zeta}\dfrac{\partial \psi}{\partial r} \end{vmatrix} = 0 .$$

Let us consider the envelope of $\psi(s, u | r, \xi, \zeta) = 0$ [subject to $A(r, \xi, \zeta) = 0$] under the transformation of parameters

$$\begin{aligned} r &= +(x^2 + y^2 + z^2)^{1/2} \\ \xi &= z / +(x^2 + y^2 + z^2)^{1/2} , \\ \zeta &= +\left(\frac{x + iy}{x - iy}\right)^{1/2} . \end{aligned}$$

We realize that, for $X = (x, y, z) \ni R^3$, the Jacobian cannot vanish and hence the transformation is one-to-one. However, as may be confirmed by direct computation

$$\frac{\partial(r, \xi, \zeta)}{\partial(x, y, z)} \neq 0, \text{ for all } X \in C^3 ,$$

which are a finite distance from the origin.

Under this transformation our family of complex surfaces becomes

$$\{\psi(s, u | r, \xi, \zeta) = 0\} \to \{\chi(s, u | x, y, z) = 0\} ,$$

with the auxiliary condition

$$\{A(r, \xi, \zeta) = 0\} \to \{P(x, y, z) = 0\} .$$

Now, for a point (s, u) to lie on the envelope to $\chi = 0$, the first variation must vanish, i.e.

$$\begin{aligned} \delta\chi &= \frac{\partial \chi}{\partial x}\delta x + \frac{\partial \chi}{\partial y}\delta y + \frac{\partial \chi}{\partial z}\delta z = 0 \\ &= \left(\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)\delta x \\ &\quad + \left(\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right)\delta y \\ &\quad + \left(\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial \psi}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)\delta z \\ &= \frac{\partial \psi}{\partial r}\left(\frac{\partial r}{\partial x}\delta x + \frac{\partial r}{\partial y}\delta y + \frac{\partial r}{\partial z}\delta z\right) \\ &\quad + \frac{\partial \psi}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\delta x + \frac{\partial \xi}{\partial y}\delta y + \frac{\partial \xi}{\partial z}\delta z\right) \\ &\quad + \frac{\partial \psi}{\partial \zeta}\left(\frac{\partial \zeta}{\partial y}\delta x + \frac{\partial \zeta}{\partial y}\delta y + \frac{\partial \zeta}{\partial z}\delta z\right) = 0 . \end{aligned}$$

From our auxiliary condition we have

$$\begin{aligned}\Big[\frac{\partial r}{\partial x}\delta x + \frac{\partial r}{\partial y}\delta y + \frac{\partial r}{\partial z}\delta z\Big] \\ = -\frac{\partial A}{\partial \xi}\Big[\frac{\partial \xi}{\partial x}\delta x + \frac{\partial \xi}{\partial y}\delta y + \frac{\partial \xi}{\partial z}\delta z\Big]\Big/\frac{\partial A}{\partial r} \\ -\frac{\partial A}{\partial \zeta}\Big[\frac{\partial \zeta}{\partial x}\delta x + \frac{\partial \zeta}{\partial y}\delta y + \frac{\partial \zeta}{\partial z}\delta z\Big]\Big/\frac{\partial A}{\partial r}\ ,\end{aligned}$$

which, together with $\delta\chi = 0$, yields

$$\Big[\frac{\partial A}{\partial r}\frac{\partial \psi}{\partial \xi} - \frac{\partial A}{\partial \xi}\frac{\partial \psi}{\partial r}\Big][\delta\xi(x, y, z)] + \Big[\frac{\partial A}{\partial r}\frac{\partial \psi}{\partial \zeta} - \frac{\partial A}{\partial \zeta}\frac{\partial \psi}{\partial r}\Big][\delta\zeta(x, y, z)] = 0\ .$$

We conclude that under a one-to-one, continuous transformation of parameters the envelope is invariant and we have the following corollary to Theorem 2. Let $F(x, y, z) = 0$ be a representation of the singularities of $H^{\infty}(X)$, $X \in C^3$. Then the function $G(s, u)$, which generates $H^{\infty}(X)$ under the Whittaker operator, can only have singularities on the envelope, $E(s, u) = 0$, to the family of complex surfaces

$$\chi(s, u\,|\,x, y, z) \equiv \Big[-(x - iy)\frac{u}{2} + z + (x + iy)\frac{1}{2u}\Big] - s = 0\ ,$$

where the parameters (x, y, z) are subject to the auxiliary condition $F(x, y, z) = 0$.

To illustrate the use of Theorem 1, we consider the case where $(1/u)f(t, u)$ has the particular form

$$\frac{1}{u}f(t, u) = F\Big[t^{-1}\Big(u - \frac{1}{u}\Big)\Big]\ ;$$

$F(x)$ is an arbitrary function of x singular at $x = \beta$. This choice of $(1/u)f(t, u)$ generates an $H(X)$ having a simple type of singularity. Since the singularities of $(1/u)f(t, u)$ satisfy $u - (1/u) = t\beta$, we represent the singularity manifold as

$$S(x, y, z, u) = -u[\beta(x - iy) + 2] + 2\beta z + \frac{1}{u}[\beta(x + iy) + 2]\ .$$

Eliminating u between $S = 0$, and $\partial S/\partial u = 0$, we obtain the locus $(x + 2/\beta)^2 + y^2 + z^2 = 0$, for the singularities of $H(X)$.

When β is real this reduces to a point singularity in R^3. However, if β is complex the singularities in R^3 are given by

$$x = -\frac{2}{|\beta|^2}\Re\beta\ ,$$

$$y^2 + z^2 = \frac{4}{|\beta|^4}(\mathfrak{Im}\bar{\beta})^2 .$$

We note that these are only the possible singularities of $H(X)$. To find the actual singularities we make use of our inverse Whittaker operator to find which of the possible singularities of $H(X)$ correspond to singularities of $(1/u)f(t, u)$.

Let us consider the locus of

$$\left(x + \frac{2}{\beta}\right)^2 + y^2 + z^2 = 0 \text{ in } R^3, \text{ that is}$$

$$x = -\frac{2}{|\beta|^2}\mathfrak{Re}\beta ,$$

and

$$y^2 + z^2 = \frac{4}{|\beta|^4}(\mathfrak{Im}\beta)^2 .$$

If we wish to find which singularities of $(1/u)f(t, u)$ correspond to this real locus, we eliminate two parameters from χ and consider the first variation with respect to the remaining parameter. Doing this,

$$\chi = -\frac{x}{2}\left(u - \frac{1}{u}\right) + \frac{iy}{2}\left(u + \frac{1}{u}\right) + z - s = 0, \text{ becomes}$$

$$\chi = 2\frac{\mathfrak{Re}\beta}{|\beta|^2}\left(u - \frac{1}{u}\right) \pm \frac{i}{2}\sqrt{\frac{4}{|\beta|^4}(\mathfrak{Im}\bar{\beta})^2 - z^2}\left(u + \frac{1}{u}\right) + z - s = 0 .$$

The first variation is then

$$\frac{\partial\chi}{\partial z} = \frac{\pm\frac{i}{2}(-z)\left(u + \frac{1}{u}\right)}{\sqrt{\frac{4}{|\beta|^2}(\mathfrak{Im}\bar{\beta})^2 - z^2}} + 1$$

Eliminating z, between χ and $\partial\chi/\partial z$ yields

$$\pm \mp i(\mathfrak{Im}\beta)\left(u + \frac{1}{u}\right)^2 + (\mathfrak{Re}\beta)\left(u - \frac{1}{u}\right) \pm 4i(\mathfrak{Im}\beta)$$
$$= s|\beta|^2\left(u - \frac{1}{u}\right) .$$

By choosing suitable signs this is recognized readily as

$$\left(u - \frac{1}{u}\right) = \beta s .$$

REMARK. In concluding we note, that as in the case of harmonic functions regular at the origin, a connection will exist between the coefficients of the series development for $f(t, u)$ and the singularities of $H(X)$[3]. Hence, it would be of interest to investigate whether a relation exists between singularities as predicted by Theorem 1 of this paper, and the corresponding coefficients of the series development for $f(t, u)$. Such an investigation should lead to a classification of harmonic functions in terms of their pole-like singularities in three-dimensional complex space.

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[3] References to Integral Operators in the Theory of Linear Partial Differential Equations, see Bergman [3] and Kreyszig [6].