# ON EXPANSIVE HOMEOMORPHISMS

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**1. Introduction.** A homeomorphism $\phi$ of a compact metric space $X$ onto $X$ is said to be *expansive* provided there exists $d > 0$ such that if $x, y \in X$ with $x \neq y$, then there exists an integer $n$ such that $\rho(x\phi^n, y\phi^n) > d$ (see [1] and [3]). The question arises as to the possibility of extending the results concerning expansive homeomorphisms to compact uniform spaces. The extension is possible, although trivial in light of the corollary to Theorem 1.

In §§ 3 and 4 the setting is a compact metric space $X$. Theorem 2 is stronger than Theorem 10.36 of [1] in that we do not require $X$ to be self-dense. Also, the lemmas of which Theorem 2 is a consequence are perhaps of some interest in themselves. In § 4 we show that if $X$ is self-dense, then for each $x \in X$ and each $\varepsilon > 0$ there exists $y \in U(\varepsilon, x)$ such that $x$ and $y$ are not doubly asymptotic.

**2.** A homeomorphism $\phi$ of a compact uniform space $(X, \mathscr{U})$ onto $(X, \mathscr{U})$ is said to be expansive provided there exists $U \in \mathscr{U}$ such that $U \neq \varDelta$ (the diagonal) and if $x, y \in X$ with $x \neq y$, then there exists an integer $n$ such that $(x\phi^n, y\phi^n) \notin \bar{U}$. For uniform spaces we use the notation of [2], but following Weil [4] we suppose $(X, \mathscr{U})$ is Hausdorff; i.e., $\cap\{U: U \in \mathscr{U}\} = \varDelta$. We also suppose that each $U \in \mathscr{U}$ is symmetric.

THEOREM 1. *Let $(X, \mathscr{U})$ be a compact uniform space which is not metrizable and let $\phi$ be a homeomorphism of $X$ onto $X$. If $U \in \mathscr{U}$, then there exist $x, y \in X$ with $x \neq y$ such that $(x\phi^n, y\phi^n) \in U$ for each integer $n$. (Compare with Theorem* 10.30 *of* [1].)

*Proof.* Select $V \in \mathscr{U}$ such that $V \circ V \circ V \subset U$ and $\bar{V} \subset U$ (see [2], p. 180). Since $\phi^n$, for each integer $n$, is uniformly continuous, we may choose $U_1 \in \mathscr{U}$ with $U_1 \subset V$ such that $(p, q) \in U_1$ implies $(p\phi^k, q\phi^k) \in V$ for $k = \pm 1$. For $i > 1$, choose $U_i \in \mathscr{U}$ with $U_i \subset U_{i-1}$ such that $(p, q) \in U_i$ implies $(p\phi^k, q\phi^k) \in V$ for $k = \pm i$. Since $(X, \mathscr{U})$ is not metrizable, the countable set $\{U_i \mid i = 1, 2, \cdots\}$ is not a base for the uniformity $\mathscr{U}$([4], p. 16). Thus there exists $W \in \mathscr{U}$ with $W \subset U$ such that $i \geqq 1$ implies $U_i \cap \operatorname{comp} W \neq 0$. Choose, for each $i, (x_i, y_i) \in U_i \cap \operatorname{comp} W$. Since $X \times X$ is a compact Hausdorff space, there exists $(x, y)$ such that each neighborhood of $(x, y)$ contains $(x_i, y_i)$ for an infinite number of positive integers $i$. Let $n$ be an arbitrary positive integer, then there exists $m > n$ such that $(x_m, y_m) \in U_n(x) \times U_n(y)$. Hence $(x, x_m) \in U_n$ and $(y, y_m) \in U_n$;

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therefore $(x\phi^k, x_m\phi^k) \in V$ and $(y\phi^k, y_m\phi^k) \in V$ for $k = \pm n$. Also $(x_m, y_m) \in U_m \subset U_n$ so that $(x_m\phi^k, y_m\phi^k) \in V$ for $k = \pm n$. Hence $(x\phi^k, y\phi^k) \in V \circ V \circ V \subset U$ for $k = \pm n$. Each $(x_i, y_i) \in U_i \subset V$ and $\bar{V} \subset U$; hence $(x, y) \in U$. Finally, $x \neq y$. For otherwise we could choose $S \in \mathscr{U}$ such that $S \circ S \subset W$; then $(x_k, y_k) \in S(x) \times S(x)$ for some $k$, and hence $(x, x_k) \in S$, $(x, y_k) \in S$ so that $(x_k, y_k) \in W$. This completes the proof.

An immediate consequent of the theorem is the following

COROLLARY. *If $(X, \mathscr{U})$ is a compact uniform space on which it is possible to define an expansive homeomorphism, then $(X, \mathscr{U})$ is metrizable.*

3. The author is indebted to the referee for suggesting the arrangement of the material in this section. In the original version, Lemma 2 had a slightly stronger hypothesis and Lemma 3 was essentially contained in the proof of Theorem 2. In this section we suppose that $X$ is an infinite compact metric space and (with the exception of Lemma 3) that $\phi$ is an expansive homeomorphism (with expansive constant $d$) of $X$ onto $X$.

LEMMA 1. *If $x \neq y$ and if there is an integer $N$ such that $n > N$ $\{n < N\}$ implies $\rho(x\phi^n, y\phi^n) \leqq d$, then $x$ and $y$ are positively $\{$negatively$\}$ asymptotic under $\phi$.*

*Proof.* If $x$ and $y$ are not positively asymptotic under $\phi$, then there exist $\varepsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $\rho(x\phi^{n_i}, y\phi^{n_i}) \geqq \varepsilon$ with $\lim_{i\to+\infty} x\phi^{n_i} = u$ and $\lim_{i\to+\infty} y\phi^{n_i} = v$. Obviously $u \neq v$. Let $m$ be an arbitrary integer. For all $i$ sufficiently large $n_i + m > N$; hence $\rho(x\phi^{n_i+m}, y\phi^{n_i+m}) \leqq d$. Since $\lim_{i\to+\infty} x\phi^{n_i+m} = u\phi^m$ and $\lim_{i\to+\infty} y\phi^{n_i+m} = v\phi^m$, it is clear that $\rho(u\phi^m, v\phi^m) \leqq d$ for each integer $m$. This contradicts the hypothesis that $\phi$ is expansive. The alternative statement may be proved by a similar argument.

LEMMA 2. *If $\omega(x)\{\alpha(x)\}$ contains a periodic point $p$ and $\omega(x)\{\alpha(x)\}$ is not identical with the orbit of $p$, then there exist $w$ and $z$ in $\omega(x)$ $\{\alpha(x)\}$ such that $w$ and $p$ are positively asymptotic and $z$ and $p$ are negatively asymptotic.*

*Proof.* Suppose $p$ is of period $k$. There exist positive integers $n_1 < n_2 < \cdots$ such that $\lim_{i\to+\infty} x\phi^{n_i} = p$. Let $k_i$ be the smallest nonnegative integer such that $n_i + k_i$ is a multiple of $k$. Since $0 \leqq k_i < k$, there exists $m$ such that $k_i = m$ for an infinite number of integers $i$. Thus there are positive integers $m_1 < m_2 < \cdots$ such that

$$\lim_{i\to+\infty} x\phi^{m_i+m} = \lim_{i\to+\infty} x\phi^{kj_i} = p\phi^m .$$

Denote $\phi^k$ by $\theta$ (with expansive constant $d_1$) and $p\phi^m$ by $q$ (see [1], p. 86). Thus $\lim_{i\to+\infty} x\theta^{j_i} = q$ and $q\theta = q$. We can assume that $\rho(x\theta^{j_i}, q)<d_1$ for each $i$.

The points $x$ and $q$ are not positively asymptotic under $\theta$, since otherwise $\omega(x)$ under $\phi$ would consist of the $k$ points in the orbit of $p$. Hence, by Lemma 1, there exist arbitrarily large integers $r$ such that $\rho(x\theta^r, q)>d_1$. Therefore we can assume that $s_1 < s_2\cdots$ are positive integers where $s_i$ is the smallest positive integer such that $\rho(x\theta^{j_i+s_i}, q) > d_1$ and $\lim_{i\to+\infty} x\theta^{j_i+s_i} = u \in \omega(x)$. Let $-a$ be an arbitrary negative integer, then for all $i$ sufficiently large $0 < s_i - a < s_i$. Hence $\rho(x\theta^{j_i+s_i-a}, q) \leqq d_1$, and therefore $\rho(u\theta^{-a}, q) \leqq d_1$ for each negative integer $-a$. Thus, by Lemma 1, $u$ is negatively asymptotic to $q$ under $\theta$ and hence under $\phi$([1], p. 85). We can assume $j_i < j_i + s_i < j_{i+1}$ and hence that there exist positive integers $t_2<t_3<\cdots$ where $t_i$ is the smallest positive integer such that $\rho(x\theta^{j_i-t_i}, q) > d_1$ and $\lim_{i\to+\infty} x\theta^{j_i-t_i} = v \in \omega(x)$. By an argument similar to the above, $v$ is positively asymptotic to $q$ under $\phi$. Since $\alpha(x)$ under $\phi$ coincides with $\omega(x)$ under $\phi^{-1}$, this completes the proof.

In the following lemma we do not require $\phi$ to be expansive.

LEMMA 3. *If $x$ is not periodic and $\omega(x)\{\alpha(x)\}$ is the orbit of a periodic point $p$, then there exists a point $q$ in the orbit of $p$ such that $q$ and $x$ are positively $\{$negatively$\}$ asymptotic.*

*Proof.* Let $p \in \omega(x)$ and, as in the first paragraph of the proof of Lemma 2, select positive integers $j_1 < j_2 < \cdots$ such that $\lim_{i\to+\infty} x\theta^{j_i} = q = p\phi^m$ and $q\theta = q$, $\theta = \phi^k$. If $x$ and $q$ are not positively asymptotic under $\theta$, then there exists a positive constant $\alpha$ and a sequence $n_1<n_2<\cdots$ of integers such that $\rho(x\theta^{n_i}, q) > \alpha$. Let $\varepsilon > 0$ and choose $\beta > 0$ such that $\beta < \varepsilon$, $\beta < \alpha$, and $\rho(z, w) \leqq \beta$ implies $\rho(z\theta, w\theta) < \varepsilon$. We can assume that $\rho(x\theta^{j_i}, q) < \beta$. Let $s_i$ be the smallest positive integer such that $\rho(x\theta^{j_i+s_i}, q) > \beta$. Then for each $i$, $\beta < \rho(x\theta^{j_i+s_i}, q) < \varepsilon$. But the sequence $\{x\theta^{j_i+s_i}\}$ has a convergent subsequence. Let $s$ be the limit of such a convergent subsequence, then $s \neq q$, $s \in \omega(x)$ and $\rho(s, q) \leqq \varepsilon$. Thus $\omega(x)$ is not finite, contrary to hypothesis. It follows that $x$ and $q$ are positively asymptotic under $\theta$, and hence under $\phi$.

Similarly, if $\alpha(x)$ is the orbit of a periodic point $p$, then there exists a point $q$ in the orbit of $p$ such that $q$ and $x$ are negatively asymptotic under $\phi$.

THEOREM 2. *There exist $a, b, c, d \in X$ such that $a$ and $b$ are positively asymptotic under $\phi$ and $c$ and $d$ are negatively asymptotic under $\phi$.*

*Proof.* There exists a minimal set $N \subset X$ ([1], p. 15). If $N$ is infinite, then $N$ is self-dense and the conclusion follows from Theorem 10.36 of [1]. Henceforth, suppose each minimal set in $X$ is finite and thus is a periodic orbit.

Since $X$ is compact and infinite, there exists a non-isolated point $r$. If $r$ is not periodic, let $r = p$. If $r$ is periodic, then there exists $x \neq r$ such that $x$ and $r$ are asymptotic ([1], p. 87). But then $x$ is not periodic and we let $x = p$.

There exists a minimal set $N \subset \omega(p)$, and a minimal set $M \subset \alpha(p)$. Both $N$ and $M$ are periodic orbits. If $N \neq \omega(p)$ or $M \neq \alpha(p)$ the conclusion of the theorem follows from Lemma 2. If $N = \omega(p)$ and $M = \alpha(p)$, the conclusion of the theorem follows from Lemma 3.

4. In addition to the standing hypothesis of § 3 we require $X$ to be self-dense.

LEMMA 4. *If $y \in U(\varepsilon, x)$ implies that each neighborhood of $y$ contains $z$ such that $\rho(y\phi^n, z\phi^n) > d/2$ for some positive {negative} $n$, then there exists $w \in U(\varepsilon, x)$ such that $w$ and $x$ are not positively {negatively} asymptotic.*

*Proof.* Let $0 < \alpha < \varepsilon$, then there exist $x_1 \in U(\alpha, x)$ and a positive integer $n_1$ such that $\rho(x_1\phi^{n_1}, x\phi^{n_1}) > d/2$. Suppose $x_1$ and $x$ are positively asymptotic (otherwise the lemma holds); hence there exists $m_1 > n_1$ such that $n > m_1$ implies $\rho(x_1\phi^n, x\phi^n) < d/8$. Choose $\alpha_1 > 0$ such that $U(\alpha_1, x_1) \subset U(\alpha, x)$ and $\rho(p, q) < \alpha_1$ implies $\rho(p\phi^n, p\phi^n) < d/8$ for $0 \leqq n \leqq m_1$. For $i > 1$ we select $x_i, n_i, m_i$, and $\alpha_i > 0$ such that $x_i \in U(\alpha_{i-1}, x_{i-1})$, $n_i > m_{i-1}$, $\rho(x_i\phi^{n_i}, x_{i-1}\phi^{n_i}) > d/2$, $m_i > n_i$, $n > m_i$ implies $\rho(x_i\phi^n, x\phi^n) < d/8$, $U(\alpha_i, x_i) \subset U(\alpha_{i-1}, x_{i-1})$, and $\rho(p, q) < \alpha_i$ implies $\rho(p\phi^n, q\phi^n) < d/8$ for $0 \leqq n \leqq m_i$. We can suppose $\lim_{i \to +\infty} x_i = w \in \overline{U(\alpha, x)} \subset U(\varepsilon, x)$ and $w \neq x$. If $i > 1$, then $n_i > m_{i-1}$ and hence $\rho(x_{i-1}\phi^{n_i}, x\phi^{n_i}) < d/8$. But $\rho(x_i\phi^{n_i}, x_{i-1}\phi^{n_i}) > d/2$, and the triangle inequality implies $\rho(x_i\phi^{n_i}, x\phi^{n_i}) > 3d/8$. If $j > i$, then $x_j \in U(\alpha_i, x_i)$ and, since $m_i > n_i$, $\rho(x_j\phi^{n_i}, x_i\phi^{n_i}) < d/8$. Therefore $\rho(x_j\phi^{n_i}, x\phi^{n_i}) > d/4$ for $j \geqq i$. If $i > 1$ is fixed, then $\rho(x_j\phi^{n_i}, w\phi^{n_i})$ is arbitrarily small for $j$ sufficiently large. Hence $\rho(x\phi^{n_i}, w\phi^{n_i}) \geqq d/4$. Since $\{n_i\}$ is an increasing sequence of positive integers, $w$ and $x$ are not positively asymptotic. This proof establishes the alternative statement by using $\phi^{-1}$ rather than $\phi$.

THEOREM 3. *For each $x \in X$ and each $\varepsilon > 0$ there exists $y \in U(\varepsilon, x)$ such that $x$ and $y$ are not doubly asymptotic.*

*Proof.* Suppose there exist $x \in X$ and $\varepsilon > 0$ such that $z \in U(\varepsilon, x)$ implies $x$ and $z$ are positively asymptotic. Suppose $\varepsilon < d/2$, then, by the above lemma, there exist $y \in U(\varepsilon, x)$ and $\alpha > 0$ such that $U(\alpha, y) \subset U(\varepsilon, x)$ and $t \in U(\alpha, y)$ implies that $\rho(t\phi^n, y\phi^n) \leqq d/2$ for $n \geqq 0$. Therefore $u, v \in U(\alpha, y)$ implies $\rho(u\phi^n, v\phi^n) \leqq d$ for $n \geqq 0$. Thus, since $\phi$ is expansive, $u, v \in U(\alpha, y)$ implies $\rho(u\phi^n, v\phi^n) > d$ for some negative $n$. By the alternative form of the lemma above, there exists $w \in U(\alpha, y)$ such that $w$ and $y$ are not negatively asymptotic. Therefore either $w$ and $x$ are not negatively asymptotic or $y$ and $x$ are not negatively asymptotic, which establishes the theorem.

If $X$ is an infinite minimal set, then a stronger statement can be made. Since $X$ is pointwise almost periodic under $\phi$([1], p. 31), $\varepsilon > 0$ implies $\rho(x, x\phi^n) < \varepsilon$ for some $n \neq 0$. It is easy to show that $x$ and $x\phi^n$ are neither positively nor negatively asymptotic.

If $X$ is not self-dense, then, as shown by the following example, each pair of distinct points may be both positively and negatively asymptotic. Let $X$ consist of the real numbers $0, 1/n \{n = \pm 1, \pm 2, \cdots\}$, and let

$$x\phi = \begin{cases} 0 \quad \text{if} \quad x = 0\,. \\ 1/(n+1) \quad \text{if} \quad x = 1/n \quad \text{and} \quad n \neq -1\,. \\ 1 \quad \text{if} \quad x = -1. \end{cases}$$

## References

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