POSITIVE OPERATORS COMPACT IN AN AUXILIARY TOPOLOGY

F. F. BONSALL

Of the several generalizations to infinite dimensional spaces of the Perron-Frobenius theorem on matrices with non-negative elements, two are outstanding for their freedom from ad hoc conditions.

THEOREM A (Krein and Rutman [3] Theorem 6.1). *If the positive cone K in a partially ordered Banach space E is closed and fundamental, and if T is a compact linear operator in E that is positive (i.e., $TK \subset K$) and has non-zero spectral radius ρ, then ρ is an eigenvalue corresponding to positive eigenvectors of T and of T^*.*

THEOREM B ([4] p. 749 [1] p. 134). *If the positive cone K in a partially ordered normed space E is normal*[1] *and has interior points, and if T is a positive linear operator in E, then the spectral radius is an eigenvalue of T^* corresponding to a positive eigenvector.*

In [2], we have proved the following generalization of Theorem A.

THEOREM C. *Let the positive cone K in a normed and partially ordered space E be complete, and let T be a positive linear operator in E that is continuous and compact in K. If the partial spectral radius μ of T is non-zero, then μ is an eigenvalue of T corresponding to a positive eigenvector.*

Also in [2], we have developed a single method of proof of Theorems A, B, C which exploits the fact that the resolvent operator is a geometric series, and thus avoids the use of complex analysis or any other deep method.

In [5] (Theorems (10.4), (10.5)), Schaefer has further extended these results by showing that (A) and (C) remain valid for operators in locally convex spaces, with suitable definitions of spectral radius and partial spectral radius.

Our aim in the present article is to unify these theorems still further. We prove a single theorem (Theorem 1) that contains Theorem C (and hence A), and also contains Theorem B except in the case $\rho = 0$, for which an extra gloss is needed (Theorem 2). The central idea is that instead of being compact in K in the norm topology, T maps the part of the unit ball in K into a set that is compact with respect to a

Received February 19, 1960.

[1] K is said to be a *normal* cone if there exists a positive constant κ such that
$$\|x + y\| \geqq \kappa \|x\| \qquad (x, y \in K) .$$

second linear topology, this topology being related to the norm topology in a certain way. This idea is, in essence, derived from the recent paper [6] of Schaefer, though his conditions are too restrictive for our purpose. Again we use only elementary real analysis of the kind used in [2]. After proving our two main theorems, we exhibit a number of examples of situations in which these theorems are applicable.

NOTATION. We suppose that E is a normed and partially ordered real linear space with norm $\|\cdot\|$, norm topology τ_N, and positive cone K; i.e., K is a non-empty set satisfying the axiom:

(i) $x, y \in K$, $\alpha \geqq 0$ imply $x + y$, $\alpha x \in K$,

(ii) $x, -x \in K$ imply $x = 0$.

We write $x \leqq y$ or $y \geqq x$ to denote that $y - x \in K$.

We suppose that K is complete with respect to the norm. However, we do not require that E be complete, so that there is no real loss of generality in supposing that $E = K - K$, and we shall therefore suppose that this is the case. We exclude the trivial case in which $K = (0)$.

We denote by B the intersection of K with the closed unit ball in E, i.e., $B = \{x : x \in K \text{ and } \|x\| \leqq 1\}$, and suppose that T is a linear operator in E that is positive $(TK \subset K)$ and partially bounded (i.e., $\|Tx\|$ is bounded on B). We denote the partial bound of T by $p(T)$ i.e.,

$$p(T) = \sup \{\|Tx\| : x \in B\} ,$$

and by μ the partial spectral radius

$$\mu = \lim_{n\to\infty} \{p(T^n)\}^{1/n} .$$

We are indebted to H. H Schaefer for several helpful suggestions, and in particular for pointing out that substantial simplification can be obtained by introducing a second norm q into E defined as follows. Let B_0 denote the convex symmetric hull of B, i.e.,

$$B_0 = \{\alpha x + \beta y\colon\ x, y \in B, |\alpha| + |\beta| = 1\} ,$$

and let q be the gauge functional of B_0,

$$q(x) = \inf \{\lambda\colon\ \lambda > 0 \text{ and } \lambda^{-1}x \in B_0\} .$$

It is easily verified that q is a norm in E, that $q(x) \geqq \|x\|$ $(x \in E)$, and that $q(x) = \|x\|$ $(x \in K)$. Also the completeness of K with respect to the given norm implies that E and K are complete with respect to q.

Given a positive operator T, the partial bound and the partial spectral radius are the usual operator norm and spectral radius for the operator T in the Banach space (E, q). For $\lambda > \mu$, the resolvent operator

$R_\lambda = (\lambda I - T)^{-1}$ is given by the series

$$R_\lambda = \frac{1}{\lambda} I + \frac{1}{\lambda^2} T + \frac{1}{\lambda^3} T^2 + \cdots .$$

which converges in the operator norm for (E, q), and is a partially bounded positive operator.

We suppose that we are given a second linear topology τ in E, such that K is (τ)-closed and T is (τ)-continuous in K.

DEFINITION. Given a subset A of K, we say that τ is *sequentially stronger than* τ_N *at* 0 *relative to* A if 0 is a (τ_N)-cluster point of each sequence of points of A of which it is a (τ)-cluster point.

THEOREM 1. *If* TB *is contained in a* (τ)*-compact set,* τ *is sequentially stronger than* τ_N *at* 0 *relative to* TB, *and* $\mu > 0$, *then there exists a non-zero vector* u *in* K *with* $Tu = \mu u$.

THEOREM 2. *If* B *is contained in a* (τ)*-compact set, and* τ *is sequentially stronger than* τ_N *at* 0 *relative to* B, *then there exists a non-zero vector* u *in* K *with* $Tu = \mu u$.

Since $TB \subset p(T)B$, Theorem 2 is contained in Theorem 1 except when $\mu = 0$.

The proofs of these theorems will depend on the following two lemmas. Lemma 1, which is needed in the proof of Lemma 2, is repeated from [2] in order to make the present paper self-contained.

LEMMA 1. *Let* $\{a_n\}$ *be an unbounded sequence of non-negative real numbers. Then there exists a subsequence* $\{a_{n_k}\}$ *such that*

(i) $$a_{n_k} > k \qquad (k = 1, 2, \cdots),$$

(ii) $$a_{n_k} > a_j \qquad (j < n_k,\ k = 1, 2, \cdots).$$

Proof. By induction. With $n_1, \cdots, n_{k-1}$ chosen to satisfy (i) and (ii), let n_k be the smallest positive integer r with $a_r > a_{n_{k-1}} + k$.

LEMMA 2. *If* TB *is contained in a* (τ)*-compact set, and* τ *is sequentially stronger than* τ_N *at* 0 *relative to* TB, *then*

$$\lim_{\lambda \to \mu + 0} p(R_\lambda) = \infty .$$

Proof. Suppose that the conditions of the lemma are satisfied, but that $p(R_\lambda)$ does not tend to infinity as λ decreases to μ. Then there exists a positive constant M such that $p(R_\nu) \leqq M$ for some ν greater than and arbitrarily close to μ.

The case $\mu = 0$ is easily settled. For if $\mu = 0$, then

$$\lambda R_\lambda x \geqq x \qquad (\lambda > 0,\ x \in K),$$

and letting λ tend to zero through values for which $p(R_\lambda) \leqq M$, we obtain $-x \in K$, $K = (0)$. This is the trivial case that we have excluded.

Suppose now that $\mu > 0$. Then we may choose λ, ν with

$$0 < \lambda < \mu < \nu < \lambda + M^{-1}$$

and with $p(R_\nu) \leqq M$. With this choice of λ, ν the series

$$R_\nu + (\nu - \lambda)R_\nu^2 + (\nu - \lambda)^2 R_\nu^3 + \cdots$$

converges in operator norm for the Banach space (E, q) to a partially bounded positive operator S with

$$Sx = \lambda^{-1}x + \lambda^{-1}TSx \qquad (x \in K).$$

Thus

$$Sx \geqq \lambda^{-1}TSx \qquad (x \in K),$$

and therefore

$$Sx \geqq \lambda^{-(n+1)}T^n x \qquad (x \in K,\ n = 1, 2, \cdots). \tag{1}$$

Since $\lim_{n\to\infty} p(\lambda^{-(n+1)}T^n) = \infty$, and since the partial bound of a positive operator coincides with its operator norm in (E, q), the principle of uniform boundedness implies that there exists a point $x \in E$ with $q(\lambda^{-(n+1)}T^n x)$ unbounded. Since $E = K - K$, it follows that there exists $w \in K$ for which the sequence $(\| \lambda^{-(n+1)}T^n w \|)$ is unbounded. Therefore, by Lemma 1, there exists a subsequence such that

$$\lim_{k\to\infty} \| \lambda^{-(n_k+1)}T^{n_k} w \| = \infty\ , \tag{2}$$

$$\| \lambda^{-(n_k+1)}T^{n_k} w \| \geqq \| \lambda^{-n_k}T^{n_k-1} w \|\ . \tag{3}$$

Since

$$\| T^{n_k} w \| \leqq p(T) \| T^{n_k-1} w \|\ ,$$

we also have

$$\lim_{k\to\infty} \| \lambda^{-n_k}T^{n_k-1} w \| = \infty\ . \tag{4}$$

Let $y_k = \| T^{n_k-1}w \|^{-1} T^{n_k-1}w$. Then, by (1), there exists $z_k \in K$ with

$$\| \lambda^{-n_k}T^{n_k-1}w \|^{-1} Sw = \lambda^{-1}Ty_k + z_k \qquad (k = 1, 2, \cdots). \tag{5}$$

By (4) and (5), we have

$$\lambda^{-1}Ty_k + z_k \to 0 \quad (\tau)\ . \tag{6}$$

Since $y_k \in B$ and TB is contained in a (τ)-compact set, the sequence $(\lambda^{-1}Ty_k)$ has a (τ)-cluster point y in K. By (6), $-y$ is a (τ)-cluster point of (z_k), and since $z_k \in K$ and K is (τ)-closed, $-y \in K$. Thus $y = 0$, and 0 is a (τ)-cluster point of (Ty_k). But τ is sequentially stronger than τ_N at 0 relative to TB, and so 0 is a (τ_N)-cluster point of (Ty_k). But this is absurd, for, by (3),

$$\| Ty_k \| \geqq \lambda \| y_k \| = \lambda .$$

Proofs of Theorems 1 *and* 2. Since $TB \subset p(T)B$, Lemma 2 is available under the conditions of each theorem, and gives

$$\lim_{\lambda \to \mu + 0} p(R_\lambda) = \infty .$$

Then, applying the principle of uniform boundedness as in the proof of Lemma 2, we see that there exists a sequence (λ_n) converging decreasingly to μ, and a point w in K with $\| w \| = 1$ and

$$\lim_{n \to \infty} \| R_{\lambda_n} w \| = \infty ,$$

and we may suppose that $R_{\lambda_n} w \neq 0$ $(n = 1, 2, \cdots)$. Let $\alpha_n = \| R_{\lambda_n} w \|^{-1}$, and $u_n = \alpha_n R_{\lambda_n} w$. Then

$$\mu u_n - T u_n = (\mu - \lambda_n) u_n + \alpha_n w . \tag{8}$$

Under the conditions of Theorem 2, the proof is easily completed. For, since $u_n \in B$ and B is contained in a (τ)-compact set, it follows from (8) that

$$\mu u_n - T u_n \to 0 \quad (\tau) .$$

Also (u_n) has a (τ)-cluster point u in K, and since T is (τ)-continuous in K, we have

$$\mu u - T u = 0 .$$

We have $u \neq 0$, for otherwise 0 is a (τ_N)-cluster point of (u_n), which is absurd, since $\| u_n \| = 1$.

Finally, suppose that the conditions of Theorem 1 are satisfied. Then, by (8),

$$(\mu I - T) T u_n = T(\mu I - T) u_n = (\mu - \lambda_n) T u_n + \alpha_n T w .$$

Since TB is contained in a (τ)-compact set, it follows that

$$(\mu I - T) T u_n \to 0 \quad (\tau) ,$$

and (Tu_n) has a (τ)-cluster point v in K. Therefore, by the (τ)-continuity of T,

$$(\mu I - T)v = 0 .$$

If $v = 0$, then 0 is a (τ_N)-cluster point of (Tu_n). But, by (8),

$$\mu u_n - Tu_n \to 0 \quad (\tau_N) ,$$

and so 0 is a (τ_N)-cluster point of (μu_n). Since $\mu \neq 0$ and $\|u_n\| = 1$, this is absurd. Hence $v \neq 0$, and the proof is complete.

It will be noticed that the preceding theorems and lemmas remain true if compactness is replaced by countable compactness, no change in the proofs being required. It may be of interest to remark that under the conditions of Theorem 2, K is a normal cone. However, since this fact is not needed for our main purpose, we omit its proof.

EXAMPLE 1. Taking $\tau = \tau_N$ in Theorem 1, we obtain Theorem C, and hence, as we have seen in [2], Theorem A also.

EXAMPLE 2. Suppose that there exists a subset A of K with the following properties:

(i) Given $x \in E$ with $\|x\| \leqq 1$, there exists $a \in A$ with $-a \leqq x \leqq a$.

(ii) TA is contained in a (τ_N)-compact set.[2]

Let E^* denote the usual dual space of continuous linear functionals on the normed space E, and let K^* denote the dual cone of all elements of E^* that are non-negative on K. Then K^* is a norm complete positive cone in E^*, and we denote by B^* the intersection of K^* with the closed unit ball in E^*.

For each φ in E^*, let $T^*\varphi$ be defined as usual by

$$(T^*\varphi)(x) = \varphi(Tx) \qquad (x \in E).$$

Since T is not necessarily a bounded operator in E, $T^*\varphi$ may fail to belong to E^*. However, $T^*K^* \subset K^*$, and T^* is a partially bounded operator in $K^* - K^*$. For, given $\varphi \in B^*$ and $x \in E$ with $\|x\| \leqq 1$, there exists $a \in A$ with $-a \leqq x \leqq a$, and therefore

$$-\varphi(Ta) \leqq \varphi(Tx) \leqq \varphi(Ta) .$$

Since TA is contained in a (τ_N)-compact set, the set $\{\|Ta\| : a \in A\}$ has a finite upper bound M and so $|\varphi(Tx)| \leqq M$, $\|T^*\varphi\| \leqq M$, $T^*B^* \subset MB^*$, T^* is partially bounded. It is easily seen that T^* is weak*-continuous in K^* and that K^* is weak*-closed.

We shall show that if the partial spectral radius μ^* of T^* is not zero, then Theorem 1 is applicable to the operator T^* in the space $K^* - K^*$ with the weak* topology as the auxiliary topology τ. This will prove the existence of a non-zero element ψ of K^* with

[2] In Examples 2, 3 no auxiliary topology is needed in E, but an auxiliary topology will appear in the dual space.

$$T^*\psi = \mu^*\psi .$$

Since T^* maps B^* into the weak*-compact set MB^*, we need only prove that the weak* topology is sequentially stronger than the norm topology at 0 relative to T^*B^*. To prove this, let $\varphi_n \in B^*$ $(n = 1, 2, \cdots)$, and suppose that 0 is a weak*-cluster point of the sequence $(T^*\varphi_n)$. Since TA is contained in a (τ_N)-compact set, given $\varepsilon > 0$, there exist $a_1, \cdots, a_r$ in A such that for each point a in A there is some k $(1 \leqq k \leqq r)$ with

$$\| Ta - Ta_k \| < \varepsilon/2 . \tag{9}$$

Since 0 is a weak*-cluster point of $(T^*\varphi_n)$, there exists an infinite set $\varLambda$ of positive integers such that

$$|(T^*\varphi_n)(a_k)| < \varepsilon/2 \qquad (k = 1, \cdots, r;\ n \in \varLambda), \tag{10}$$

i.e.,

$$|\varphi_n(Ta_k)| < \varepsilon/2 \qquad (k = 1, \cdots, r;\ n \in \varLambda).$$

By (9) and (10), we have

$$|\varphi_n(Ta)| < \varepsilon \qquad (a \in A,\ n \in \varLambda). \tag{11}$$

Given $x \in E$ with $\| x \| \leqq 1$, there exists $a \in A$ with $-a \leqq x \leqq a$, and so, by (11),

$$|(T^*\varphi_n)(x)| = |\varphi_n(Tx)| \leqq \varphi_n(T_a) < \varepsilon \qquad (n \in \varLambda),$$
$$\| T^*\varphi_n \| \leqq \varepsilon \qquad (n \in \varLambda).$$

Therefore 0 is a norm-cluster point of $(T^*\varphi_n)$, and we have proved that Theorem 1 is applicable.

EXAMPLE 3. Suppose that there exists a subset A of K with the following properties:

(i) Given $x \in E$ with $\| x \| \leqq 1$, there exists $a \in A$ with $-a \leqq x \leqq a$.

(ii) A is contained in a (τ_N)-compact set.

Let K^*, B^*, T^* be defined as in Example 2. Given $\varphi \in B^*$ and $x \in E$ with $\| x \| \leqq 1$, there exists $a \in A$ with $-a \leqq x \leqq a$, and therefore

$$|\varphi(Tx)| \leqq \varphi(Ta) \leqq \| Ta \| \leqq p(T) \| a \| .$$

Since A is contained in a (τ_N)-compact set, $\| a \|$ is bounded on A, and T^* is a partially bounded mapping of K^* into itself.

We show that Theorem 2 is applicable to the operator T^*. Since K^* is weak*-closed, B^* is weak*-compact, and T^* is weak*-continuous in K^*, we need only prove that the weak* topology is sequentially stronger than the norm topology at 0 relative to B^*. This is proved by an argument similar to that in Example 2, but using A in place of TA.

It follows that there exists a non-zero element ψ of K^* with $T^*\psi = \mu^*\psi$, where μ^* is the partial spectral radius of T^*.

In particular, the conditions of this example are satisfied with A consisting of a single point if K contains an interior point in the normed space E. Thus Theorem B is contained in this example, and hence in Theorem 2.

EXAMPLE 4. Theorem 1 of Schaefer [6] is a case of our Theorem 2. In this case the topology τ is given, and Schaefer constructs a norm in $K - K$ in such a way that

$$\|x\| = f(x) \qquad (x \in K),$$

where f is a certain (τ)-continuous linear functional. Since f is (τ)-continuous, it is easily verified that τ is sequentially stronger than τ_N at 0 relative to B.

REFERENCES

1. F. F. Bonsall, *Endomorphisms of partially ordered vector spaces*, J. London Math. Soc., **30** (1955), 133-144.

2. ———, *Linear operators in complete positive cones*, Proc. London Math. Soc., (3) **8** (1958), 53-75.

3. M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space* (Russian), Uspehi Mat. Nauk (N.S.), 3, No. 1, **23** (1948), 3-95, English translation: American Math. Soc., Translation 26.

4. M. Krein, *Sur les opérations lineáires transformant un certain ensemble conique en lui-même*, C.R. (Doklady) Acad. Sci. U.R.S.S. (N.S.), **23** (1939), 749-752.

5. H. Schaefer, *Halbgeordnete lokalkonvexe Vektorräume*. II, Math. Annalen **138** (1959), 259-286.

6. ———, *Some spectral properties of positive linear operators*, Pacific J. Math., **10** (1960), 1009-1019.

DURHAM UNIVERSITY
NEWCASTLE-ON-TYNE