CRITERION FOR rTH POWER RESIDUACITY

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The Law of Quadratic Reciprocity in the rational integers states: If p, q are two distinct odd primes, then q is a square $(\bmod p)$ if and only if $(-1)^{(p-1)/2}p$ is a square $(\bmod q)$.

One of the classical generalizations of the law of reciprocity is of the following type. Let r be a fixed positive integer, $\phi(r)$ denotes the number of positive integers $\leqq r$ which are relatively prime to r; p, q are two distinct primes and $p \equiv 1 \pmod r$. Then can we find rational integers $a_1(p), a_2(p), \cdots, a_h(p)$ determined by p, such that q is an rth power $(\bmod p)$ if and only if $a_1(p), \cdots, a_h(p)$ satisfy certain conditions $(\bmod q)$.

The Law of Quadratic Reciprocity states that for $r = 2$, we may take $a_1(p) = (-1)^{(p-1)/2}p$.

Jacobi and Gauss solved this problem for $r = 3$ and $r = 4$, respectively. Mrs. E. Lehmer gave another solution recently [2].

In this paper I would like to develop the theory when r is a prime and $q \equiv 1 \pmod r$. I then show that q is an rth power $(\bmod p)$ if and only if a certain linear combination of $a_1(p), \cdots, a_{r-1}(p)$ is an rth power $(\bmod q)$. $a_1(p), \cdots, a_{r-1}(p)$ are determined by solving several simultaneous Diophantine equations. This determination appears mildly formidable and to make the actual numerical computations would certainly be so for a large r. (See Theorem B below.) Also given is a criterion for when r is an rth power $(\bmod p)$ in terms of a linear combination of $a_1(p), \cdots, a_{r-1}(p) \pmod{r^2}$. (See Theorem A below.)

It is possible by the methods developed in this paper to eliminate the conditions that r is a prime and $q \equiv 1 \pmod r$. This would complicate the paper a great deal, and the cases given clearly indicate the underlying theory.

Consider the following Diophantine equations in the rational integers:

$$r \sum_{j=1}^{r-1} X_j^2 - \left(\sum_{j=1}^{r-1} X_j\right)^2 = (r-1)p^{r-2} \tag{1}$$

$$\textstyle\sum_1^{(1)} X_{j_1}X_{j_2} = \sum_i^{(1)} X_{j_1}X_{j_2} \qquad i = 2, \cdots, \dfrac{r-1}{2}\,, \tag{2}$$

where $\sum_i^{(k)}$ denotes the sum over all $j_1, \cdots, j_{k+1} = 1, 2, \cdots, r-1$, with the condition $j_1 + \cdots + j_k - kj_{k+1} \equiv i \pmod r$.

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$$(3) \qquad 1+\sum_{j=1}^{r-1} X_j \equiv \sum_{j=1}^{r-1} jX_j \equiv 0 \pmod r$$

(4) not all of the $X_j \equiv 0 \pmod p$ and

$$\sum\nolimits_i^{(k)} X_{j_1} \cdots X_{j_{k+1}} - \sum\nolimits_0^{(k)} X_{j_1} \cdots X_{j_{k+1}} \equiv 0 \pmod{p^{r-k-1}}$$

for $k = 2, \cdots, r-2$; $i = 1, 2, \cdots, r-1$.

We shall prove in § II that there exist exactly $r-1$ distinct integral solutions of the equations (1) through (4). In particular let $\{X_j = a_j, j = 1, \cdots, r-1\}$ be a solution. Then we prove that the $a_j(p) = a_j$ satisfy our residuacity criterion, namely

THEOREM A. *r is an rth power $\pmod p$ if and only if*

$$\sum_{j=1}^{r-1} ja_j + \frac{1}{2} ra_{r-1} \equiv 0 \pmod{r^2} .$$

THEOREM B. *If $q \equiv 1 \pmod r$ and h is any integer such that h^r is the least power of h which is $\equiv 1 \pmod q$, then q is an rth power $\pmod q$ if and only if $\sum_{j=1}^{r-1} a_j h^j$ is an rth power $\pmod q$.*

At the end of § II various special cases are considered.

In particular, for $q = 2$, $r = 5$, then 2 is a quintic power $\pmod p$ if and only if $a_j \equiv a_{5-j} \pmod 2$, $j = 1, 2$.

For $q = 2$, $r = 7$, then 2 is a 7th power $\pmod p$ if and only if $a_j \equiv 1 \pmod 2$, $i = 1, \cdots, 6$.

Let $r = 3$. Then the solutions to the Diophantine equations (1) to (4) are (a_1, a_2) and (a_2, a_1), where

$$(5) \qquad p = a_1^2 - a_1a_2 + a_2^2,\ a_1 \equiv a_2 \equiv 1 \pmod 3 .$$

Multiplying (5) by 4 and grouping terms gives

$$4p = (a_1 + a_2)^2 + 3(a_1 - a_2)^2 .$$

Let $L = -a_1 - a_2$, $M = (a_1 - a_2)/3$. This gives the representation which Lehmer employs:

$$4p = L^2 + 27M^2,\ L \equiv 1 \pmod 3 .$$

Theorem A states that 3 is a cubic residue $(\text{mod } p)$ if and only if $a_1 \equiv a_2 \pmod 9$. This, in turn, is equivalent to M being divisible by 3, the condition quoted by Lehmer.

I. **Notation.** r denotes a prime number, ζ_r a primitive rth root of unity, Q the rational numbers, $Q(\zeta_r)$ the cyclotomic field over Q generated by ζ_r. For $j = 1, 2, \cdots, r-1$, σ_j are the automorphisms of $Q(\zeta_r)/Q$

such that $\sigma_j(\zeta_r) = \zeta_r^j$. $\sigma^{-1}(\zeta_r) = \zeta_r^{j'}$, where $jj' \equiv 1 \pmod r$. p denotes a positive rational prime $\equiv 1 \pmod r$, and $\chi_p = \chi$ will be any primitive rth power character $(\mathrm{mod}\, p)$.

$$g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta_p^n$$

will be the Gaussian sum associated with χ_p. $\langle\alpha\rangle$ denotes the fractional part of α; i.e., $\langle\alpha\rangle = \alpha - [\alpha]$.

LEMMA 1. (i) $|g(\chi^k)|^2 = p$,
(ii) $g(\chi)^k g(\chi^{-k}) \in Q(\zeta_r)$,
(iii) $g(\chi)^r \in Q(\zeta_r)$, and
(iv) $\sigma_k(g(\chi)^r) = g(\chi^k)^r$
for $k = 1, 2, \cdots, r-1$.

Proof. (i) is the classical result about the absolute value of $g(\chi)$ and can easily be deduced from the definition of $g(\chi)$. (ii), (iii) and (iv) follow from Galois Theory using the relation $\sum_{n=1}^{p-1} \chi(n)\zeta_p^{nt} = \chi(t)^{-1}g(\chi)$ for any integer t prime to p.

LEMMA 2. *There exists a prime ideal $\mathfrak{p}$ in $Q(\zeta_r)$ dividing p such that* $(g(\chi^k)^r) = \sum_{j=1}^{r-1}\sigma_j^{-1}\mathfrak{p}^{r\langle kj/r\rangle}$.

Conversely, given any prime ideal $\mathfrak{p}_1$ in $Q(\zeta_r)$ dividing p, there exists a k such that

$$(g(\chi^k)^r) = \sum_{j=1}^{r-1} \sigma_j^{-1}\mathfrak{p}_1^j \,.$$

Proof. Lemma 2 is a result of Stickelberger. For a proof see Davenport and Hasse [1]. See especially the elegant proof on page 181–2. In $Q(\zeta_r)$, the ideal $(r) = (1-\zeta_r)^{r-1}$,

LEMMA 3. $(1-\zeta_r^t)(1-\zeta_r)^{-1} \equiv t \pmod{(1-\zeta_r)}$ and $r(1-\zeta_r^t)^{-r+1} \equiv -1 \pmod{(1-\zeta_r)}$ for $(t, r) = 1$.

Proof. The first fact follows as

$$(1-\zeta_r^t)(1-\zeta_r)^{-1} = \sum_{j=0}^{t-1}\zeta_r^j \equiv \sum_{j=0}^{t-1} 1 \equiv t \pmod{(1-\zeta_r)} \,.$$

The second follows from Wilson's Theorem as

$$\begin{aligned} r(1-\zeta_r^t)^{-r+1} &= \Big(\prod_{j=1}^{r-1}(1-\zeta_r^{jt})\Big)(1-\zeta_r^t)^{-r+1} \\ &= \prod_{j=1}^{r-1}(1-\zeta_r^{jt})(1-\zeta_r^t)^{-1} \equiv (r-1)! \equiv -1 \pmod{(1-\zeta_r)} \,. \end{aligned}$$

THEOREM 1. *For any t not divisible by r,*

$$g(\chi^t)^r + 1 \equiv r(1 - \chi(r)^{-t}) \pmod{(1-\zeta_r)^{r+1}},$$

and consequently, $\chi(r) = 1$ if and only if

$$g(\chi^t)^r + 1 \equiv 0 \pmod{(1-\zeta_r)^{r+1}}.$$

Proof. As

$$g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta_p^n ,$$

the binomial theorem yields

$$\begin{aligned} -g(\chi)^r &= \Big(-\sum_{n=1}^{p-1}\zeta_p^n + \sum_{n=1}^{p-1}(1-\chi(n))\zeta_p^n\Big)^r = (1 + \sum_n (1-\chi(n))\zeta_p^n)^r \\ &\equiv 1 + r\sum_n (1-\chi(n))\zeta_p^n + \sum_n (1-\chi(n))^r \zeta_p^{rn} \pmod{(1-\zeta_r)^{r+1}}, \end{aligned}$$

as all other terms are divisible by at least $r(1-\zeta_r)^2$. By Lemma 3, if $\chi(n) \neq 1$, $(1-\chi(n))^{r-1} \equiv -r \pmod{(1-\zeta_r)^r}$, and clearly, if $\chi(n) = 1$,

$$(1-\chi(n))^r \equiv -r(1-\chi(n)) \pmod{(1-\zeta_r)^{r+1}}.$$

Thus,

$$\begin{aligned} -g(\chi)^r &\equiv 1 + r\Big(\sum_{n=1}^{p-1}(1-\chi(n))\zeta_p^n - (1-\chi(n))\zeta_p^{rn}\Big) \\ &\equiv 1 + r\sum_n (1-\chi(n))\zeta_p^n - (1-\chi(n)\chi(r)^{-1})\zeta_p^n \\ &\equiv 1 - r(1-\chi(r)^{-1})\sum_n \chi(n)\zeta_p^n \\ &\equiv 1 - r(1-\chi(r)^{-1})\sum_n \zeta_p^n \\ &\equiv 1 + r(1-\chi(r)^{-1}) \pmod{(1-\zeta_r)^{r+1}}. \end{aligned}$$

By (iv) of Lemma 1,

$$-g(\chi^t)^r = -\sigma_t(g(\chi)^r) \equiv 1 + r(1-\chi(r)^{-t}) \pmod{(1-\zeta_r)^{r+1}},$$

which completes the first statement of Theorem 1. The second statement in Theorem 1 then follows immediately.

Let q denote any positive rational prime other than r, f the least positive integer such that $q^f \equiv 1 \pmod r$, and $ef = r - 1$. Then in $Q(\zeta_r)$ the ideal $(q) = \mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_e$, where the $\mathfrak{A}_j$ are prime ideals and

$$\text{(6)} \qquad \operatorname*{Norm}_{Q(\zeta_r), Q}(\mathfrak{A}_j) = q^f .$$

In the following let $\mathfrak{A}$ be any of the e prime divisors $\mathfrak{A}_j$, $j = 1, \cdots, e$.

THEOREM 2. *Let q, p, and r be distinct.*

Then

$$(7) \qquad g(\chi)^{q^f-1} \equiv \chi(q)^{-f} \pmod q .$$

Consequently $\chi(q) = 1$ *if and only if*

$$(8) \qquad g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}} \text{ for some } \beta \in Q(\zeta_r) .$$

Proof.
$$\begin{aligned} g(\chi)^{q^f} &= \left(\sum_{n=1}^{p-1} \chi(n)\zeta_p^n\right)^{q^f} \\ &\equiv \sum_{n=1}^{p-1} \chi(n)^{q^f}\zeta_p^{nq^f} \pmod q \\ &\equiv \sum_n \chi(n)\zeta_p^{nq^f} \pmod q), \text{ as } r \mid q^f - 1 , \\ &\equiv \chi(q)^{-f} g(\chi) \pmod q . \end{aligned}$$

Multiplying both sides of the above congruence by $\overline{g(\chi)}$, and noting (i) of Lemma 1, yields

$$pg(\chi)^{q^f-1} \equiv \chi(q)^{-f}p \pmod q \text{ or } g(\chi)^{q^f-1} \equiv \chi(q)^{-f} \pmod q ,$$

as p and q are distinct primes. Hence, we have proved (7).

Note that as $r \mid q^f - 1$, (7) becomes a congruence in $Q(\zeta_r)$. As $f \mid r-1$, $(f, r) = 1$, we have by (7) that $\chi(q) = 1$ if and only if $g(\chi)^{q^f-1} \equiv 1 \pmod{\mathfrak{A}}$.

(Note that $1 - \zeta_r^t \not\equiv 0 \pmod{\mathfrak{A}}$ unless $\zeta_r^t = 1$.)

If $g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}}$ for some $\beta \in Q(\zeta_r)$, then

$$g(\chi)^{q^f-1} \equiv \beta^{q^f-1} \equiv 1 \pmod{\mathfrak{A}}$$

by (6).

Conversely, if $g(\chi)^{q^f-1} \equiv 1 \pmod{\mathfrak{A}}$ then $(g(\chi)^r)^{(q^f-1)/r} \equiv 1 \pmod{\mathfrak{A}}$. By Lemma 1, $g(\chi)^r \in Q(\zeta_r)$. By (6) this implies $g(\chi)^r \equiv \beta^r \pmod{\mathfrak{A}}$. (Euler's Criterion for rth powers.)

In the above argument we must bear in mind that $g(\chi) \notin Q(\zeta_r)$.

II. In the last section we have developed a criterion for rth power residuacity in $Q(\zeta_r)$. From this we derive a criterion in the rational numbers Q, which is the purpose of Theorems A and B.

First let us assume that there is a rational integral solution $X_j = a_j$ of equations (1), (2), (3) and (4). In $Q(\zeta_r)$ define the algebraic integer $\alpha = \sum_{j=1}^{r-1} a_j\zeta_r^j$. We shall prove that α satisfies

$$(9) \qquad |\sigma_k(\alpha)|^2 = p^{r-2} , \qquad k = 1, 2, \cdots, r-1 .$$

$$(10) \qquad (p\alpha)^k\sigma_k(p\alpha)^{-1}$$

is also an algebraic integer in $Q(\zeta_r)$, for $k = 1, 2, \cdots, r-1$.

To prove (9) we note that

$$\begin{aligned}|\alpha|^2 &= \Big(\sum_j a_j\zeta_r^j\Big)\Big(\sum_i a_i\zeta^{r-i}\Big)\\ &= \sum_{j,i} a_j a_i \zeta_r^{j-i}\\ &= \sum_{j=1}^{r-1} a_j^2 + \sum_{i=1}^{r-1}\big(\textstyle\sum_i^{(1)} a_{j_1}a_{j_2}\big)\zeta_r^i\ .\end{aligned}$$

By (2) all of the coefficients of ζ_r^i are equal, since for any i, the sums corresponding to i and $r-i$ are identical. Thus

$$\begin{aligned}|\alpha|^2 &= \sum_j a_j^2 - \textstyle\sum_1^{(1)} a_{j_1}a_{j_2}\\ &= \sum_j a_j^2 - (r-1)^{-1}\sum_{i=1}^{r-1}\textstyle\sum_i^{(1)} a_{j_1}a_{j_2}\\ &= r(r-1)^{-1}\sum_j a_j^2 - (r-1)^{-1}\sum_{i=0}^{r-1}\textstyle\sum_i^{(1)} a_{j_1}a_{j_2}\\ &= r(r-1)^{-1}\sum_{j=1}^{r-1} a_j^2 - (r-1)^{-1}\Big(\sum_{j=1}^{r} a_j\Big)^2\\ &= p^{r-2}\end{aligned}$$

by (1). Similarly $|\sigma_k(\alpha)|^2 = p^{r-2}$. Thus (1) and (2) imply (9).

Let k be a fixed integer $2 \leqq k \leqq r-1$. Then

$$\begin{aligned}(p\alpha)^k\sigma_k(p\alpha)^{-1} &= p^{k-1}\alpha^k\sigma_k(\alpha)^{-1}\\ &= p^{k-1}\alpha^k\sigma_{-k}(\alpha)\,|\,\sigma_k(\alpha)\,|^{-2}\\ &= p^{-r+k+1}\alpha^k\sigma_{-k}(\alpha)\end{aligned} \tag{11}$$

by (10). Now

$$\begin{aligned}\alpha^k\sigma_{-k}(\alpha) &= \big(\textstyle\sum a_j\zeta_r^j\big)^k\big(\textstyle\sum a_j\zeta_r^{-jk}\big)\\ &= \sum_{i=0}^{r-1}\big(\textstyle\sum_i^{(k)} a_{j_1}\cdots a_{j_{k+1}}\big)\zeta_r^i\\ &= \sum_{i=1}^{r-1}\big(\textstyle\sum_i^{(k)} - \sum_0^{(k)}\big)\zeta_r^i\ .\end{aligned} \tag{12}$$

Condition (4) implies that each coefficient of ζ_r^i in (12) is divisible by p^{r-k-1}. Placing this information in (11) states that $(p\alpha)^k\sigma_k(p\alpha)^{-1}$ is an integer; thus proving (10).

(4) also tells us that p, but not p^2, divides $p\alpha$, as not all the coefficients of ζ_r^j in $\alpha = \sum_{j=1}^{r-1} a_j\zeta_r^j$ are divisible by p.

If we restate the above facts in terms of ideals, we have that $(p\alpha)$ is an integral ideal in $Q(\zeta_r)$ divisible only by the prime ideals which divide p.

There exists one prime ideal, say $\mathfrak{p}$, dividing p, which divides $p\alpha$ but $\mathfrak{p}^2$ does not divide $p\alpha$. All other prime factors of p in $Q(\zeta_r)$ are of the form $\sigma_i^{-1}\mathfrak{p}$. Hence,

(13) $$(p\alpha) = \sum_{i=1}^{r-1} \sigma_i^{-1}\mathfrak{p}^{d_i} \text{ where } d_1 = 1, d_i > 0 .$$

By (9)

$$\begin{aligned}(p\alpha)(\sigma_{-1}(p\alpha)) &= (p^2 \,|\, \alpha\,|^2) = p^r \\ &= \Big(\prod_i \sigma_i^{-1}\mathfrak{p}^{d_i}\Big)\Big(\prod_i \sigma_{-1}\sigma_i^{-1}\mathfrak{p}^{d_i}\Big) \\ &= \prod_i \sigma_i^{-1}\mathfrak{p}^{d_i+d_{r-i}}\end{aligned}$$

or

(14) $$d_i + d_{r-i} = r .$$

By (10), $(p\alpha)^k\sigma_k(p\alpha)^{-1}$ is integral, or

$$\begin{aligned}(p\alpha)^k(\sigma_k(p\alpha))^{-1} &= \prod_i \sigma_i^{-1}\mathfrak{p}^{d_ik} \prod_i \sigma_k\sigma_i^{-1}\mathfrak{p}^{-d_i} \\ &= \prod \sigma_i^{-1}\mathfrak{p}^{d_ik-d_{ik}}\end{aligned}$$

is an integral ideal. (The index of d_{ik} is interpreted mod r.) Hence, $kd_i \geqq d_{ik}$.

As $d_1 = 1$, $k \geqq d_k$ for $k = 2, 3, \cdots, r-2$. By (14) this yields that $d_k = k$. By Lemma 2, we arrive at the fact that in terms of ideals

(15) $$(p\alpha) = (g(\chi^t)^r) \text{ for some } 1 \leqq t < r .$$

In proving (15) we have used (1), (2) and (4). We wish to prove that $p\alpha = g(\chi^t)^r$. To do this we now utilize (3). By (15) we have that for some unit $\eta \in Q(\zeta_r)$, $g(\chi^t)^r = \eta p\alpha$, or

(16) $$g(\chi^{tk})^r = \sigma_k(\eta p\alpha) = \sigma_k(\eta)\sigma_k(p\alpha) .$$

Taking the absolute value of both sides of (16) and utilizing (i) of Lemma 1 and (9) gives $p^r = |\,\sigma_k(\eta)\,|^2p^r$, or $|\,\sigma_k(\eta)\,|^2 = 1$. By a Theorem of Dirichlet on units (See [3] Theorem IV 9, A pp. 174), any unit which has all of its conjugates with absolute value 1 is then a root of unity. As $\eta \in Q(\zeta_r)$, $\eta = \pm\zeta_r^s$.

Now

$$\begin{aligned}\alpha = \sum_{j=1}^{r} a_j\zeta_r^j &= \sum_j a_j - \sum_j a_j(1-\zeta_r^j) \\ &\equiv \sum_j a_j - \sum_j ja_j(1-\zeta_r) \pmod{(1-\zeta_r)^2} ,\end{aligned}$$

by Lemma 3. As $p \equiv 1 \pmod r$, $p \equiv 1 \pmod{(1-\zeta_r)^2}$. By (3),

$$1 + \sum_j a_j \equiv \sum_j ja_j \equiv 0 \pmod r .$$

Hence, $p\alpha \equiv -1 \pmod{(1-\zeta_r)^2}$. By Theorem 1, $g(\chi^t)^r \equiv -1 \pmod{1-\zeta_r}^2$. Therefore, $\eta \equiv 1 \pmod{(1-\zeta_r)^2}$. But $\eta = \pm\zeta_r^s \equiv \pm(1+s(1-\zeta_r)) \pmod{(1-\zeta_r)^2}$; i.e., $s \equiv 0 \pmod r$ and the $+$ sign holds. Hence, $\eta = 1$.

Therefore, if the a_j are any integral solution of (1), (2), (3) and (4), there exists an integer $1 \leqq t \leqq r-1$ such that

$$p \sum_{j=1}^{r-1} a_j \zeta_r^j = g(\chi^t)^r . \tag{17}$$

Conversely, given any integer $t, 1 \leqq t \leqq r-1$, and writing

$$g(\chi^t)^r = p \sum_{j=1}^{r-1} a_j \zeta_r^j ,$$

we can prove that the a_j are rational integers which satisfy (1), (2), (3), and (4). The proof is merely reversing the above steps we used in proving (17). By Lemma 2 the prime factorizations of $(g(\chi^s)^r)$ and $(g(\chi^t)^r)$, $1 \leqq s < t \leqq r-1$, are distinct, and thus $g(\chi^s)^r \neq g(\chi^t)^r$. Hence, we have shown that there are precisely $r-1$ rational integral solutions of (1), (2), (3), and (4).

We are now in a position to prove Theorems A and B. First for Theorem A.

Let a_j be an integral solution of (1) through (4). Then we have shown that $p \sum_{j=1}^{r-1} a_j \zeta_r^j = g(\chi^t)^r$ for some integer t relatively prime to r. By Theorem 1, the above states that $\chi(r) = 1$ if and only if $p \sum_j a_j \zeta_r^j \equiv -1 \pmod{(1-\zeta_r)^{r+1}}$.

Define $b_s, s = 0, 1, \cdots, r-2$, by $b_0 = -pa_{r-1}, b_s = p(a_s - a_{r-1}), s = 1, 2, \cdots, r-2$. Then

$$p \sum_{j=1}^{r-1} a_j \zeta_r^j = \sum_{s=0}^{r-2} b_s \zeta_r^s .$$

Further let

$$C_i = (-1)^i \sum_{s=i}^{r-2} \binom{s}{i} b_s ,$$

where $\binom{s}{i}$ is the binomial coefficient. Then

$$\begin{aligned} p \sum_{j=1}^{r-1} a_j \zeta_r^j &= \sum_{s=0}^{r-2} b_s \zeta_r^s = \sum_s b_s (1-(1-\zeta_r))^s \\ &= \sum_s b_s \sum_{i=0}^{s} (-1)^i \binom{s}{i} (1-\zeta_r)^i \\ &= \sum_{i=0}^{r-2} C_i (1-\zeta_r)^i . \end{aligned}$$

The first statement in Theorem 1 states that $g(\chi^t)^r + 1 \equiv 0 \pmod{(1-\zeta_r)^r}$. Hence,

$$\begin{aligned} \sum_{i=0}^{r-2} C_i (1-\zeta_r)^i + 1 &\equiv (C_0 + 1) + \sum_{i=1}^{r-2} C_i (1-\zeta_r)^i \\ &\equiv 0 \pmod{(1-\zeta_r)^r} \end{aligned}$$

This implies that $C_0 + 1 \equiv 0 \pmod{r^2}$. Hence,

$$\sum_{i=0}^{r-2} C_i(1-\zeta_r)^i \equiv C_1(1-\zeta_r) \pmod{(1-\zeta_r)^{r+1}}$$

or that $\chi(r) = 1$ if and only if

$$C_1 \equiv 0 \pmod{r^2}\ . \tag{18}$$

Now

$$\begin{aligned} C_1 &= (-1)\sum_{s=1}^{r-2}\binom{s}{1}b_s = -\sum_{s=1}^{r-2} sb_s \\ &= -p\sum_{s=1}^{r-2} s(a_s - a_{r-1}) \\ &= -p\sum_{s=1}^{r-2} sa_s + \frac{1}{2}p(r-2)(r-1)a_{r-1} \\ &\equiv -p\left(\sum_{s=1}^{r-1} sa_s + \frac{1}{2}ra_{r-1}\right) \pmod{r^2}\ . \end{aligned} \tag{19}$$

Equations (18) and (19) complete the proof of Theorem A.

Theorem B is also derived immediately from Theorem 2. If $q \equiv 1 \pmod r$, q a positive rational prime, then in $Q(\zeta_r)$, $(q) = \mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_{r-1}$, where $\mathfrak{A}_j$ are prime ideals and $\mathrm{Norm}_{Q(\zeta_r),Q}\,\mathfrak{A}_j = q$.

We may take $0, 1, 2, \cdots, q-1$ as a set of residues $(\bmod\ \mathfrak{A}_1)$. Hence, as $1 - \zeta_r^t \not\equiv 0 \pmod{\mathfrak{A}_1}$, unless $\zeta_r^t = 1$, $\zeta_r \equiv h \pmod{\mathfrak{A}_1}$, where h is a rational integer such that $h^r \equiv 1 \pmod q$.

Thus by Theorem 2, $\chi(q) = 1$ if and only if there is a $\beta \in Q(\zeta_r)$ such that $g(\chi^t)^r = p\sum_j a_j\zeta_r^j \equiv p\sum_j a_jh^j \equiv \beta^r \pmod{\mathfrak{A}_1}$.

We may take $\beta = b \in Q$ by the above remarks.

Hence, $\chi_p(q) = 1$ if and only if $\chi_q(p\sum_j a_jh^j) = 1$ where χ_q is a primitive rth power character $(\bmod\ q)$.

If we had chosen another h_1 whose order was $r \pmod q$, then $h_1 \equiv h^t \pmod{\mathfrak{A}_1}$, and

$$p\sum_j a_jh_1^j \equiv p\sum_j a_j\zeta_r^{jt} \equiv g(\chi^t)^r \pmod{\mathfrak{A}_1}\ .$$

Thus, any h whose order $(\bmod\ q)$ is r works equally well in Theorem B.

There are several special cases one can derive when $q \not\equiv 1 \pmod r$, in particular, when $q = 2$, and $r = 5, 7$.

If $q = 2$, $r = 5$, then in $Q(\zeta_r)$, 2 remains a prime because 2^4 is the least power of 2 congruent to 1 (mod 5). One can easily compute that the only elements in $Q(\zeta_5)$ which are fifth powers (mod 2) are $1 = -\sum_{j=1}^{4}\zeta_5^j$, $\zeta_5 + \zeta_5^{-1}$, and $\zeta_5^2 + \zeta_5^{-2}$ (mod 2). Hence, for $r = 5$, $\chi_p(2) = 1$ if and only if $a_j \equiv a_{5-j} \pmod 2$.

For $q = 2$, $r = 7$, then $2^3 \equiv 1 \pmod 7$. Hence, in $Q(\zeta_7)$, $(2) = \mathfrak{A}_1\mathfrak{A}_2$ where $\mathrm{Norm}\,\mathfrak{A}_i = 8$. For $\alpha \equiv \beta^7 \pmod{\mathfrak{A}_1}$, $\beta \not\equiv 0 \pmod{\mathfrak{A}_1}$, and $\beta \in Q(\zeta_7)$

implies $\alpha \equiv 1 \pmod{\mathfrak{A}_1}$. Hence, for $r = 7$, $\chi_p(2) = 1$ if and only if $a_j \equiv 1 \pmod 2$ for $j = 1, \cdots, 6$.

One could easily generalize this to the case when $r = 2^s - 1$. Then $\chi_p(2) = 1$ if and only if $a_j \equiv 1 \pmod 2$ for $j = 1, \cdots, r-1$.

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