ON SIMILARITY INVARIANTS OF CERTAIN OPERATORS IN L_p

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The purpose of this paper is to extend the result of Corollary, Theorem 2 of the author's paper on Volterra operators (Annals of Math., 66, 1957, pp. 481–494 quoted as A; we shall use the definitions and notations of that paper) to the most general situation applicable: We are dealing with operators T_F where $F(x, y) = (y - x)^{m-1}\, aG(x, y)$ is a function defined on the triangle $0 \leqq x \leqq y \leqq 1$, where m is a positive integer, a a complex number of absolute value 1, G is a complex valued function which is continuously differentiable and $G(x, x)$ is positive real. We recall that if $f \in L_p\,[0, 1]$, then $(T_F)(f)(x) = \int_x^1 F(x, y)f(y)dy$ is again in $L_p\,[0, 1]$. The only difference from A is the presence of the constant a which affects none of results except Theorem 2 and its Corollary. Theorems 1 and 2 of the present paper fill the gap. Theorem 3 shows that differentiability conditions imposed on F cannot be abandoned entirely—and also that the integral equation (1) of A cannot be solved unless K (which corresponds to our F) has at least first derivatives near $y = x$.

If c is constant and E is the function identically equal to 1, we define T_E^c as T_H which $H(x, y) = (y - x)^{c-1}/\Gamma(c)$ (fractional integration of order c).

THEOREM 1. *Let c_1 and c_2 be complex numbers and let r_1 and r_2 be real numbers such that $r_i \geqq 1$, then $c_1 T_E^{r_1}$ is similar to $c_2 T_E^{r_2}$ if and only if $c_1 = c_2$ and $r_1 = r_2$.*

Proof. The first part of the Proof of Theorem 2 of A applies and implies that $r_1 = r_2$ $(= r)$ and $|c_1| = |c_2|$. Thus suppose that $c_1 T_E^r$ is similar to $c_2 T_E^r$ or that cT_E^r is similar to

$$(1) \qquad T_E^r = PcT_E^r P^{-1} \text{ for } |c| = 1$$

where P is a bounded linear transformation of $L_p\,[0, 1]$ onto itself with the bounded linear inverse P^{-1}. If T is similar to $S = PTP^{-1}$, then $f(T)$ is similar to

$$(2) \qquad f(S) = Pf(T)P^{-1}$$

for polynomials and even analytic functions f. Let

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$$f(z) = \sum_{i=0}^{\infty} a_i z^{i+1}$$

Then

$$f(cT_E^r) = \sum_{i=0}^{\infty} a_i c^{i+1} T_E^{r(i+1)} = T_{g_1(y-x)}$$

where $g_1(t) = ct^{r-1}g(ct^r)$ where we have written t for $y - x$ and where

$$g(z) = \sum_{i=0}^{\infty} b_i z^i$$

with $b_i = a_i/\Gamma(r(i+1))$. Equations (1) and (2) imply that $\| f(T_E^r) \| \leqq \| P \| \| P^{-1} \| \| f(cT_E^r) \|$. The definition of the norm of a linear transformation in a Banach space implies the following inequality:

$$\| f(T_E^r) \| = \| T_{t^{r-1}g(t^r)} \| \geqq \left\| \int_x^1 (y - x)^{r-1} g((y - x)^r) k(y) dy \right\|_p$$

for all $k \in L_p\,[0, 1]$ such that $\| k \|_p = 1$. On the other hand, Lemma 2 of A implies that

$$\| T_{ct^{r-1}g(t^r)} \| \leqq \| ct^{r-1}g(ct^r) \|_1 = \| t^{r-1}g(ct^r) \|_1 \, .$$

Thus if $k(y) = 1$, we obtain

$$\begin{aligned} L = \left\| \int_x^1 (y - x)^{r-1} g((y - x)^r) dy \right\|_p &\leqq \| f(T_E^r) \| \\ &\leqq \| P \| \| P^{-1} \| \| f(cT_E^r) \| \\ &\leqq \| P \| \| P^{-1} \| \| t^{r-1} g(ct^r) \|_1 = R \, . \end{aligned} \tag{3}$$

We shall find a family of functions g_v (and correspondingly f_v) depending on a positive parameter v such that if we use the notations L_v and R_v for the corresponding left and right hand sides of (3), $L_v \to \infty$ and $R_v \to 0$ as $v \to \infty$ contradicting the inequality (3): this contradiction then proves our theorem.

Let us first consider the case where the real part of c, $Re(c)$, is less than 0. Let $g_v(t) = \exp(vt)$. Since T_E^r is generalized nilpotent for $r \geqq 1$, the corresponding function $f_v(T_E^r)$ exists and (1) indeed implies (2) for $S = T_E^r$ and $T = cT_E^r$. Then

$$R_v = \| t^{r-1} g_v(ct^r) \|_1 = \int_0^1 | t^{r-1} \exp(vct^r) | \, dt$$

and $R_v \to 0$ as $v \to \infty$. On the other hand

$$L_v = (1/r^p) \int_0^1 (\exp(v(1 - x)) - 1/v)^p dx \to \infty$$

as $v \to \infty$. If finally $Re(c) \geqq 0$ and $c \neq 1$, then there exist a positive integer n such that $Re(c^n) < 0$. But then (1) implies that $c^n T_E^{nr}$ is similar to $T_E^{nr} = Pc^n T_E^{nr} P^{-1}$ which contradicts the preceding result and the proof of the theorem is complete.

THEOREM 2. *Let $F(x, y) = (y - x)^{m-1} aG(x, y)$ satisfy, in addition to the general hypotheses stated above, one of the following:*

(1) *G is analytic in a suitable region and m is arbitrary;*

(2) *$G(x, y) = G(y - x)$, $G(0) \neq 0$, $G \in C^2$ and m is arbitrary;*

(3) *$G \in C^2$ and $m = 1$. Let A be a complex number. Then $AI + T_F$ and $AI + T_F^*$ are similar to the unique operator $AI + caT_E^m$ and $AI + c\bar{a}T_E^m$ respectively where $c = \left(\int_0^1 (G(u, u)^{1/m} du\right)^m$.*

Here I is the identity operator and T_K^*, the adjoint of T_K, is defined by

$$(T_K^*)(f)(x) = \int_0^x \overline{K(y, x)} f(y) dy .$$

Proof. Note first that A implies that $AI + T_F$ is similar to $AI + caT_E^m$ and that $AI + T_F^*$ is similar to $AI + c\bar{a}T_E^{*m}$ (see Cor. Theorem 2 of A). Observe next that $T_E^* f(x) = \int_0^x f(y) dy$ and

$$T_E^{*m} f(x) = (1/\Gamma(m)) \int_0^x (x - y)^{m-1} f(y)\, dy$$

and that if $(S_{1-x} f)(x) = f(1 - x)$ then S_{1-x} is an isometry of $L_p[0, 1]$ onto itself and $S_{1-x} T_E^m S_{1-x}^{-1} = T_E^{*m}$. It remains to show uniqueness. Suppose that $A_1 I + c_1 a_1 T_E^{m_1}$ is similar to $A_2 I + c_2 a_2 T_E^{m_2}$. Then $A_1 = A_2$ (because of the complete continuity of T_E) and $c_1 a_1 T_E^{m_1}$ is similar to $c_2 a_2 T_E^{m_2}$ which by Theorom 1 implies that $c_1 = c_2$, $a_1 = a_2$, $m_1 = m_2$.

THEOREM 3. *The linear transformation $T_E + T_E^{1+a}$ where $0 < a < 1$ of $L_p[0, 1]$ into itself is not similar to any linear transformation cT_E^r for complex c and real $r \geqq 1$.*

Proof. Preliminaries. 1. If two linear transformations S and T are similar, i.e., if there exists P such that $S = PTP^{-1}$, then there exists a constant K such that

$$1/K \leqq \| T^n \| / \| S^n \| \leqq K , \tag{4}$$

for all positive integers n. It suffices to take $K = \| P \| \| P^{-1} \|$.

2. The following inequality is a consequence of the fact that if $0 \leqq F_1(x, y) \leqq F_2(x, y)$ then $\| T_{F_1} \| \leqq \| T_{F_2} \|$:

$$(5) \qquad \|(T_E + T_E^{1+a})^n\| \geqq n \| T_E^{n+a}\|$$

for all positive integers n.

3. Our next task is to find estimates for $\| T_E^n \|$. An estimate from above is the following:

$$(6) \qquad \| T_E^n \| \leqq 1/(n\Gamma(n)p^{1/p})$$

for all positive integers n. An estimate from below is furnished by the following Proposition:

Given the real positive number e there exists a positive number $K = K(e)$ and a positive integer $N = N(e)$ such that for all integers $n \geqq N$,

$$(7) \qquad \| T_E^n \| \geqq K/(n^{1+e}\Gamma(n)) .$$

Proof of (6). If $f \in L_p[0, 1]$,

$$T_E^n f(x) = \int_x^1 [(y - x)^{n-1}/\Gamma(n)] f(y) dy .$$

If $(1/p) + (1/q) = 1$, Hölder's inequality yields

$$\int_x^1 (y - x)^{n-1} f(y) dy \leqq \left(\int_x^1 (y - x)^{(n-1)q} dy\right)^{1/q} \| f \|_p$$
$$= (1 - x)^{((n-1)q+1)/q} \| f \|_p / (((n - 1)q + 1)^{1/q})$$

so that

$$\begin{aligned} &\| T_E^n f \|_p^p \\ &= \int_0^1 | (T_E^n f)(x) |^p dx \\ &= (1/\Gamma(n))^p \int_0^1 \left| \int_x^1 (y - x)^{n-1} f(y) dy \right|^p dx \\ &\leqq (1/\Gamma(n))^p (1/((n - 1)q + 1)^{p/q}) \int_0^1 (1 - x)^{((n-1)p+(p/q))} dx \, \| f \|_p^p \\ &= (1/\Gamma(n))^p (1/((n - 1)q + 1)^{p/q}) (1/((n - 1)p + (p/q) + 1)) \, \| f \|_p^p \end{aligned}$$

which implies that

$$\| T_E^n \| \leqq (1/\Gamma(n))(1/((n - 1)q + 1)^{1/q})(1/((n - 1)p + (p/q) + 1)^{1/p})$$

which in turn implies (6).

Proof of (7). We first observe that elementary considerations concerning the gamma function imply that given c such that $0 < c < 1$ and given a positive real number d there exists an integer N depending on c and d such that for all integers $n \geqq N$

(8) $$\Gamma(n+c) < (n+c)^{c+d}\Gamma(n) .$$

Consider next the function $f(x) = r(1-x)^{-s} \in L_p\,[0, 1]$ such that $\|f\|_p = 1$, i.e., $r^p = 1 - sp$ and $0 < s < 1/p$. Then

$$T_E^n f(x) = r\Gamma(1-s)(1-x)^{n-s}/\Gamma(n+1-s)$$

and

$$\| T_E^n \| \geqq r\Gamma(1-s)/\Gamma(n+1-s)(p(n-s)+1)^{1/p} .$$

We now choose s (and hence r) such that for the positive real number e of (7), $0 < (1/p) - s < e$ and then we choose d such that $0 < d < e + s - (1/p)$ and finally by virture of (8) we obtain N as a function of e such that for all integers $n \geqq N$, $\Gamma(n+1-s) < (n+1-s)^{1-s+d}\Gamma(n)$ whence

$$\| T_E^n \| \geqq r\Gamma(1-s)/(n+1-s)^{1-s+d}\Gamma(n)(p(n-s)+1)^{1/p}$$

which upon choosing $K = K(e)$ properly implies (7).

After these preliminaries, we turn to the proof of the theorem. We distinguish several cases. Let $T = T_E + T_E^{1+a}$.

Case 1. $|c| \leqq 1$. Consider

$$h_n = \| (cT_E^r)^n \| / \| T^n \| \leqq \| T_E^n \| / (n \| T_E^{n+a} \|)$$

where we have used (5) and the fact that $r \geqq 1$. Take now positive real numbers e and d such that $a + e + d < 1$. Then there exists by (7) a positive constant K and an integer N such that for all integers $n \geqq N$

(9) $$\begin{aligned} h_n &\leqq (n+a)^{1+e}\Gamma(n+a)/(n^2\Gamma(n)p^{1/p}K) \\ &\leqq (n+a)^{1+e+a+d}\Gamma(n)/(n^2\Gamma(n)p^{1/p}K) \end{aligned}$$

where we have made use of (8) and (6). The last inequality implies that $h_n \to 0$ which in conjunction with (4) implies the truth of our theorem in the case under consideration.

Case 2. $r < 1$. Using the notations and making similar choices as under Case 1, (9) becomes

$$h_n \leqq |c|^n (n+a)^{1+e+a+d}\Gamma(n)/(n^2 r\Gamma(rn)p^{1/p}K)$$

which, since $|c|^n\Gamma(n)/\Gamma(rn)$ is bounded (in fact converges to 0) for $r > 1$ as $n \to \infty$, again proves the truth of the theorem in the present case.

Case 3. $r = 1$, $|c| > 1$. This time we consider the quotient

$$
\begin{aligned}
k_n &= \| T^n \| / \| (cT_E)^n \| \\
&\leqq \sum_{i=0}^{n} \binom{n}{i} \| T_E^{n+a(n-i)} \| / (| c |^n \| T_E^n \|) \qquad (10) \\
&\leqq ((n^{1+e}\Gamma(n)/(| c |^n K p^{1/p})) \sum_{i=0}^{n} \binom{n}{i} / (\Gamma(n + a(n - i) + 1)) ,
\end{aligned}
$$

which is valid for sufficiently large n; again we used (6) and (7).

In order to complete the proof of our theorem, we need the following fact:

Given any positive real number e and given the positive real number $a < 1$, there exists an integer $N = N(e; a)$ such that for all integers i and n such that $0 \leqq i \leqq n \leqq N$

$$\Gamma(n)/\Gamma(n + a(n - i) + 1) \leqq 2e^{n-i} . \tag{11}$$

Proof. The case $i = 0$ results from elementary considerations about the gamma function. If $i = 1$, we find N_1 so that (11) is valid for $i = 0$ and $n \geqq N_1$. We then find N_2 so that (8) is true for some arbitrary but fixed d, for $c = a$ and for $n \geqq N_2$. Then $\Gamma(n)/\Gamma(n+(n-1)a+1) \leqq (\Gamma(n)/\Gamma(n + na + 1))/(n + na + 1)^{a+d}$ which for $n \geqq \max(N_1, N_2, e^{-1/a}) = N_3$ implies (11) for $i = 2$ and $n \geqq N_3$. The remaining cases are settled by induction (except $i = n$ which is obvious); note that we never have to go above N_3 at any point. This completes the proof of (11).

The proof is now completed by substituting (11) into (10):

$$k_n \leqq 2n^{1+c}(1 + e_1)^n / | c |^n K p^{1/p}$$

where e_1 is the constant e of (11). Thus $k_n \to 0$ upon proper choice of e_1 and our theorem is again true in view of (4). This completes the proof of Theorem 3.

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