MANIFOLDS WITH POSITIVE CURVATURE

THEODORE FRANKEL

0. Introduction and a conjecture. In 1936 J. L. Synge [10] proved that an even dimensional orientable compact manifold M_n with positive sectional curvature is simply connected. His proof was an application of a formula for the second variation of arc length derived by him in an earlier article.[1] In the present paper we continue the study of positively curved manifolds again using the ideas of Synge and applying them to an only slightly different situation, namely to the "position" of certain submanifolds of M.

Theorem 1 states that two compact totally geodesic (see § 2 for definitions) submanifolds V_r and W_s of M_n must necessarily intersect if their dimension sum is at least that of M, i.e. if $r + s \geqq n$. As remarked above the proof is a straightforward continuation of Synge's method. Unfortunately totally geodesic submanifolds are not a too common occurrence.

If M_n is a Kähler manifold[2] the situation is much more satisfactory. There, instead of requiring V and W to be totally geodesic, we need only ask that they be complex analytic submanifolds (Theorem 2).

Examples of compact Riemannian manifolds of positive sectional curvature are the spheres, the real, complex and quaternionic projective spaces and the Cayley plane. Rauch [8] has shown that if the sectional curvatures do not differ too much from that of the sphere and if the space is simply connected, then it is itself topologically a sphere (see also the recent improvements by W. Klingenberg, Über kompakte Riemannsche Mannigfaltigkeiten, Math. Ann., **137** (1959), pp. 351–61). Berger [2] has shown that if M_n is an even dimensional, simply connected manifold and if the sectional curvature K satisfies $1/4 \leqq K \leqq 1$, then the manifold is one of the spaces listed above.

In the list the only Kähler manifolds are the complex projective n-spaces $P_n(\boldsymbol{C})$ with the usual Fubini metric. If (e_1, e_2) is a pair of orthonormal tangent vectors to $P_n(\boldsymbol{C})$, then the sectional curvature $K(e_1, e_1)$ satisfies $1/4 \leqq K(e_1, e_2) \leqq 1$ with $K = 1$ if and only if the plane $e_1 \wedge e_2$ is a "complex direction." It may very well be that

CONJECTURE. *The positively curved Kähler manifolds of complex dimension n are analytically homeomorphic to $P_n(\boldsymbol{C})$.* The Gauss Bonnet

Received January 11, 1960. Work supported in part by the National Science Foundation.

[1] For completeness we include in § 1 a derivation of the second variation formula.

[2] Since the Ricci curvature of a positively curved manifold is positive, the Kähler manifold is a "Hodge manifold" and Kodaira's theorem [6] states that the manifold is algebraic.

theorem shows that this is true for $n = 1$. Using Theorem 2, A. Andreotti has shown that the conjecture is true for $n = 2$ and his proof is presented in Theorem 3. It relies heavily on the known classification of algebraic surfaces.[2]

Difficulties in attempting to construct counter examples stem from the fact that the product of two positively curved manifolds has only *nonnegative* curvature (in the product metric). If $e_1 \wedge e_2$ is a product plane (e.g., if e_1 is "horizontal" and e_2 is "vertical"), then $K(e_1, e_2) = 0$ and this is the only time 0 curvature can occur. Our results in general do not apply to such spaces.

The last section is devoted to proving the existence of fixed points for certain maps, thus showing further similarities with $P_n(C)$.

I should like to thank A. Andreotti, E. Calabi and N. Hawley for discussions of the results.

1. Second variation of arc length. Our notation is as follows. M_n is a complete n dimensional Riemannian manifold and V_r and W_s are submanifolds of dimension r and s respectively. $\mathscr{C}(t)$ is a geodesic going from $\mathscr{C}(0) = P \in V$ to $\mathscr{C}(l) = Q \in W$ striking V and W orthogonally; t represents arc length along $\mathscr{C}$. X_t is a unit vector field that is displaced parallel along $\mathscr{C}$ and is tangent to V and W at P and Q respectively; X_t (if it exists) is thus orthogonal to $\mathscr{C}$ for all t. Finally T_t is the unit tangent vector to $\mathscr{C}$.

We construct a "variation" of the geodesic $\mathscr{C}$ as follows. We pass a small "ribbon" of surface through $\mathscr{C}$ that is tangent to X_t at $\mathscr{C}(t)$ for all t such that $0 \leqq t \leqq l$. This ribbon cuts V and W in two curves. We now pass curve segments on the ribbon tangent to X_t at $\mathscr{C}(t)$, the curves varying smoothly from V to W. The ribbon is chosen so "thin" that no two segments intersect. On each segment we use the directed arc length α from $\mathscr{C}$ as parameter and we may suppose that $-\varepsilon \leqq \alpha \leqq +\varepsilon$. Each point on the ribbon carries two coordinates (t, α) and we have two systems of coordinate curves $t =$ constant and $\alpha =$ constant (the original geodesic is of course $\alpha = 0$). We have two coordinate vector fields $T = \partial/\partial t$ and $X = \partial/\partial\alpha$ defined on the ribbon with $T = T_t$ at $(t, 0)$ and $X = X_t$ at this same point. The problem is to investigate the lengths of the curves $\alpha =$ constant.

We recall some facts and notation of Riemannian geometry (our notation follows [7]). We let $g(Y, Z)$ denote the Riemannian scalar product of two vectors Y and Z; if $(x_1, \cdots, x_n)$ are local coordinates for M, then $g(Y, Z) = \sum_{ij} g_{ij} Y^i Z^j$. If Y is a vector at a point and if f is a function, then the covariant derivative of f with respect to Y, written $\nabla_Y(f)$, is the directional derivative of f in the direction Y. If Z is a vector field, the covariant derivative of Z with respect to Y is again a vector, written $\nabla_Y Z$. If Y is also a vector field, the Lie or commutator bracket of Y and Z is given by $[Y, Z] = YZ - ZY = \nabla_Y Z - \nabla_Z Y$. In particular, if Y and Z are coordinate vectors $\nabla_Y Z - \nabla_Z Y = [Y, Z] = 0$. Hence in the case of our particular vectors we have

$$\nabla_X T = \nabla_T X . \tag{1}$$

Next we have the Ricci operator identity

$$\nabla_Y \nabla_Z - \nabla_Z \nabla_Y = R(Y, Z) + \nabla_{[Y,Z]}$$

where $R(Y, Z)$ is, for each pair (Y, Z), a linear transformation on tangent vectors. $R(Y, Z)$ is constructed from the Riemann curvature tensor and in terms of coordinates the transformation of vectors $U \to R(Y, Z)U$ is given by

$$\sum_i U^i \frac{\partial}{\partial x^i} \to \sum_i \Big(\sum_{jkl} - R^i_{jkl} Y^k Z^l U^j \Big) \frac{\partial}{\partial x^i} .$$

$R(Y, Z)$ is skew symmetric; $R(Y, Z) = -R(Z, Y)$. In our case the Ricci identity becomes

$$\nabla_X \nabla_T - \nabla_T \nabla_X = R(X, T) . \tag{2}$$

The Riemannian sectional curvature corresponding to the 2-plane $T \wedge X$ is given by

$$K(T, X) = g(R(X, T)T, X) = -g(R(X, T)X, T) . \tag{3}$$

Finally we recall that the scalar product is "covariant constant," i.e.

$$\frac{\partial}{\partial \alpha} g(Y, Z) = \nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) .$$

The length of the curve $\alpha =$ constant is given by

$$L(\alpha) = \int_0^l g(T, T)^{1/2} dt .$$

LEMMA ([9]). *The first and second variations of arc length are*

$$\begin{cases} L'_X(0) = \left.\dfrac{dL}{d\alpha}\right|_0 = 0 \\ L''_X(0) = \left.\dfrac{d^2L}{d\alpha^2}\right|_0 = g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P - \displaystyle\int_0^l K(T, X) dt . \end{cases}$$

Proof.

$$L'(\alpha) = \int_0^l \frac{\partial}{\partial \alpha} g(T, T)^{1/2} dt = \int_0^l \nabla_X g(T, T)^{1/2} dt ,$$

thus

$$L'(\alpha) = \int_0^l \frac{g(\nabla_X T, T)}{g(T, T)^{1/2}} dt \ . \tag{4}$$

But $g(T, T) \equiv 1$ along $\alpha = 0$ (T is unit tangent to $\mathscr{C}(t)$) and so from (1) we get

$$L'(0) = \int_0^l g(\nabla_X T, T)dt = \int_0^l g(\nabla_T X, T)dt = 0$$

since $\nabla_T X = 0$ for a parallel displaced X.

For the second variation we continue from (4)

$$L''(\alpha) = \int_0^l \nabla_X \left\{ \frac{g(\nabla_T X, T)}{g(T, T)^{1/2}} \right\} dt$$

which expands to

$$L''(0) = \int_0^l \nabla_X g(\nabla_T X, T)dt - \int_0^l g(\nabla_T X, T)^2 dt \ .$$

But X is displaced parallel along $\mathscr{C}$; $\nabla_T X = 0$ and so the second integral vanishes. Thus

$$L''(0) = \int_0^l g(\nabla_X \nabla_T X, T)dt + \int_0^l g(\nabla_T X, \nabla_X T)dt$$

but again the second integral vanishes. Using (2) the first term becomes

$$L''(0) = \int_0^l g(\nabla_T \nabla_X X, T)dt + \int_0^l g(R(X, T)X, T)dt \ .$$

The first integral transforms by means of

$$g(\nabla_T \nabla_X X, T) = \nabla_T g(\nabla_X X, T) - g(\nabla_X X, \nabla_T T) = \frac{\partial}{\partial t} g(\nabla_X X, T)$$

and using (3) we get the desired second variation.

The end terms in the second variation are interpreted as follows. $B_T(X)_P \equiv g(\nabla_X X, T)$ is the second fundamental form for V at P corresponding to the normal vector T, evaluated at the tangent vector X.

2. Real manifolds with positive curvature. A submanifold V of a Riemannian M_n is *totally geodesic* if any geodesic of M that is tangent to V at a point lies wholly in V. This implies that every geodesic of V (in the naturally induced metric from M) is at the same time a geodesic of M.

THEOREM 1. *Let M_n be a complete*[3] *connected manifold with positive Riemannian sectional curvature and let V_r and W_s be compact totally geodesic submanifolds. If $r + s \geqq n$ then V_r and W_s have a non-empty intersection.*

Proof. At first we assume that V_r and W_s are *any* compact submanifolds. We suppose they do not intersect. Then there is a shortest geodesic $\mathscr{C}(t)$, say of length $l > 0$, from V to W and let P and Q be the points $\mathscr{C}(0)$ and $\mathscr{C}(l)$ respectively. Since $\mathscr{C}$ is the shortest geodesic from V to W it strikes V and W orthogonally. We will arrive at a contradiction by exhibiting a variation X for which $L''_X(0) < 0$, thus showing that $\mathscr{C}$ cannot be minimizing.

Let $\mathscr{V}_0$ be the tangent space to V_r at P. By parallel translation along $\mathscr{C}$ we get a subspace $\mathscr{V}_l$ of $\mathscr{M}$, the tangent space to M_n at Q. Since $\mathscr{V}_0$ is orthogonal to $\mathscr{C}$ at P, $\mathscr{V}_l$ is also orthogonal to $\mathscr{C}$ at Q. Let $\mathscr{W}$ be the tangent space to W_s at Q. Then $\mathscr{V}_l$ and $\mathscr{W}$ are two subspaces of the linear space $\mathscr{M}$; moreover, both $\mathscr{V}_l$ and $\mathscr{W}$ are orthogonal to $\mathscr{C}$ at Q. Thus the dimension of their intersection is

$$\dim(\mathscr{V}_l \cap \mathscr{W}) \geqq r + s - (n-1) \geqq 1 \tag{5}$$

and thus $\mathscr{V}_l$ and $\mathscr{W}$ have at least a one dimensional subspace in common. But this simply means that there is a unit vector X_0 tangent to V at P whose parallel translate is tangent to W at Q. Let X_t be the parallel translate of X_0 along $\mathscr{C}$. The term $-\int_0^l K(T, X)dt$ of the second variation formula is strictly negative by the curvature assumption.

So far V and W were arbitrary. To evaluate the end terms in the second variation we use the fact that V and W are totally geodesic. The variation vector X_t is given. For the construction of the "ribbon" we can choose geodesics of M through each X_t; since X_0 is tangent to V at P and since V is totally geodesic, the geodesic through X_0 will lie entirely in V. Likewise the geodesic through X_l will lie entirely in W. Thus the curves $\alpha =$ constant will have their endpoints on V and W as required for the variation. But since X_0 and X_l are tangent vectors to geodesics of M we have $\nabla_X X = 0$ at P and Q. Hence the end terms of the second variation formula vanish and we have

$$L''_X(0) = -\int_0^l K(T, X)dt < 0 \qquad \text{Q.E.D.}$$

as desired.

We note that $g(\nabla_X X, T)_P = g(\nabla_X X, T)_Q = 0$ is merely the statement that all second fundamental forms for a totally geodesic submanifold vanish identically.

[3] If the curvature is bounded away from 0, $K \geqq \varepsilon > 0$, the classical result of Bonnet-Myers states that M_n is necessarily compact.

There is at least one situation when totally geodesic submanifolds arise "naturally." If $f\colon M_n \to M_n$ is an *isometric* map of a Riemannian manifold into itself, then the set of fixed points $F = \{P \in M \mid f(P) = P\}$ has as components totally geodesic submanifolds (see [4]). Hence

COROLLARY. *If $f\colon M_n \to M_n$ is an isometry of a compact connected Riemannian manifold with positive curvature, then no two fixed set components can have dimension sum $\geqq n$.*

3. Kähler manifolds with positive curvature. A Kähler manifold M is a special type of Riemannian manifold whose underlying space is a complex manifold. There is a linear transformation J on each tangent space that sends any vector Y into a vector JY orthogonal to Y (J represents multiplication by $(-1)^{1/2}$). J has the properties $J^2 = -I$ and $g(JY, JZ) = g(Y, Z)$ for all vectors Y and Z (this last property states that g is a "Hermitian" metric). From J we construct the Kähler exterior 2-form ω, defined by

$$\omega(Y, Z) = g(JY, Z) .$$

ω is exterior because $\omega(Y, Z) = -\omega(Z, Y)$. All that has been said so far holds for any Hermitian manifold. The further condition defining a Kähler manifold can be stated as requiring that ω be covariant constant, $\nabla_U \omega = 0$ for all vectors U; i.e., for any vector fields Y and Z we have

$$\nabla_U \omega(Y, Z) = \omega(\nabla_U Y, Z) + \omega(Y, \nabla_U Z) .$$

Since g is also covariant constant we conclude that J is also, i.e., we have the operator equation

$$\nabla_U \circ J = J \circ \nabla_U \tag{6}$$

for any vector U.

A linear subspace $\mathscr{V}$ of the tangent space to a complex manifold at a point is said to be *complex* if it is invariant under J, $J\colon \mathscr{V} \to \mathscr{V}$. A submanifold is *complex analytic* if its tangent space at each point is complex.

When dealing with complex manifolds dimension subscripts will denote complex dimension.

The following result is easily true for $P_n(C)$ since it holds for the linear subspaces.

THEOREM 2. *Let M_n be a complete, connected Kähler manifold with positive sectional curvature and let V_r and W_s be compact complex analytic submanifolds. If $r + s \geqq n$, then V_r and W_s we have a non-empty intersection.*

Proof. The proof is again by contradiction, starting exactly as in Theorem 1. We again arrive at a variation vector X_t, parallel displaced along $\mathscr{C}$ and tangent to V and W at P and Q respectively. Now, however, we have additional information. Since V and W are complex analytic the vector field $J(X_t)$ is tangent to V and W at P and Q respectively. Further, from (6) we have $\nabla_T J(X_t) = J\nabla_T X_t = 0$ since X_t is parallel displaced. Thus $J(X_t)$ is also parallel displaced and gives us the same type of variation vector as X_t. We claim

$$\begin{cases}\text{the second variation corresponding to at least one of}\\ \text{the fields } X_t \text{ or } JX_t \text{ is strictly negative}\end{cases}$$

again giving a contradiction.

To prove our claim we suppose

$$L''_X(0) = g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P - \int_0^l K(T, X)dt \geqq 0 . \tag{7}$$

By the hypothesis of positive curvature we conclude that

$$g(\nabla_X X, T)_Q - g(\nabla_X X, T)_P > 0 .$$

We will be finished if we can show $g(\nabla_{JX} JX, T)_Q - g(\nabla_{JX} JX, T)_P < 0$. But this is actually the case as follows from the fact that every second fundamental form of a complex analytic submanifold of a Kähler manifold is skew-hermitian,[4] i.e.

$$\begin{cases} g(\nabla_{JX} JX, T)_P = -g(\nabla_X X, T)_P & \text{for } V \\ g(\nabla_{JX} JX, T)_Q = -g(\nabla_X X, T)_Q & \text{for } W . \end{cases} \tag{8}$$

The proof of this is simple and we include it here for completeness.

Let $\mathscr{R}$ be a complex analytic curve (real dimension 2) on V tangent to X_0 and JX_0 at P. Then X_0 can be extended to a tangent vector field X on $\mathscr{R}$ and of course JX is an extension of JX_0. Since X and JX are tangent vector fields to $\mathscr{R}$ the commutator bracket $[JX, X]$ is again a vector field tangent to $\mathscr{R}$, and thus orthogonal to T at P. Using $[JX, X] = \nabla_{JX} X - \nabla_X JX$ and (6) and $J^2 = -I$ we get at P

$$\begin{aligned} g(\nabla_{JX} JX, T) &= g(J\nabla_{JX} X, T) = g(J[JX, X] + J\nabla_X JX, T) \\ &= g(J[JX, X], T) - g(\nabla_X X, T) . \end{aligned}$$

Since $[JX, X]$ is tangent to $\mathscr{R}$, so is $J[JX, X]$ and so the first term vanishes and the result follows. Q.E.D.

[4] This is a reflection of the fact that Kähler submanifolds of a Kähler manifold are minimal submanifolds in the sense of the calculus of variations. Thus their mean curvatures vanish for all normal directions.

4. Kähler surfaces with positive curvature. We now consider the case of Kähler surfaces M_2 (real dimension 4). We noticed previously[2] that by Kodaira's theorem such a surface is necessarily algebraic.

We recall that an *exceptional curve* (of the first kind) arises in the following fashion. There is a surface N_2 and a point $P \in N_2$ such that M_2 is a quadratic transform [3] of N_2 and the exceptional curve is the quadratic transform of p. Thus exceptional curves result from blowing up a point p of a surface by means of the Hopf σ-process; i.e., the point p is replaced by the complex projective line $P_1(\boldsymbol{C})$ of complex directions at p. Since there clearly are curves that do not intersect the exceptional curve (hyperplane section of N_2 for example) we conclude from Theorem 2 that a positively curved compact Kähler surface has no exceptional curves (of the first kind).

THEOREM 3. *A compact Kähler surface M_2 with positive sectional curvature is complex analytically homeomorphic to $P_2(\boldsymbol{C})$.*

Proof (Andreotti). As mentioned before[2] the Ricci curvature of a positively curved Kähler M_n is positive. The negative of the exterior Ricci form represents the characteristic class of the canonical bundle K over M. By Kodaira's "vanishing theorem" [5] we conclude $H^p(M_n; \Omega^0(K^1)) = 0$, $p \neq n$, where K^i is the line bundle $K \otimes \cdots \otimes K$, i factors and where $\Omega^0(K^i)$ is the sheaf of germs of holomorphic sections of K^i. Thus the plurigenera $P_i = \dim H^0(M_n; \Omega^0(K^i))$ all vanish and since M_2 is simply connected the arithmetic genus $p_a = P_1 - h^{1,0} = 0$ also. We now apply results in the classification theory of *surfaces*, i.e., $n = 2$. By a theorem of Castelnuovo-Enriques (for references see, for example, Zariski's book, *Introduction to the problem of minimal models in the theory of algebraic surfaces*, Math. Soc. of Japan, 1958, p. 84) we conclude that M_2 is rational. As we have just seen M_2 can have no exceptional curves (of the first kind). By a result of Andreotti [1] M_2 is either birationally equivalent, without exceptions, to $P_2(\boldsymbol{C})$ or else it is a ruled surface. Since the rulings would be compact curves that do not intersect, Theorem 2 eliminates this last possibility. Q.E.D.

5. Correspondences. A (holomorphic) correspondence of a complex manifold N_n with itself is a complex analytic n dimensional submanifold of $N_n \times N_n$.

A holomorphic map $f\colon N_n \to N_n$ gives rise to a correspondence, the graph $G(f)$ of f; $G(f) = \{(p, fp) \mid p \in N_n\}$. $G(f)$ is of course a special type of correspondence since f is single valued. Let $\Delta = \{(p, p) \mid p \in N_n\}$ be the diagonal of $N_n \times N_n$. It is clear that a map f will have a fixed point whenever $G(f)$ intersects the diagonal Δ. A correspondence will be said to have a fixed point if it intersects the diagonal.

THEOREM 4. *Every (holomorphic) correspondence of a connected compact Kähler manifold N_n with positive curvature has a fixed point.*

Proof. Again this is a simple known property of $P_n(\boldsymbol{C})$.

The correspondence is a complex analytic submanifold V_n of $N_n \times N_n$. The same is true for the diagonal Δ. We need only show that V_n and Δ intersect, and this almost follows from Theorem 2. However, as pointed out in the introduction, $N_n \times N_n$ has only nonnegative curvature; product planes give 0 sectional curvature. This, however, is easily mended as follows.

In our previous notation $V_n = V$, $\Delta = W$ and $N_n \times N_n = M$. In the proof of Theorem 2 positive curvature occurs only in the statement $\int_0^l K(T, X)dt > 0$. Now we can only say.

$$\begin{cases} L''_X(0) = (\nabla_X X, T)_Q - (\nabla_X X, T)_P - \int_0^l K(T, X)dt \\ \qquad\qquad\qquad\qquad \int K(T, X)dt \geqq 0 \end{cases}$$

Again we suppose $L''_X(0) \geqq 0$.

Case 1. $(\nabla_X X, T)_Q - (\nabla_X X, T)_P > 0$. Then from (8) we $L''_{JX}(0) < 0$ and we are finished.

Case 2. $(\nabla_X X, T)_Q = (\nabla_X X, T)_P$ and $\int_0^l K(T, X)dt = 0$. We will then be finished if we can show $\int_0^l K(T, JX)dt > 0$. Now $\int_0^l K(T, X)dt = 0$ means $T \wedge X$ is a product plane along $\mathscr{C}$, in particular at $Q \in W = \Delta$. Choose a real basis for the tangent space to $N_n \times N_n$ at Q consisting of the $2n$ "horizontal" orthonormal vectors $e_1, Je_1, \cdots, e_n, Je_n$ and the $2n$ "vertical" orthonormal vectors $f_1, Jf_1, \cdots, f_n, Jf_n$. Since $T \wedge X$ is a product plane the basis can be so chosen that

$$X = (\cos\theta)e_1 + (\sin\theta)f_1$$
$$T = -(\sin\theta)e_1 + (\cos\theta)f_1 .$$

Thus

$$JX = (\cos\theta)Je_1 + (\sin\theta)Jf_1 .$$

This means that the only possibilities for $T \wedge JX$ to be a product plane are either $\cos\theta = 0$ or $\sin\theta = 0$, i.e., either $T = \pm e_1$ or $T = \pm f_1$. But e_1 and f_1 being respectively horizontal and vertical cannot be orthogonal to the diagonal $W = \Delta$ while the geodesic tangent T must be. We thus conclude that if $T \wedge X$ is a product plane then $T \wedge JX$ *cannot* be. Hence $\int_0^l K(T, JX)dt > 0$. Q.E.D.

The isometries (rotations) of the 3-sphere without fixed points show that there is no real analogue of Theorem 4.

REFERENCES

1. A. Andreotti, *On the complex structures of a class of simply connected manifolds*, in the Lefschetz symposium volume *Algebraic geometry and topology*, Princeton, 1957.
2. M. Berger, *Les variétés riemanniennes* (1/4)-*pincées*. C. R. Acad. Sci. Paris, **250** (1960), 442-44.
3. H. Hopf, *Zur Topologie der komplexen Mannigfaltigkeiten*, Studies and Essays presented to R. Courant, New York, (1948), 167-85.
4. S. Kobayashi, *Fixed points of isometries*, Nagoya Math. J., **13** (1958) 63-68.
5. K. Kodaira, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 1268-73.
6. ———, *On Kähler varieties of restricted type*, Ann. of Math., **60** (1954), 28-48.
7. K. Nomizu, *Lie groups and differential geometry*, Math, Soc. Japan (1956).
8. H. E. Rauch, *A contribution to differential geometry in the large*, Ann. of Math., **54** (1951), 38-55.
9. J. L. Synge, *The first and second variations of length in Riemannian space*, Proc. London Math. Soc., **25** (1926).
10. ———, *On the connectivity of spaces of positive curvature*, Quart. J. Math. (Oxford series) **7** (1936), 316-20.

STANFORD UNIVERSITY