HARDY'S INEQUALITY AND ITS EXTENSIONS

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1. Introduction. In this paper we are concerned with a systematic and uniform treatment of some analogues and extensions of Hardy's inequality for integrals. This result we state as

THEOREM 1. *If* $p > 1$, $f(x) \geqq 0$, *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p dx$$

unless $f \equiv 0$. *The constant is the best possible.*

This theorem was first proved by Hardy [1], and various alternative proofs have been given by other authors. (For reference to these, see [3, 240—243].) Theorem 1, together with the following generalization of this result (also due to Hardy, [2] and [3, Th. 330]) may be regarded as models of the class of inequalities with which this paper deals.

THEOREM 2. *If* $p > 1$, $r \neq 1$, $f(x) \geqq 0$, *and* $F(x)$ *is defined by*

$$F(x) = \begin{cases} \int_0^x f(t)dt & (r > 1)\,, \\ \int_x^\infty f(t)dt & (r < 1)\,, \end{cases}$$

then

$$\int_0^\infty x^{-r} F^p dx < \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf)^p dx$$

unless $f \equiv 0$. *Again the constant is the best possible.*

Our integral inequalities will be of the form

$$\int_a^b s(x) F^p dx \leqq \int_a^b r(x) f^p dx \tag{1.1}$$

where $p > 1$ (or $p < 0$), and F is defined (as in Theorem 2) as a suitable integral of $f(x)$. For $0 < p < 1$, we obtain inequalities of the form (1.1), but with the inequality sign reversed. Our method of proof differs from those referred to above. We make use of the Euler-Lagrange differential equations

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$$\frac{d}{dx}\{r(x)y'^{p-1}\} + s(x)y^{p-1} = 0 \ , \tag{1.2}$$

$$\frac{d}{dx}\{r(x)(-y')^{p-1}\} - s(x)y^{p-1} = 0 \ , \tag{1.3}$$

associated with the minimum problem (1.1). Here, (1.2) applies if $F' = f$, while (1.3) applies if $F' = -f$. Nevertheless, the method is not a variational method, the difficulties involved in such an approach being considerable. (cf. [3, p. 181], where a variational proof of Theorem 1 is sketched.) Rather, we make use of certain Riccati-like equations associated with (1.2), (1.3) leading to integral identities. Aside from this, the main tools used are Hölder's inequality and two special, simple cases of the theorem of the arithmetic-geometric means.

In § 2 we begin by disposing of several lemmas on the "order of a zero" of a function. There will be needed in § 3, where we deal with the inequalities (1.1); this arrangement avoids interrupting the main thread of the argument. Finally, in § 4, we consider the case that p is a positive, even integer, so that (1.2), (1.3) are the same, and we may allow f (and F) to change sign.

2. Preliminary lemmas.

Throughout this paper our integrals may be interpreted either in the Lebesgue sense, or as (absolutely convergent) improper Riemann integrals, with statements such as $f(x) \equiv g(x)$ to be interpreted accordingly. We always use the letters p and q to denote conjugate exponents, i.e., $p^{-1} + q^{-1} = 1$.

LEMMA 2.1. *Let $r(x)$ be positive and continuous on $a < x < b$, and suppose that $\int_a^b r|f|pdx < \infty$, where $p > 1$. Set*

$$F_1(x) = \int_a^x f(t)dt \ , \qquad (a \leqq x < b) \ ; \qquad F_2(x) = \int_x^b f(t)dt \ , \qquad (a < x \leqq b) \ .$$

If $r(x) = O[(x-a)^{p-1}]$, or if $r^{q/p}(x)\int_a^x r^{-q/p}(t)dt = O(x-a)$, then

$$r(x)\,|\,F_1(x)\,|^p = o[(x-a)^{p-1}] \ \text{as} \ x \to a+ \ . \tag{2.1}$$

If $r(x) = O[(b-x)^{p-1}]$, or if $r^{q/p}(x)\int_x^b r^{-q/p}(t)dt = O(b-x)$, then

$$r(x)\,|\,F_2(x)\,|^p = o[(b-x)^{p-1}] \ \text{as} \ x \to b- \ . \tag{2.2}$$

Either a or b may be infinite, the order conditions being modified appropriately.

Proof. We prove only (2.1), the proof of (2.2) being the same.

First, note that $F_1(x) = o(1)$, so that if $r(x) = O[(x-a)^{p-1}]$, then (2.1) follows. If the alternative hypothesis holds, then

$$|F_1(x)| \leqq \int_a^x |f|\,dt = \int_a^x (r^{1/p}|f|)(r^{-1/p})dt \leqq \left(\int_a^x r|f|^p dt\right)^{1/p} \cdot \left(\int_a^x r^{-q/p}dt\right)^{1/q}$$

by Hölder's inequality. Hence

$$r(x)|F_1(x)|^p \leqq o(1)\left\{r^{q/p}(x)\int_a^x r^{-q/p}dt\right\}^{p/q} = o(1)\cdot O[(x-a)^{p/q}] = o[(x-a)^{p-1}] ,$$

as asserted.

We remark that (2.1) is well-known in the case $r(x) \equiv 1$, where the assertion is simply that $f \in L_p(a, x)$ implies $|F_1(x)|^p = o[(x-a)^{p-1}]$. (cf. [3, Th. 222].) That $r(x)$ must satisfy some restriction in order to assure (2.1) for all F_1 such that $\int r|f|^p dx$ converges is easily seen by taking $a = 0$, $f \equiv 1$, $F \equiv x$. Finally, we note that the two hypotheses assuring (2.1) are mutually exclusive. For, $r(x) \leqq k(x-a)^{p-1}$ implies $r^{-q/p}(x) \geqq k(x-a)^{-1}$, so that $\int_a^x r^{-q/p}dt$ does not exist.

COROLLARY 2.1. *The hypotheses for* (2.1) *are satisfied if either*

(2.1.1) $\qquad 0 < c_1 \leqq r(x) \leqq c_2$ *on* $(a, x]$, *or*

(2.1.2) $\qquad r(x)$ *is nonincreasing on* $(a, x]$, *and* a *is finite.*

For (2.2), *the same result holds, with "nonincreasing" replaced by "nondecreasing,"* (*and* b *is finite*).

If $a = -\infty$, then (2.1.1) implies that $r(x)$ is bounded on $(-\infty, x]$, and hence that $r(x) = o(|x|^{p-1})$ as $x \to -\infty$. If a is finite, then

$$r^{p/q}(x)\int_a^x r^{-q/p}dt \leqq c_2^{q/p}c_1^{-q/p}\int_a^x dt = O(x-a) ,$$

or

$$r^{q/p}(x)\int_a^x r^{-q/p}dt \leqq \int_a^x dt = O(x-a) ,$$

according as (2.1.1) or (2.1.2) holds.

The next lemma, although not strictly required in the sequel, may shed some light on the question as to whether the inequality (1.1) can be "improved," in a given case. The notation is that of Lemma 2.1, and we again assume that $\int_a^b r|f|^p dx < \infty$, and $p > 1$.

LEMMA 2.2. *If either* $r^{q/p}(x)\int_a^x r^{-q/p}dt = o(b-x)$, *or if both* $r(x) = o[(b-x)^{p-1}]$ *and* $r^{q/p}(x)\int_k^x r^{-q/p}dt = O(b-x)$, *where* $k \geqq a$, *then*

(2.3) $$r(x)\,|F_1(x)|^p = o[(b-x)^{p-1}] \text{ as } x \to b- .$$

Similarly, if $r^{q/p}(x)\int_x^b r^{-q/p}dt = o(x-a)$, *or if both* $r(x) = o[(x-a)^{p-1}]$ *and* $r^{q/p}(x)\int_x^k r^{-q/p}dt = O(x-a)$, *where* $k \leqq b$, *then*

(2.4) $$r(x)\,|F_2(x)|^p = o[(x-a)^{p-1}] \text{ as } x \to a+ .$$

Proof. Again we shall prove only the first half of this lemma. If the first hypothesis is valid, then from the proof of (2.1) we have

$$r(x)\,|F_1(x)|^p \leqq \int_a^b r\,|f|^p dx \cdot \left(r^{q/p}(x)\int_a^x r^{-q/p}dt\right)^{p/q} = o[(b-x)^{p-1}] .$$

Now, suppose the alternative hypotheses are valid. Then

$$r^{q/p}(x)\int_k^x r^{-q/p}dt \leqq K(b-x)$$

for x near b. Given $\varepsilon > 0$ there corresponds $X \geqq k$ such that

$$\left(\int_X^b r\,|f|^p dt\right)^{1/p} \cdot K^{1/q} < \varepsilon .$$

Proceeding now as in Lemma 2.1, we have

$$|F_1(x)| - |F_1(X)| \leqq \int_X^x |f|\,dt \leqq \left(\int_X^x r\,|f|^p dt\right)^{1/p} \cdot \left(\int_X^x r^{-q/p}dt\right)^{1/q} ,$$

$$r^{1/p}(x)\,|F_1(x)| \leqq r^{1/p}(x)\,|F_1(X)| + \left(\int_X^b r\,|f|^q dt\right)^{1/p} \cdot \left(r^{q/p}(x)\int_X^x r^{-q/p}dt\right)^{1/q} .$$

Hence

$$r^{1/p}(x)\,|F_1(x)| \leqq r^{1/p}(x)\,|F_1(X)| + \varepsilon(b-x)^{1/q} ,$$

so

$$\frac{r^{1/p}(x)\,|F_1(x)|}{(b-x)^{(p-1)/p}} \leqq \frac{r^{1/p}(x)}{(b-x)^{1/q}} \cdot |F_1(X)| + \varepsilon .$$

Letting $x \to b-$, we obtain

$$\overline{\lim_{x \to b-}} \frac{r^{1/p}(x)\,|F_1(x)|}{(b-x)^{(p-1)/p}} \leqq \varepsilon .$$

Since ε is arbitrary, (2.3) is established.

We note, without proof, that if $b = \infty$ then (2.3) is valid if either $0 < c_1 \leqq r(x) \leqq c_2$ or if $r(x)$ is nonincreasing on $[x, \infty)$.

LEMMA 2.3. *Let $r(x)$ be positive and continuous on $a < x < b$, and suppose that $\int_a^b rf^p dx < \infty$, where $f > 0$ and $p < 0$. Set*

$$F_1(x) = \int_a^x f(t)dt\,, \qquad (a \leqq x < b)\,; \qquad F_2(x) = \int_x^b f(t)dt, \qquad (a < x \leqq b)\,.$$

If

$$\left\{r^{q/p}(x)\int_a^x r^{-q/p}dt\right\}^{-1} = O[(x-a)^{-1}]\,,$$

then

$$r(x)F_1^p(x) = o[(x-a)^{p-1}]$$

as $x \to a+$. Similarly, if

$$\left\{r^{q/p}(x)\int_x^b r^{-q/q}dt\right\}^{-1} = O[(b-x)^{-1}]\,,$$

then

$$r(x)F_2^p(x) = o[(b-x)^{p-1}]$$

as $x \to b-$. If a or b is infinite, the result is still valid, with $(x-a)$ or $(b-x)$ replaced by $|x|$.

Proof. This time we shall give the proof of the second half of the lemma. Proceeding as in Lemma 2.1, and noting that Hölder's inequality is reversed for $p < 0$, we have

$$F_2(x) = \int_x^b (r^{1/p}f)(r^{-1/p})dt \geqq \left(\int_x^b rf^p dt\right)^{1/p}\left(\int_x^b r^{-q/p}dt\right)^{1/q},$$

$$F_2^p(x) \leqq \left(\int_x^b rf^p dt\right)\left(\int_x^b r^{-q/p}dt\right)^{p/q}.$$

Hence

$$r(x)F_2^p(x) \leqq o(1)\cdot\left\{r^{q/p}(x)\int_x^b r^{-q/p}dt\right\}^{p/q} = o\,[(b-x)^{p-1}]\,,$$

since $p/q = p - 1 < 0$.

We point out that the existence of the integrals $\int_x^b rf^p dt$, $\int_x^b f dt$ assures the existence of $\int_x^b r^{-q/p}dt$ in this case $(p < 0)$. Finally, we note that the appropriate hypothesis of the lemma is satisfied: if $a \neq -\infty$, and either $0 < c_1 \leqq r(x) \leqq c_2$ or $r(x)$ is nonincreasing on $(a, x]$; or if $b < \infty$, and either $0 < c_1 \leqq r(x) \leqq c_2$ or $r(x)$ is nondecreasing on $[x, b)$.

LEMMA 2.4. *With the same notation as in Lemma 2.3 $(p < 0,\ \int_a^b rf^p dx < \infty)$. If either*

$$\left\{r^{q/p}(x)\int_a^x r^{p/q-}dt\right\}^{-1} = o[(b-x)^{-1}] \,,$$

or

$$r(x) = o[(b-x)^{p-1}] \,,$$

then

$$r(x)F_1^p(x) = o[(b-x)^{p-1}] \text{ as } x \to b- \,.$$

If either

$$\left\{r^{q/p}(x)\int_x^b r^{-q/p}dt\right\}^{-1} = o[(x-a)^{-1}] \,,$$

or

$$r(x) = o[(x-a)^{p-1}] \,,$$

then

$$r(x)F_2^p(x) = o[(x-a)^{p-1}] \text{ as } x \to a+ \,.$$

Again, we shall prove only the second assertion. Since F_2 increases as $x \to a+$, F_2^p decreases; hence if $r(x) = o[(x-a)^{p-1}]$, the conclusion follows.

If the alternative hypothesis holds, then from the proof of Lemma 2.3, we have

$$r(x)F_2^p(x) \leqq \int_a^b rf^p dx \cdot \left\{r^{q/p}(x)\int_x^b r^{-q/p}dt\right\}^{p/q} = o[(x-a)^{p-1}] \,.$$

Finally, we note that if $a \neq -\infty$, the second assertion of the lemma will be valid if $r(x)$ is bounded near $x = a$.

LEMMA 2.5. *Let $r(x)$, $s(x)$ be positive and continuous for $a < x < b$. Suppose $F_1(x)$ is nonnegative and nondecreasing on $a < x < b$, and that*

$$\int_a^b s(x)F_1^p(x)dx < \infty \,, \qquad 0 < p < 1 \,.$$

If $r(x) = O\left\{(x-a)^{p-1}\cdot\int_x^{(3x-a)/2} s(t)dt\right\}$, *then* $r(x)F_1^p(x) = o[(x-a)^{p-1}]$ *as* $x \to a+$.

If $r(x) = O\left\{(b-x)^{p-1}\int_x^{(b+x)/2} s(t)dt\right\}$, *then* $r(x)F_1^p(x) = o[(b-x)^{p-1}]$ *as* $x \to b-$.

If $a = -\infty$, or $b = \infty$, the assertions should be

$$r(x) = O\left\{|x|^{p-1}\int_x^{x/2} s(t)dt\right\} \text{ implies } r(x)F_1^p(x) = o(|x|^{p-1}) \text{ as } x \to -\infty \,,$$

or

$$r(x)=O\left\{x^{p-1}\int_{x}^{2x}s(t)dt\right\} \text{ implies } r(x)F_1^p(x)=o(|x|^{p-1}) \text{ as } x\to\infty.$$

Proof. Since F_1^p is nondecreasing we have

$$\int_{x}^{(3x-a)/2}s(t)F_1^p(t)dt \geqq F_1^p(x)\int_{x}^{(3x-a)/2}s(t)dt \geqq Kr(x)F_1^p(x)(x-a)^{1-p}.$$

The result now follows from the fact that the left term of this inequality converges to zero as $x\to a+$. The second assertion of the lemma follows in the same way. Finally we note that if $a\neq-\infty$, and $r(x)$ is bounded near $x=a$, then $r(x)F_1^p(x)=o[(x-a)^{p-1}]$ is immediately valid.

LEMMA 2.6. *With the same hypotheses as in Lemma* 2.5, *except* $F_2(x)$ *is supposed nonnegative and nonincreasing on* $a<x<b$:

If $r(x)=O\left\{(x-a)^{p-1}\int_{(x+a)/2}^{x}s(t)dt\right\}$, *then* $r(x)F_2^p(x)=o[(x-a)^{p-1}]$ *as* $x\to a+$.

If $r(x)=O\left\{(b-x)^{p-1}\int_{(3x-b)/2}^{x}s(t)dt\right\}$, *then* $r(x)F_2^p(x)=o[(b-x)^{p-1}]$ *as* $x\to b-$.

This is proved in precisely the same way as Lemma 2.5.

3. Integral inequalities with p real. Let p be a real parameter ($p\neq 0$, $p\neq 1$). Consider the pair of second-order, nonlinear differential equations

$$\frac{d}{dx}\{r(x)y'^{p-1}\}+s(x)y^{p-1}=0, \tag{3.1}$$

$$\frac{d}{dx}\{r(x)(-y')^{p-1}\}-s(x)y^{p-1}=0, \tag{3.2}$$

where $s(x)$, $r(x)$, $r'(x)$ *are assumed continuous on an interval* $a<x<b$, *and* $r(x)>0$ *on this interval.* Here either a or b, or both, may be infinite. We note that these two equations are identical if p is an even integer. In particular, when $p=2$, these equations reduce to the self-adjoint linear equation

$$(r(x)y')'+s(x)y=0.$$

Let $y(x)$ be a solution of (3.1) for which $y(x)>0$, $y'(x)>0$ on (a,b) and set $h(x)=[y'(x)/y(x)]^{p-1}$. Then $h(x)$ satisfies the Riccati-like equation

$$\frac{d}{dx}(rh)+(p-1)rh^q=-s(x), \qquad q=p/(p-1). \tag{3.1*}$$

Similarly, if $y(x)$ is a solution of (3.2) such that $y(x) > 0, y'(x) < 0$ on (a, b), and we set $h(x) = [-y'(x)/y(x)]^{p-1}$, then h satisfies

$$\frac{d}{dx}(rh) - (p-1)rh^q = s(x) \tag{3.2)*$$

Now, suppose $f_1(x), f_2(x)$ are nonnegative, measurable functions on (a, b). With the pair of differential equations (3.1), (3.2), we shall associate the functions

$$F_1(x) = \int_a^x f_1(t)dt\ , \qquad a \leqq x < b\ , \tag{3.3}$$

$$F_2(x) = \int_x^b f_2(t)dt\ , \qquad a < x \leqq b\ , \tag{3.4}$$

Notice, in particular, that our notation implies $F_1(a) = F_2(b) = 0$, and that f_1, f_2 are integrable on any closed subinterval of (a, b). Since r, h and F_i are all continuous and $h \neq 0$ on such a closed subinterval, it follows that if $a < a' < b' < b$, then the integrals

$$I_i(a', b') = \int_{a'}^{b'} r\{f_i^p + (p-1)h^q F_i^p - phf_iF_i^{p-1}\}dx \tag{3.5}$$

exist, provided $f_i \in L_p(a', b')$. In the case $0 < p < 1$ this latter condition follows from the fact that $f_i \in L(a', b')$. If $p < 0$, we must also insist that f_i be strictly positive. Taking $i = 1$ and integrating by parts the last term of (3.5), we obtain

$$I_1(a', b') = \int_{a'}^{b'} rf_1^p dx + (p-1)\int_{a'}^{b'} rh^q\ F_1^p dx + \int_{a'}^{b'} (rh)'F_1^p dx - rhF_1^p\Big|_{b'}^{a'}\ ,$$

or using (3.1)*,

$$I_1(a', b') = \int_{a'}^{b'} rf_1^p dx - \int_{a'}^{b'} sF_1^p dx + r(a')h(a')F_1^p(a') - r(b')h(b')F_1^p(b')\ . \tag{3.6}$$

Proceeding in the same way for I_2, and using (3.2)*, we obtain

$$I_2(a', b') = \int_{a'}^{b'} rf_2^p dx - \int_{a'}^{b'} sF_2^p dx + r(b')h(b')F_2^p(b') - r(a')h(a')F_2^p(a')\ . \tag{3.7}$$

We now use the fact that $I_i(a', b')$ is nonnegative if $p > 1$ or $p < 0$, and nonpositive if $0 < p < 1$. Indeed, this follows from the well-known inequalities [3, Th. 41]

$$x^p + (p-1)y^p - pxy^{p-1} \geqq 0\ , \qquad (p < 0 \text{ or } p > 1)\ , \tag{3.8}$$

$$x^p + (p-1)y^p - pxy^{p-1} \leqq 0\ , \qquad (0 < p < 1)\ . \tag{3.9}$$

Here, x and y are nonnegative (positive if $p < 0$), and in both cases strict inequality holds unless $y = x$. Setting $x = f_i$, $y = h^p F_i$ in (3.8), and recalling that $r(x) > 0$ on (a', b') ,we see from (3.5) that $I_i(a', b') \geqq 0$, with strict inequality unless $f_i/F_i \equiv \pm y'/y$, i.e., unless $f_i \equiv cy'$. Similarly, in the case $0 < p < 1$, we may apply (3.9) to prove $I_i(a', b') \leqq 0$. Hence from (3.6) and (3.7) we obtain

$$\int_{a'}^{b'} sF_1^p dx \lessgtr \int_{a'}^{b'} rf_1^p dx + r(x)h(x)F_1^p(x)\Big|_{b'}^{a'}, \tag{3.10}$$

$$\int_{x'}^{b'} sF_2^p dx \lessgtr \int_{a'}^{b'} rf_2^p dx + r(x)h(x)F_2^p(x)\Big|_{a'}^{b'}, \tag{3.11}$$

where, in both cases, the upper inequality sign holds if $p < 0$ or $p > 1$, and the lower sign holds if $0 < p < 1$. If $p < 0$ or $p > 1$ our hypothesis will be $\int_a^b rf_i^p dx < \infty$. This will incidentally assure $f_i \in L_p(a', b')$. Finally, we note that if inequality holds in (3.10) or (3.11) for any (a', b'), then (assuming the existence of the corresponding limits) inequality will also hold when $a' = a$, $b' = b$. This follows from the fact that $|I_i(a', b')|$ does not decrease as the interval (a', b') expands.

We must now consider separately the three cases $p > 1$, $p < 0$, $0 < p < 1$, as the details differ in the three cases.

3.1. *The case* $p > 1$. Here we have two theorems of which we prove only the first, the remaining theorem following by the same arguments.

THEOREM 3.1.1. *Suppose the differential equation* (3.1) (*where* $p > 1$) *has a solution* $y(x)$ *such that* $y(x) > 0$, $y'(x) > 0$ *on* $a < x < b$, *and that*

$$y'(x)/y(x) = O[(x - a)^{-1}] \text{ as } x \to a+ . \tag{3.1.1}$$

If $r(x) = O[(x - a)^{p-1}]$, *or* $r^{q/p}(x)\int_a^x r^{-q/p}dt = O(x - a)$, *and* $\int_a^b rf_1^p dx < \infty$, *then*

$$\int_a^b s(x)F_1^p dx \leqq \int_a^b r(x)f_1^p(x)dx . \tag{3.1.2}$$

Proof. Setting $h(x) = [y'(x)/y(x)]^{p-1}$, we have $h = O[(x - a)^{1-p}]$ by (3.1.1). Moreover, by Lemma 2.1, we have $rF_1^p = o[(x - a)^{p-1}]$. Hence, letting $a' \to a$ and $b' \to b$ in (3.10), we obtain

$$\int_a^b sF_1^p dx \leqq \int_a^b rf_1^p dx - \overline{\lim_{x\to b}}\, r(x)h(x)F_1^p(x) , \tag{3.1.3}$$

which certainly implies (3.1.2). By our previous remarks, equality can hold in (3.1.3) only if $F_1(x) = cy(x)$, so that equality can hold in (3.1.2) only if both this condition holds (so $y(a)$ must be zero), and

$$\lim_{x\to b} c^p r(x)y(x)y'^{p-1}(x) = 0 .$$

On the other hand, even if this holds for $c \neq 0$, equality may hold in (3.1.2) only for $f_1 \equiv 0$, since $\int ry'^p dx$ may diverge. Indeed, by Lemma 2.1, this integral will diverge unless

$$\lim_{x\to a} r(x)y(x)y'^{p-1}(x) = 0 . \tag{3.1.4}$$

In summary then, *equality holds in* (3.1.2) *only if* $f_1 \equiv cy'(x)$ *where* $c = 0$ *unless all of the conditions*

$$y(a) = 0 , \quad \lim_{x\to a} ryy'^{p-1} = 0 , \quad \int_a^b ry'^p dx < \infty , \tag{3.1.5}$$

hold. In particular, c must be zero unless (3.1.4) is satisfied.

The inequality (3.1.2) is certainly sharp (i.e, the unit constant on the right side cannot be reduced) if the conditions (3.1.5) all hold. Suppose that $\int_a^b ry'^p dx < \infty$, and at least one of the remaining conditions of (3.1.5) does not hold. Then, in general, (3.1.2) is not sharp. This is easily seen by taking $p = 2$, $r(x) = s(x) = 1$, $y(x) = \sin x$, with $0 < a < b \leqq \pi/2$; in this case the unit constant can be reduced to $4(b-a)^2/\pi^2$.

Finally, *if* $\int_a^b ry'^p dx = \infty$, *and* $s(x) \geqq 0$, *then* (3.1.2) *is sharp if*

$$\varliminf_{x\to a} r(x)y(x)y'^{p-1}(x) < \infty \text{ and } \varliminf_{x\to b} r(x)y(x)y'^{p-1}(x) < \infty . \tag{3.1.6}$$

In fact, if $\int_a^x ry'^p dx = \infty$, the first of conditions (3.1.6) is sufficient, and if $\int_x^b ry'^p dx = \infty$, the second of conditions (3.1.6) is sufficient to ensure the sharpness of (3.1.2). To prove this assertion, we take

$$f_1(x) = \begin{cases} 0 , & a \leqq x \leqq a' , \\ y'(x) , & a' < x < b' , \\ 0 , & b' \leqq x \leqq b , \end{cases}$$

where a', b' will be fixed later. Then $F_1(x) = y(x) - y(a')$ for $a' < x < b'$, and

$$F_1^p(x) = y^p(x)\left\{1 - \frac{y(a')}{y(x)}\right\}^p \geqq y^p(x)\left\{1 - p\frac{y(a')}{y(x)}\right\} , a' < x < b' . \tag{3.1.7}$$

This inequality is the special case of (3.8) obtained by taking $x = 1 - y(a')y^{-1}(x)$, $y = 1$. Using (3.1.7) and (3.6) we now have

$$\begin{aligned} \int_{a'}^{b'} sF_1^p dx &\geqq \int_{a'}^{b'} sy^p dx - py(a')\int_{a'}^{b'} sy^{p-1}dx \\ &= \int_{a'}^{b'} ry'^p dx + rhy^p \Big|_{b'}^{a'} - py(a')\int_{a'}^{b'} sy^{p-1}dx . \end{aligned}$$

From (3.1), we have

$$\int_{a'}^{b'} sy^{p-1}dx = ry'^{p-1}\Big|_{b'}^{a'} .$$

Hence

$$\int_a^b sF_1^p dx > \int_{a'}^{b'} rf_1^p dx - r(b')y(b')y'^{p-1}(b') - py(a')r(a')y'^{p-1}(a')$$
$$> (1-\delta)\int_{a'}^{b'} rf_1^p dx = (1-\delta)\int_a^b rf_1^p dx ,$$

provided that

$$r(b')y(b')y'^{p-1}(b') + pr(a')y(a')y'^{p-1}(a') < \delta\int_{a'}^{b'} rf_1^p dx .$$

By (3.1.6) this inequality can be satisfied for any $\delta > 0$ by selecting a' or b', or both, appropriately close to a or b. Hence (3.1.2) is sharp.

It is of interest to note that the sharpness of (3.1.2) implies only that

$$\inf_{f_1}\left\{\overline{\lim_{x\to b}}\, r(x)h(x)F_1^p(x)\right\} = 0 ,$$

where the infimum is taken over all admissible $f_1 \not\equiv 0$. Hence, in general, (3.1.3) certainly states more than (3.1.2) even when (3.1.2) is sharp. On the other hand, if $r(x)$ satisfies the order conditions of Lemma 2.2 at $x = b$, and if $y'(x)y^{-1}(x) = O[(b-x)^{-1}]$ as $x \to b-$, then (3.1.2) and (3.1.3) are the same.

Finally, *we note that if* $p \geqq 2$, *then* (3.1.3) *can be improved to*

$$\text{(3.1.8)} \qquad \int_a^b sF_1^p dx \leqq \int_a^b rf_1^p dx - \overline{\lim_{x\to b}}\, r(x)h(x)F_1^p(x) - \int_a^b r\,|f_1 h^{1/(p-1)}F_1|^p dx .$$

This follows from the fact that the inequality (3.8) can be improved slightly to give

$$\text{(3.1.9)} \qquad x^p + (p-1)y^p - pxy^{p-1} \geqq |x-y|^p , \qquad p \geqq 2 .$$

THEOREM 3.1.2. *Suppose the differential equation* (3.2) (*with* $p > 1$) *has a solution* $y(x)$ *such that* $y(x) > 0$, $y'(x) < 0$ *on* $a < x < b$, *and that*

$$\text{(3.1.10)} \qquad y'(x)/y(x) = O[(b-x)^{-1}] \qquad \textit{as } x \to b- .$$

If $r(x) = O[(b-x)^{p-1}]$, *or* $r^{q/p}(x)\int_x^b r^{-q/p}dt = O(b-x)$, *and* $\int_a^b rf_2^p dt < \infty$, *then*

$$\text{(3.1.11)} \qquad \int_a^b s(x)F_2^p(x)dx \leqq \int_a^b r(x)f_2^p(x)dt ,$$

where $f_2 \geqq 0$, $F_2(x) = \int_x^b f_2 dt$. *Equality holds in* (3.1.11) *only if* $f_2 \equiv cy'(x)$ *where* $c = 0$ *unless all of the conditions*

$$(3.1.12) \qquad y(b) = 0\ , \qquad \lim_{x \to a} ry(-y')^{p-1} = 0\ , \qquad \int_a^b r(-y')^p dx < \infty$$

hold. If $s(x) \geqq 0$, *and* $\int_a^b r(-y')^p dx = \infty$, *then* (3.1.11) *is sharp provided*

$$\varliminf_{x \to a} ry(-y')^{p-1} < \infty,\ \textit{and}\ \varliminf_{x \to b} ry(-y')^{p-1} < \infty\ .$$

Finally, if $p \geqq 2$, (3.1.11) *may be improved to*

$$(3.1.13) \quad \int_a^b sF_2^p dx \leqq \int_a^b rf_2^p dx - \varlimsup_{x \to a} r(x)h(x)F_2^p(x) - \int_a^b r\,|\,f_2 - h^{1/(p-1)}F_2\,|^p dx\ ,$$

where $h = [(-y')/y]^{p-1}$.

Theorem 1 is the special case of Theorem 3.1.1 obtained by setting $a = 0, b = \infty, y(x) = x^{(p-1)/p}$. More generally, Theorem 2 is obtained from Theorem 3.1.1. (for $r > 1$) and Theorem 3.1.2 (for $r < 1$) by taking $y(x) = x^{(r-1)/p}$. In this case, we have

$$r(x) = kx^{p-r}\ , \qquad r^{q/p}(x) = k_1 x^{(p-r)/(p-1)}\ , \qquad \left(k = \left(\frac{p}{|\,r-1\,|}\right)^p\right)$$

so that $r^{q/p}(x)\int_0^x r^{-q/p}dt = k_2 x$, or $r^{q/p}(x)\int_x^\infty r^{-q/p}dt = k_2 x$ according as $r > 1$ or $r < 1$. Since $\int_0^x r\,|\,y'\,|^p dx = \infty$, equality can hold in (3.1.2) only for $f_1 \equiv 0$, ond in (3.1.11) only for $f_2 \equiv 0$. On the other hand,

$$r(x)y(x)\,|\,y'(x)\,|^{p-1} \equiv K$$

so that the corresponding inequality is sharp. Finally, we note that $r(x) = o(x^{p-1})$ and $r^{q/p}(x)\int_0^x r^{-q/p}dt = O(x)$ as $x \to \infty$, for the case $r > 1$, and $r(x) = o(x^{p-1})$, $r^{q/p}(x)\int_x^\infty r^{-q/p}dt = O(x)$ as $x \to 0$, for the case $r < 1$. Hence, according to Lemma 2.2, we have $r(x)F_i^p(x) = o(x^{p-1})$ or $F_i^p(x) = o(x^{r-1})$ as $x \to 0$ (for $i = 2$, $r < 1$), or as $x \to \infty$ (for $i = 1$, $r > 1$). Since $y'/y = kx^{-1}$ in both cases, it follows that (3.1.3) reduces to (3.1.2), with a similar remark holding in case $r < 1$.

As another example for Theorem 3.1.1 we have the following inequality (cf. [3, Th. 256]): *If* $p > 1, y' \geqq 0, y(x) = \int_0^x y'dt$, *then*

$$(3.1.14) \qquad \int_0^{\pi/2} y^p dx \leqq \frac{1}{p-1}\left(\frac{p}{2}\sin\frac{\pi}{p}\right)^p \int_0^{\pi/2} y'^p dx$$

equality holding only if $y = cy(x)$ *where* $y(x)$ *is the unique solution of the equation*

$$x = \frac{p}{2} \sin \frac{\pi}{p} \int_0^y (1 - t^p)^{-1/p} dt \, , \qquad 0 \leqq y \leqq 1 \, .$$

We conclude this section with three examples similar to Theorem 2. *We suppose that* $p > 1$, *and* $i = 1$ *or* 2 *according as* $\alpha > 0$ *or* $\alpha < 0$ *in the first two inequalities, while* $i = 1$ *or* 2 *according as* α, β *are both positive, or both negative in the third inequality. Then*

$$\text{(3.1.15)} \quad |\alpha|^p \int_0^\infty \frac{x^{-1-\alpha}}{(1 + x^\alpha)^{p-1}} F_i^p dx < \int_0^\infty x^{p(1-\alpha)-1} f_i^p dx \qquad \textit{unless } f_i \equiv 0 \, ;$$

$$\text{(3.1.16)} \quad |\alpha|^p \left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{x^{-1+\alpha}}{(1 + x^\alpha)^p} F_i^p dx < \int_0^\infty x^{(p-1)(1-\alpha)} f_i^p dx \quad \textit{unless } f_i \equiv 0 \, ;$$

$$\text{(3.1.17)} \quad |\alpha|^{p-1}(|\alpha| + |\beta|)(p-1) \int_0^\infty \frac{x^{(p-1)(1-\alpha)-p+\beta}}{(1 + x^\beta)^p} F_i^p dx < \int_0^\infty x^{(p-1)(1-\alpha)} f_i^p dx$$

unless $F_i \equiv c x^\alpha (1 + x^\beta)^{-\alpha/\beta}$. *In all three cases, the constants are the best possible.* The (inadmissible) extremal functions for the inequalities (3.1.15), (3.1.16) are $y = 1 + x^\alpha$, $y = (1 + x^\alpha)^{1-(1/p)}$ respectively.

3.2. *The case* $p < 0$. The theorems corresponding to Theorems 3.1.1 and 3.1.2 are stated as Theorems 3.2.1 and 3.2.2, of which we prove only the second.

THEOREM 3.2.1. *Suppose the differential equation* (3.1) (*with* $p < 0$) *has a solution* $y(x)$ *such that* $y(x) > 0$, $y'(x) > 0$ *on* $a < x < b$, *and that*

$$\text{(3.2.1)} \qquad y(x)/y'(x) = O(x - a) \qquad \textit{as } x \to a+ \, ,$$

$$\text{(3.2.2)} \qquad \left\{ r^{q/p}(x) \int_a^x r^{-q/p} dt \right\}^{-1} = O[(x - a)^{-1}] \qquad \textit{as } x \to a+ \, .$$

If $f_1 > 0$, $F_1(x) = \int_a^x f_1 dt$, *and if* $\int_a^b r f_1^p dx < \infty$, *then*

$$\text{(3.2.3)} \qquad \int_a^b s(x) F_1^p dx \leqq \int_a^b r(x) f_1^p dx \, .$$

Equality holds in (3.2.3) *only if* $f_1 \equiv c y'(x)$, *and all of the conditions*

$$\text{(3.2.4)} \qquad y(a) = 0 \, , \qquad \lim_{x \to b} r y y'^{p-1} = 0 \, , \qquad \int_a^b r y'^p dx < \infty \, ,$$

hold. Finally, if $s(x) \geqq 0$, *and* $\int_a^b r y'^p dx = \infty$, *then* (3.2.3) *is sharp if*

$$\text{(3.2.5)} \quad \varliminf_{x \to b} r y y'^{p-1} < \infty , \textit{ and } \varliminf_{x \to a} r(x) [y'(x)]^{p-1} \left(\int_a^x r^{-q/p} dt \right)^{1-(1/p)} < \infty \, .$$

THEOREM 3.2.2. *Suppose the differential equation* (3.2) (*with* $p < 0$)

has a solution $y(x)$ *such that* $y(x) > 0$, $y'(x) < 0$ *on* $a < x < b$, *and that*

$$y(x)/y'(x) = O(b-x) \qquad \text{as } x \to b- \,, \tag{3.2.6}$$

$$\left\{r^{q/p}(x)\int_x^b r^{-q/p}dt\right\}^{-1} = O[(b-x)^{-1}] \qquad \text{as } x \to b- \,. \tag{3.2.7}$$

If $f_2 > 0$, $F_2(x) = \int_x^b f_2 dt$, *and if* $\int_a^b r f_2^p dt < \infty$, *then*

$$\int_a^b s(x)F_2^p dx \leqq \int_a^b r(x) f_2^p dx \,. \tag{3.2.8}$$

Equality holds in (3.2.8) *only if* $f_2 \equiv cy'(x)$, *and all of the conditions*

$$y(b) = 0 \,,\ \lim_{x\to a} ry(-y')^{p-1} = 0 \,,\ \int_a^b r(-y')^p dx < \infty \,, \tag{3.2.9}$$

hold. Finally, if $s(x) \geqq 0$, *and* $\int_a^b r(-y')^p dx = \infty$, *then* (3.2.8) *is sharp if*

$$\varliminf_{x\to a} ry(-y')^{p-1} < \infty, \text{ and } \varliminf_{x\to a} r(x)[-y'(x)]^{p-1}\left(\int_x^b r^{-q/p}dt\right)^{1-(1/p)} < \infty \,. \tag{3.2.10}$$

To prove Theorem 3.2.2, we set $h(x) = [-y'(x)/y(x)]^{p-1}$. Since $p < 1$, we have $h = O[(b-x)^{1-p}]$, and by Lemma 2.3 we also have $rF_2^p = o[(b-x)^{p-1}]$. Hence, from (3.11) we obtain

$$\int_a^b sF_2^p dx \leqq \int_a^b r f_2^p dx - \varlimsup_{x\to a} r(x)h(x)F_2^p(x) \,, \tag{3.2.11}$$

where equality can hold only if $F_2(x) \equiv cy(x)$. Comparing this with (3.2.8), we verify that conditions (3.2.9) are necessary and sufficient for equality (for an admissible function).

To prove the assertion concerning the sharpness of (3.2.8), we must modify the procedure used in Theorem 3.1.1 in view of our requirement $f_2 > 0$. Here, we set

$$f_2(x) = \begin{cases} g(x) \,, & a \leqq x \leqq a' \,, \\ -y'(x) \,, & a' < x < b' \,, \\ Mk(x) \,, & b' \leqq x \leqq b \,, \end{cases}$$

where a', b', M are to be assigned, $g(x)$ is any (fixed) admissible function, and $k(x)$ is an admissible function to be chosen later. For $a' < x < b'$, we have

$$F_2(x) = y(x)\left\{1 - \frac{y(b')}{y(x)} + \frac{M}{y(x)}\int_{b'}^b k dt\right\} .$$

Hence, as in Theorem 3.1.1,

$$F_2^p(x) \geqq y^p(x)\left\{1 - p\frac{y(b')}{y(x)} + p\frac{M}{y(x)}\int_{b'}^b kdt\right\},$$

and

$$\begin{aligned}\int_{a'}^{b'} sF_2^p dx &\geqq \int_{a'}^{b'} sy^p dx - py(b')\int_{a'}^{b'} sy^{p-1}dx + pM\int_{b'}^b kdt\int_{a'}^{b'} sy^{p-1}dx \\ &= \int_{a'}^{b'} r(-y')^p dx + rhy^p\Big|_{a'}^{b'} - p\left\{y(b') - M\int_{b'}^b kdt\right\}\cdot\left\{r(-y')^{p-1}\right\}\Big|_{a'}^{b'} \\ &> \int_{a'}^{b'} r(-y')^p dx - r(a')y(a')[-y'(a')]^{p-1} + py(b')r(a')[-y'(a')]^{p-1} \\ &\qquad + pMr(b')[-y'(b')]^{p-1}\int_{b'}^b kdt\,.\end{aligned}$$

Hence

$$\int_a^b sF_2^p dx > (1-\delta)\int_a^b fr_2^p dx\,,$$

provided

$$\begin{aligned}&r(a')y(a')[-y'(a')]^{p-1} - py(b')r(a')[-y'(a')]^{p-1} - pMr(b')[-y'(b')]^{p-1} \\ &\qquad + (1-\delta)\left\{\int_a^{a'} rg^p dx + M^p\int_{b'}^b rk^p dx\right\} < \delta\int_{a'}^{b'} r(-y')^p dx\,.\end{aligned}$$

We first choose $M = M(b')$ so as to minimize the left side of this inequality. This is accomplished by choosing

$$M = r^{1/(p-1)}(b')\cdot[-y'(b')]\cdot\left(\int_{b'}^b kdx\right)^{1/(p-1)}\left(\int_{b'}^b rk^p dx\right)^{1/(1-p)}.$$

With this choice of M, we find (after some reduction) that we want to choose a', b', k so that

$$\begin{aligned}&r(a')y(a')[-y'(a')]^{p-1} - py(b')r(a')[-y'(a')]^{p-1} \\ &\qquad + (1-p)r^q(b')\cdot[-y'(b')]^p\left(\int_{b'}^b kdx\right)^q\left(\int_{b'}^b rk^p dx\right)^{1/(1-p)} < \delta\int_{a'}^{b'} r(-y')^p dx\,.\end{aligned}$$

An application of Hölder's inequality shows that the best possible choice for k is $k = cr^{-q/p}$, at last for x near b. Moreover, such a k is admissible since $K(x) = \int_x^b r^{-q/p}dt$ is well-defined, and since

$$\int_x^b rk^p dt = \int_x^b r^{-q/p}dt < \infty\,.$$

With this choice of k we want to choose a', b' so that

$$\begin{aligned}(3.2.12)\quad &r(a')y(a')[-y'(a')]^{p-1} - py(b')r(a')[-y'(a')]^{p-1} \\ &\qquad + (1-p)r^{p/(p-1)}(b')[-y'(b')]^p\int_{b'}^b r^{-q/p}dx < \delta\int_{a'}^{b'} r(-y')^p dx\,.\end{aligned}$$

Now, if $\int_a^x r(-y')^p dx = \infty$, we fix $b' < b$ arbitrarily. Using the first of conditions (3.2.10), together with the fact that $y(x)$ is a decreasing function, we see that (3.2.12) can be satisfied for a' appropriately close to a. Similarly, if $\int_x^b r(-y')^p dx = \infty$, we fix a' arbitrarily and, using the second of conditions (3.2.10), can choose b' so that (3.2.12) is satisfied. Hence (3.2.8) is sharp in either case.

We note that the second of conditions (3.2.10) implies (but does not seem to be equivalent to) the condition

$$\varliminf_{x\to b} ry(-y')^{p-1} < \infty .$$

By taking $a = 0$, $b = \infty$, $y(x) = x^{(r-1)/p}$ we obtain the following extension of Theorem 2 to the case $p < 0$: *if $f(x) > 0$, and $F(x)$ is defined by*

$$F(x) = \begin{cases} \int_0^x f(t)dt & (r < 1) , \\ \int_x^\infty f(t)dt & (r > 1) , \end{cases}$$

then

$$\int_0^\infty x^{-r}F^p dx < \left(\left|\frac{p}{|r-1|}\right|\right)^p \int_0^\infty x^{-r}(xf)^p dx . \tag{3.2.13}$$

The constant is the best possible. It may be of interest to note that in this case, as for the case $p > 1$, the hypotheses of Lemma 2.4 are satisfied by $r(x)$ so that (3.2.11) and (3.2.8) are identical.

As further examples of Theorem 3.2.2 we have the following:

If $\alpha > 0, 1 < \beta \leqq 1 - (1/p)$, $f_2 > 0$, $F_2 = \int_x^1 f_2 dt$, then

$$(\alpha\beta)^{p-1}\alpha(1-p)(\beta-1)\int_0^1 \frac{x^{\alpha(1-p)(\beta-1)-1}}{(1-x^\alpha)^p} F_2^p dx \leqq \int_0^1 x^{(1-p)(\alpha\beta-1)} f_2^p dx , \tag{3.2.14}$$

where strict inequality always holds if $\beta = 1 - (1/p)$, and otherwise equality holds only if $F_2 \equiv c(1-x^\alpha)^\beta$.

If $0 < \alpha < 1 - \gamma/(p-1)$, then

$$\alpha^{p-1}[\gamma + (\alpha-1)(p-1)]\int_0^1 \frac{x^{\gamma+(\alpha-1)(p-1)-1}}{(1-x^\alpha)^{p-1}} F_2^p dx < \int_0^1 x^\gamma f_2^p dx , \tag{3.2.15}$$

unless $F_2 \equiv c(1-x^\alpha)$.

If $\alpha > 0, p < 0$, then

$$\alpha^p\left(\frac{p-1}{p}\right)^p \int_0^1 \frac{x^{\alpha-1}}{(1-x^\alpha)^p} F_2^p dx < \int_0^1 x^{(1-p)(\alpha-1)} f_2^p dx . \tag{3.2.16}$$

In each case, the constant is best possible. The inadmissible extremal function for the last inequality is $y = (1 - x^\alpha)^{1-(1/p)}$. Corresponding inequalities can also be obtained involving $F_1 = \int_0^x f_1 dt$, as well as for the case $p > 1$.

3.3. *The case* $0 < p < 1$. Here again we have two theorems corresponding to the two equations (3.1), (3.2).

THEOREM 3.3.1. *Suppose the differential equation* (3.1) (*with* $0 < p < 1$ *and* $s(x) > 0$) *has a solution* $y(x)$ *such that* $y(x) > 0$, $y'(x) > 0$ *on* $a < x < b$, *and that*

(3.3.1) $$y(x)/y'(x) = O(b - x) \qquad as\ x \to b- ,$$

(3.3.2) $$(b - x)^{1-p} r(x) = O\left\{\int_x^{(b+x)/2} s(t)dt\right\}, \qquad as\ x \to b- .$$

If $f_1 \geqq 0$, $F_1(x) = \int_a^x f_1(t)dt$, *and* $\int_a^b sF_1^p dx < \infty$, *then*

(3.3.3) $$\int_a^b sF_1^p dx \geqq \int_a^b rf_1^p dx .$$

Equality holds in (3.3.3) *only if* $f_1 \equiv cy'(x)$, *where* $c = 0$ *unless*

(3.3.4) $$y(a) = 0 , \qquad \lim_{x\to a} ryy'^{p-1} = 0 , \qquad \int_a^b sy^p dx < \infty .$$

If $\int_a^b sy^p dx = \infty$, *then* (3.3.3) *is sharp if*

(3.3.5) $$\varliminf_{x\to a} ryy'^{p-1} < \infty , \qquad \varliminf_{x\to b} y^p \int_x^b sdt < \infty .$$

As in Lemma 2.5, if $b = +\infty$, then $b - x$ is replaced by x, and $(b + x)/2$ by $2x$ in the order conditions (3.3.1), (3.3.2).

To prove (3.3.3) we need only apply (3.10) and Lemma 2.5 to obtain

(3.3.6) $$\int_a^b sF_1^p dx \geqq \int_a^b rf_1^p dx + \varlimsup_{x\to a} r(x)h(x)F_1^p(x) ,$$

where equality can hold only if $F_1 \equiv cy(x)$. This proves (3.3.3) as well as the assertion concerning (3.3.4).

If $f_1 = y'$ is *not* admissible, so $\int_a^b sy^p dx = \infty$, we set

$$f_1(x) = \begin{cases} 0 , & a \leqq x \leqq a' , \\ y'(x) , & a' < x < b' , \\ 0 , & b' \leqq x \leqq b . \end{cases}$$

Proceeding as in Theorem 3.1.1 (but using (3.9) rather than (3.8)), we obtain

$$\int_a^b sF_1^p dx < \{y(b') - y(a')\}^p \int_{b'}^b s dx + \int_a^b r f_1'^p dx + (1-p) r(a') y(a') y'^{p-1}(a') + p y(a') r(b') y'^{p-1}(b')$$

$$< (1+\delta) \int_a^b r f_1'^p dx ,$$

provided

$$y^p(b') \int_{b'}^b s dx + (1-p) r(a') y(a') y'^{p-1}(a') + p y(b') r(b') y'^{p-1}(b') < \delta \int_{a'}^{b'} r y'^p dx . \tag{3.3.7}$$

Now we note that since

$$\int_{a'}^{b'} r y'^p dx = \int_{a'}^{b'} s y^p dx + r y y'^{p-1} \Big|_{a'}^{b'}$$

$$> \int_{a'}^{b'} s y^p dx - r(a') y(a') y'^{p-1}(a') ,$$

it follows that $\int_a^{b'} r y'^p dx = \infty$ if $\int_a^{b'} s y^p dx = \infty$ provided the first of conditions (3.3.5) is valid. Moreover, if $\int_{a'}^{b} s y^p dx = \infty$, then $\int_{a'}^{b} r y'^p dx = \infty$ in any case. Hence, if $\int_a^{b'} s y^p dx = \infty$, (3.3.7) can be satisfied for any $\delta > 0$ by fixing b' and letting $a' \to a$. On the other hand, if $\int_{a'}^{b} s y^p dx = \infty$, we fix a', and show that the left side of (3.3.7) remains finite for b' appropriately close to b. This is true of the first term by the second of conditions (3.3.5). For the other term of (3.3.7) involving b', we have

$$r y y'^{p-1} = r\left(\frac{y'}{y}\right)^{p-1} y^p \leqq K_1 (b-x)^{1-p} y^p r$$

$$\leqq K_2 (b-x)^{1-p} y^p (b-x)^{p-1} \int_x^{(b+x)/2} s dt ,$$

according to (3.3.1) and (3.3.2). It follows that

$$\varliminf_{x \to b} r y y'^{p-1} < \infty ,$$

and that the left side of (3.3.7) remains bounded for b' appropriately close to b.

For completeness we state the theorem corresponding to equation (3.2), but omit the proof.

THEOREM 3.3.2. *Suppose the differential equation* (3.2) (*with* $0 < p < 1$, *and* $s(x) > 0$) *has a solution* $y(x)$ *such that* $y(x) > 0$, $y'(x) < 0$ *on* $a < x < b$, *and that*

$$(3.3.8)\qquad y(x)/y'(x) = O(x-a) \qquad as\ x \to a+\ ,$$

$$(3.3.9)\qquad (x-a)^{1-p}r(x) = O\left\{\int_{(x+a)/2}^{x} s(t)dt\right\} \qquad as\ x \to a+\ .$$

If $f_2 \geqq 0$, $F_2(x) = \int_x^b f_2(t)dt$, *and* $\int_a^b sF_2^p dx < \infty$, *then*

$$(3.3.10)\qquad \int_a^b sF_2^p dx \geqq \int_a^b rf_2^p dx\ ,$$

with equality only if $f_2 \equiv cy'(x)$, *where* $c = 0$ *unless*

$$(3.3.11)\qquad y(b) = 0,\ \lim_{x\to b} ryy'^{p-1} = 0,\ \int_a^b sy^p dx < \infty\ .$$

If $\int_a^b sy^p dx = \infty$, (3.3.10) *is sharp if*

$$(3.3.12)\qquad \varliminf_{x\to b} ryy'^{p-1} < \infty\ ,\quad \varliminf_{x\to a} y^p\int_a^x sdt < \infty\ .$$

Taking $a = 0$, $b = \infty$, $y(x) = x^{(r-1)/p}$, the preceding theorems give the extension of Theorem 2 to the case $0 < p < 1$. This result is also due to Hardy ([2], and Theorem 347, [3]). By taking $y(x) = 1 + x^{\alpha}$, and $i = 1$ or 2 according as $\alpha > 0$ or $\alpha < 0$, we obtain the following analogue of (3.1.15);

$$(3.3.13)\qquad |\alpha|^p\int_0^\infty x^{-1-\alpha}(1+x^{\alpha})^{1-p}F_i^p dx > \int_0^\infty x^{p(1-\alpha)-1}f_i^p dx \qquad unless\ f_i \equiv 0\ .$$

The corresponding analogues of (3.1.16) and (3.1.17) are not valid for $0 < p < 1$. The inequality (3.3.13) is sharp although only the second of conditions (3.3.5) (or (3.3.12)) is satisfied.

4. Integral inequalities with $p = 2k$. As noted previously, if $p = 2k$ the pair of differential equations (3.1), (3.2) reduce to the single equation

$$(4.1)\qquad \frac{d}{dx}\{r(x)y'^{2k-1}\} + s(x)y^{2k-1} = 0\,.$$

If $y(x)$ is a solution of (4.1) for which $y(x) > 0$ on (a, b), and we set $h(x) = [y'(x)/y(x)]^{2k-1}$, then $h(x)$ satisfies the equation

$$(4.2)\qquad \frac{d}{dx}(rh) + (2k-1)rh^{2k/(2k-1)} = -s(x)\ .$$

We adopt a different notation from that used in (3.3), (3.4) by replacing f_i by u' and F_i by u, where *we assume throughout this section that*

$$(4.3)\qquad u(x) = \int_a^x u'(t)dt = -\int_x^b u'(t)dt\ , \qquad a \leqq x \leqq b\ ,$$

so that $u(a) = u(b) = 0$. Proceeding as in § 3 (and noting that (3.8) is valid for all real x, y when $p = 2k$) we obtain

$$\int_{a'}^{b'} su^{2k}dx \leqq \int_{a'}^{b'} ru'^{2k}dx + r(x)h(x)u^{2k}(x)\Big|_{b'}^{a'} \tag{4.4}$$

with strict inequality unless $u(x) \equiv cy(x)$. Note that

$$|u(x)| \leqq \int_a^x |u'|\,dt\,, \qquad |u(x)| \leqq \int_x^b |u'|\,dt\,.$$

It follows that Lemma 2.1 remains valid with f replaced by u' and F_i replaced by u.

We now want to weaken the hypotheses on (4.1); *in particular we want to allow y' and h to have a single discontinuity at a point c of (a, b), and to allow r to have a discontinuity or a zero at $x = c$.* Otherwise, we assume $r(x)$, $r'(x)$, $s(x)$ continuous, and $r(x) > 0$ on $a < x < b$, as in § 3. Under these hypotheses, by an *extended solution* of (4.1) we mean a function $y(x)$ positive and continuous on $a < x < b$ such that $y'(x)$ is continuous except perhaps at $x = c$, and such that rh is continuous on (a, b). Now, replacing $I_1(a', b')$ in (3.5) by $I_1(a', c - \varepsilon) + I_1(c + \varepsilon, b')$, carrying out the corresponding work following (3.5), and then letting $\varepsilon \to 0$, we again obtain (4.4), assuming the existence of $\int ru'^{2k}dx$. Finally, since $a < c < b$, Lemma 2.1 also holds.

THEOREM 4.1. *Suppose the differential equation* (4.1) *has an extended solution $y(x) > 0$ on $a < x < b$ and that*

$$\begin{aligned} y'(x)/y(x) &= O[(x-a)^{-1}] \text{ as } x \to a+\,, \\ y'(x)/y(x) &= O[(b-x)^{-1}] \text{ as } x \to b-\,, \end{aligned} \tag{4.5}$$

and both of the conditions

$$r(x) = O[(x-a)^{2k-1}]\,, \text{ or } r^{q/p}(x)\int_a^x r^{-q/p}dt = O(x-a) \text{ as } x \to a+\,, \tag{4.6}$$

$$r(x) = O[(b-x)^{2k-1}]\,, \text{ or } r^{q/p}(x)\int_x^b r^{-q/p}dt = O(b-x) \text{ as } x \to b-\,, \tag{4.7}$$

hold. If $u(x)$ satisfies (4.3), *and* $\int_a^b ru'^{2k}dx < \infty$, *then*

$$\int_a^b su^{2k}dx \leqq \int_a^b ru'^{2k}dx\,. \tag{4.8}$$

Equality holds only if $u \equiv cy(x)$, where $c = 0$ unless

$$y(a) = y(b) = 0\,, \qquad \int_a^b ry'^{2k}dx < \infty\,. \tag{4.9}$$

Moreover, if $\int_a^b ry'^{2k}dx = \infty$ *and* $s(x) \geqq 0$, *then* (4.8) *is sharp if* $y(a) = y(b)$ *and*

$$\overline{\lim_{x\to a}}\,|\,ryy'^{2k-1}\,| < \infty\ , \ \textit{and}\ \overline{\lim_{x\to b}}\,|\,ryy'^{2k-1}\,| < \infty\ . \tag{4.10}$$

The inequality (4.8), and the sufficiency of the conditions (4.9) for equality, follows from (4.4)—(4.7) together with Lemma 2.1. To prove the assertion concerning sharpness we assume that $y(a) = y(b)$, and that $\int_a^{b'} ry'^{2k}dx = \infty$, and define

$$u(x) = \begin{cases} 0\ , & a \leqq x \leqq a'\ , \\ y(x) - y(a')\ , & a' < x < b'\ , \\ 0\ , & b' \leqq x \leqq b\ . \end{cases}$$

Here a' and b' are to be chosen later, and in such a way that $y(b') = y(a')$. Thus $u(x)$ satisfies (4.3), and $\int_a^b ru^{2k}dx = \int_{a'}^{b'} ry'^{2k}dx < \infty$, so u is admissible. As in § 3.1 we find

$$\begin{aligned}\int_a^b su^{2k}dx &\geqq \int_a^b ru'^{2k}dx + (1-2k)r(a')y(a')y'^{2k-1}(a') - (1-2k)r(b')y(b')y'^{2k-1}(b') \\ &> (1-\delta)\int_a^b ru'^{2k}dx\end{aligned}$$

provided

$$(2k-1)r(a')y(a')y'^{2k-1}(a') - (2k-1)r(b')y(b')y'^{2k-1}(b') < \delta\int_{a'}^{b'} ry'^{2k}dx\ . \tag{4.11}$$

Since $(ry'^{2k-1})' = -sy^{2k-1} \leqq 0$, we see that ry'^{2k-1} is a nonincreasing function on $a < x < b$. It follows from this fact, together with $y(a) = y(b)$, that $y(x) \geqq y(a)$, $a \leqq x \leqq b$. Since $y(x) \not\equiv y(a)$ for x near a (otherwise $\int_a^{b'} ry'^{2k}dx < \infty$), $y(x)$ assumes a maximum value for $x = \alpha$, where $a < \alpha < b$. But then to each a', $a < a' < \alpha$, there corresponds at least one b', $\alpha < b' < b$, such that $y(b') = y(a')$. Choosing such a value of b' in (4.11), we see by (4.10) that for any $\delta > 0$, (4.11) can be satisfied for a' sufficiently close to a. The same proof holds if $\int_{a'}^{b} ry'^{2k}dx = \infty$.

Because of the symmetry of the extremal function, the inequality (3.1.14) can clearly be extended according to Theorem 4.1 to give: *If* $u(o) = u(\pi) = 0$, *then*

$$\int_0^\pi u^{2k}dx \leqq \frac{1}{2k-1}\left(k \sin\frac{\pi}{2k}\right)^{2k}\int_0^\pi u'^{2k}dx\ , \tag{4.12}$$

equality holding only if $u = cy(x)$, *where* $y((\pi/2) + x) = y((\pi/2) - x)$, *and for* $0 \leqq x \leqq (\pi/2)$, $y(x)$ *is the unique solution of the equation*

$$x = k \sin \frac{\pi}{2k} \int_0^y (1 - t^{2k})^{-1/2k} dt \,, \qquad 0 \leqq y \leqq 1 \,.$$

The next two inequalities are the extensions of (3.1.17) corresponding to the choices $\alpha = -(2k-1)^{-1}$, $\beta = -2k(2k-1)^{-1}$ and $\alpha = -(2k-1)^{-1}$, $\beta = -2n$ respectively.

$$\text{(4.13)} \qquad \frac{2k+1}{(2k-1)^{2k-1}} \int_{-\infty}^{\infty} \frac{(xu)^{2k} dx}{(1 + x^{2k/2k-1})^{2k}} < \int_{-\infty}^{\infty} (xu')^{2k} dx$$

$$\textit{unless } u = c(1 + x^{2k/(2k-1)})^{-1/2k} \,.$$

$$\text{(4.14)} \qquad \frac{2n(2k-1)+1}{(2k-1)^{2k-1}} \int_{-\infty}^{\infty} \frac{x^{2n(2k-1)} u^{2k} dx}{(1 + x^{2n})^{2k}} < \int_{-\infty}^{\infty} (xu')^{2k} dx$$

$$\textit{unless } u = c(1 + x^{2n})^{-1/(2n(2k-1))} \,.$$

The following examples are the extensions of the analogues (for $p > 1$) of the inequalities (3.2.14), (3.2.15).

$$\text{(4.15)} \qquad \int_{-1}^{1} \frac{u^{2k} dx}{(1 - x^{2k/(2k-1)})^{2k}} < \int_{-1}^{1} u'^{2k} dx \qquad \textit{unless } u \equiv 0 \,.$$

$$\text{(4.16)} \qquad [n(2k-1)]^{2k} \int_{-1}^{1} \frac{x^{n(2k-1)-1} u^{2k} dx}{(1 - x^{2nk})^{2k}} < \int_{-1}^{1} \frac{u'^{2k} dx}{x^{(2k-1)[n(2k-1)-1]}}$$

unless $u \equiv 0$. In (4.16), n is an odd positive integer. The inadmissible extremal functions for these inequalities are

$$(1 - x^{2k/(2k-1)})^{(2k-1)/2k}, \ (1 - x^{2nk})^{(2k-1)/2k}$$

respectively. The case $k = 1$ of (4.15) is due to Nehari [4].

$$\text{(4.17)} \quad \left(\frac{2k}{2k-1}\right)^{2k-1} (2n+1) \int_{-1}^{1} \frac{x^{2n} u^{2k} dx}{(1 - x^{2k/(2k-1)})^{2k-1}} < \int_{-1}^{1} x^{2n} u'^{2k} dx \qquad (n \geqq 0)$$

unless $u \equiv c(1 - x^{2k/(2k-1)})$.

$$\text{(4.18)} \qquad (2m)^{2k-1}[2n + (2m-1)(2k-1)] \int_{-1}^{1} \frac{x^{2n+(2m-1)(2k-1)-3} u^{2k} dx}{(1 - x^{2m})^{2k-1}}$$

$$< \int_{-1}^{1} x^{2n} u'^{2k} dx \qquad \textit{unless } u \equiv c(1 - x^{2m}) \,.$$

In this inequality, we assume $m \geqq 1$, $n \geqq 1$.

$$\text{(4.19)} \left[\frac{2(n+k)}{2k-1}\right]^{2k-1} \int_{-1}^{1} \frac{u^{2k} dx}{(1 - x^{2(n+k)/(2k-1)})^{2k-1}} < \int_{-1}^{1} \frac{u'^{2k}}{x^{2n}} dx \qquad (n \geqq 0)$$

unless $u \equiv c(1 - x^{2(n+k)/(2k-1)})$.

$$\text{(4.20)} \qquad \left(\frac{2k}{2k+1}\right)^{2k-1} \int_{-1}^{1} \frac{u^{2k} dx}{(1 - x^{2k/(2k+1)})^{2k-1}} < \int_{-1}^{1} x^{4k/(2k+1)} u'^{2k} dx$$

unless $u \equiv c(1 - x^{2k/(2k+1)})$. All of the preceding inequalities are sharp. The concept of an extended solution of (4.1) appears only in examples (4.16), (4.19) and (4.20); of these, y' has a discontinuity (at $x = 0$) only in (4.20). In examples (4.13), (4.14), $u(x)$ of course is to satisfy $u(\pm\infty) = 0$, while $u(\pm 1) = 0$ in examples (4.15)—(4.20).

A final example of Theorem 4.1 involving trigonometric functions is given by

$$(4.21) \qquad \left(\frac{2k}{2k-1}\right)^{2k-1} \int_0^\pi csc^2 x \left(\frac{u}{\sin x}\right)^{2k} dx < \int_0^\pi cot^2 x \left(\frac{u'}{\cos x}\right)^{2k} dx$$

when $u(0) = u(\pi) = 0$, *unless* $u \equiv c \sin^{2k/(2k-1)} x$.

We conclude by noting that in the case $p = 2k$, Theorems 3.1.1 and 3.1.2 (and their proofs) remain valid without the restriction $f_1 \geqq 0$, $f_2 \geqq 0$.

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