ON EQUIVALENCE OF GAUSSIAN MEASURES

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1. Introduction. When are two Gaussian processes equivalent (mutually absolutely continuous with respect to each other)? More precisely, given $\{S, B, m_i\}$, $i = 1, 2$, where S is a set of real valued functions on some interval $[a, b]$, B is a Borel field of subsets of S and m_i is a Gaussian probability measure on B, under what conditions is m_1 equivalent to m_2? This question has been investigated, by several authors. In particular, we mention Jacob Feldman, who in a recent paper [5] has shown that a certain dichotomy exists. If S is a linear space, then either m_1 and m_2 are equivalent or they are perpendicular (mutually singular). Moreover, using some results of Segal [6], he has shown that, if K is the linear span of S and the real constants, then m_1 and m_2 are equivalent if and only if the m_1-equivalence classes of K are the same as the m_2-equivalence classes of K and the identity correspondence between the $L_2(m_1)$ closure of K and the $L_2(m_2)$ closure of K is a bounded invertible operator T such that $(T^*T)^{1/2} - I$ is a Hilbert Schmidt operator.

We propose to look at this question from a somewhat different point of view. It is well known that a Gaussian process and hence its probability measure is determined by a covariance function $r(s, t)$[1]. It should therefore be possible to answer the question posed above directly in terms of conditions on the convariance functions of the two processes. We are able to do this for a rather wide class of Gaussian Markov processes (Theorem 1), and we conjecture that an answer of this type is possible in general. The crucial condition appears to be that the first derivatives of the two covariance functions have the same jump on the diagonal $s = t$. To set the stage for our main theorem, we make the following definition.

DEFINITION 1. Let $M \equiv M[a, b]$ denote the class of all Gaussian processes $\{x(t), a \leqq t \leqq b\}$ with mean function identically zero and covariance function $r(s, t)$ given by

$$r(s, t) = \begin{cases} u(s)v(t) & s \leqq t \\ u(t)v(s) & s \geqq t \end{cases},$$

where moreover,

(A) $u(a) \geqq 0$,

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[1] More correctly, it is determined by a covariance function $r(s, t)$ and a mean function $m(t)$, (see [3], p. 72). We assume that the mean function is identically zero throughout this paper.

(B) $v(t) > 0$ on $[a, b]$,

(C) u'' and v'' exist and are continuous on $[a, b]$,

(D) $v(t)u'(t) - u(t)v'(t) > 0$ on $[a, b]$[2].

Almost all sample functions of such processes are continuous (see [4], pp. 401, 402). We shall assume therefore that the space of sample functions of processes belonging to the class $M[a, b]$ is $\{C, B\}$ where $C \equiv C[a, b]$ is the set of all continuous real valued functions on $[a, b]$ and B is the Borel field of subsets of C generated by sets of the form $\{x \in C : a_k < x(t_k) \leqq b_k,\ k = 1, 2, \cdots, n,\ t_k \in [a, b]\}$.

THEOREM 1. *Let $\{x(t), a \leqq t \leqq b\}$ and $\{y(t), a \leqq t \leqq b\}$ be two Gaussian processes belonging to $M[a, b]$ with probability measures m_r and m_ρ determined by their respective covariance functions $r(s, t)$ and $\rho(s, t)$. Let*

$$r(s, t) = \begin{cases} u(s)v(t) & s \leqq t \\ u(t)v(s) & s \geqq t \end{cases}, \qquad \rho(s, t) = \begin{cases} \theta(s)\phi(t) & s \leqq t \\ \theta(t)\phi(s) & s \geqq t \end{cases}.$$

Then, necessary and sufficient conditions that m_r be equivalent to m_ρ are that

(E) $v(t)u'(t) - u(t)v'(t) = \phi(t)\theta'(t) - \theta(t)\phi'(t)$ *on* $[a, b]$,

(F) $u(a)$ *and* $\theta(a)$ *are either both zero or both non-zero.*

Moreover, if these conditions are satisfied, the Radon-Nikodym derivative of m_ρ with respect to m_r is given by

$$dm_\rho/dm_r = C_1 \exp\left\{[1/2]\left[C_2 x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\}\right]\right\},$$

where

$$C_1 = \begin{cases} \{[\phi(a)v(b)]/[\phi(b)v(a)]\}^{1/2} & \text{if } \theta(a) = 0 \\ \{[u(a)v(b)]/[\theta(a)\phi(b)]\}^{1/2} & \text{if } \theta(a) \neq 0 \end{cases},$$

$$C_2 = \begin{cases} 0 & \text{if } \theta(a) = 0 \\ [\phi(a)\theta(a) - u(a)v(a)]/[v(a)\phi(a)\theta(a)u(a)] & \text{if } \theta(a) \neq 0 \end{cases}$$

and

$$f(t) = [v(t)\phi'(t) - \phi(t)v'(t)]/[v(t)u'(t) - u(t)v'(t)]^3.$$

The "necessity" part of the proof depends on a theorem of Baxter while the "sufficiency" will be made to depend on several lemmas.

[2] Conditions (A), (B) and (D) insure that $r(s, t)$ is a covariance function. Covariance functions which factor this way are sometimes called triangular covariance functions. Gaussian processes determined by triangular covariance functions may be shown to be Markov processes.

[3] The corresponding theorem for the Wiener process on [0, 1], (i.e., for the case $r(s, t) = \min(s, t)$), was obtained by a somewhat different method in the author's doctoral dissertaton written under the direction of Professor R. H. Cameron (see [8]).

After proving the theorem, we give several examples, one of which (Example 3) implies a result previously obtained by Charlotte T. Striebel in connection with Ornstien Uhlenbeck processes.

2. Baxter's theorem and a corollary.

BAXTER'S THEOREM. *Let $\{x(t), \alpha \leqq t \leqq \beta\}$ be a Gaussian process with mean function identically zero and continuous covariance function $r(s, t)$, r having uniformly bounded second derivatives for $s \neq t$. Let*

$$f_r(t) = \lim_{s\to t^-} \frac{r(t, t) - r(s, t)}{t - s} - \lim_{s\to t^+} \frac{r(t, t) - r(s, t)}{t - s} .$$

Then with probability one,

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} [x(t_k) - x(t_{k-1})]^2 = \int_\alpha^\beta f_r(u)du \tag{2.0}$$

where $t_k = \alpha + k(\beta - \alpha)2^{-n}$, $k = 0, 1, 2, \cdots, 2^n$[4].

COROLLARY. *Let $\{x(t), a \leqq t \leqq b\}$ and $\{y(t), a \leqq t \leqq b\}$ be Gaussian processes with mean functions identically zero and covariance functions $r(s, t)$ and $\rho(s, t)$ determining probability measures m_r and m_ρ respectively. Suppose that r and ρ satisfy the conditions of the above theorem. Then, if m_r is equivalent to m_ρ, it follows that $f_r(t) = f_\rho(t)$ for all $t \in [a, b]$.*

Proof of corollary. Let S denote the common space of sample functions of the two processes and let

$$N_\beta^r = \left\{x \in S\colon \lim_{n\to\infty} \sum_{k=1}^{2^n} [x(t_k) - x(t_{k-1})]^2 = \int_a^\beta f_r(u)du\right\}$$

where $t_k = a + k(\beta - a)2^{-n}$, $k = 0, 1, 2, \cdots, 2^n$, $\beta \in [a, b]$. By Baxter's theorem $m_r\{N_\beta^r\} = 1 = m_\rho\{N_\beta^\rho\}$. Now let J denote the Radon-Nikodym derivative of m_ρ with respect m_r. Then if χ denotes the set characteristic function of N_β^r, we have

$$m_\rho\{N_\beta^r\} = E^\rho\{\chi(y)\} = E^r\{\chi(x)J(x)\} = 1^5 .$$

Hence $m_\rho\{N_\beta^r\} = m_\rho\{N_\beta^\rho\} = 1$, i.e., for each $\beta \in [a, b]$, N_β^r and N_β^ρ are sets of m_ρ measure one. It follows that $\int_a^\beta f_r(u)du = \int_a^\beta f_\rho(u)du$ for each $\beta \in [a, b]$ and, since f_r and f_ρ are continuous, $f_r(\beta) = f_\rho(\beta)$ for each $\beta \in [a, b]$.

This result gives the "necessity" almost immediately (see § 5). The

[4] This actually is a slight generalization of the theorem of Baxter (see [**1**]), the generalization being that we state the result (2.0) for the interval $[\alpha,\beta]$ rather than $[0, 1]$.

[5] E^r denotes expected value on the Gaussian process with covariance function r.

"sufficiency" is apparently harder to demonstrate. To facilitate matters we introduce some notation and prove four lemmas.

3. Notation. The following notation will be used throughout the rest of this paper.

$$r(s, t) = \begin{Bmatrix} u(s)v(t) & s \leqq t \\ u(t)v(s) & s \geqq t \end{Bmatrix},$$

$$\rho(s, t) = \begin{Bmatrix} \theta(s)\phi(t) & s \leqq t \\ \theta(t)\phi(s) & s \geqq t \end{Bmatrix},$$

r is the covariance function of the process $\{x(t), a \leqq t \leqq b\}$,
ρ is the covariance function of the process $\{y(t), a \leqq t \leqq b\}$,

$$w(s) = v(s)u'(s) - u(s)v'(s) ,$$
$$\omega(s) = \phi(s)\theta'(s) - \theta(s)\phi'(s) ,$$
$$f(s) = [v(s)\phi'(s) - \phi(s)v'(s)]/w(s) ,$$
$$m = 2^n ,$$
$$t_k = a + k(b - a)/m , \quad k = 0, 1, 2, \cdots, m; \ n > 0 ,$$

for any function g, $g_k = g(t_k)$, unless otherwise indicated,

$$\hat{w}_k = v_{k-1}u_k - u_{k-1}v_k , \quad \hat{\omega}_k = \phi_{k-1}\theta_k - \theta_{k-1}\phi_k ,$$
$$r_{jk} = r(t_j, t_k) , \quad \rho_{jk} = \rho(t_j, t_k) ,$$

R is the mxm matrix with elements r_{jk}; $j, k = 1, 2, \cdots, m$,
P is the mxm matrix with elements ρ_{jk}; $j, k = 1, 2, \cdots, m$,

$|R|$ and $|P|$ are the determinants of R and P respectively,

$$\bar{x} = (x_1, x_2, \cdots, x_m) ,$$
$$\bar{y} = (y_1, y_2, \cdots, y_m) ,$$
$$\overline{\Delta p} = (p_1 - p_0, p_2 - p_1, p_m - p_{m-1}) .$$

4. Some lemmas.

LEMMA 1. *Under* (A), (B) *and* (D) *of Definition* 1,
(a) $u(t) > 0$, $\theta(t) > 0$ *for* $t \in (a, b]$,
(b) $\hat{w}_k > 0$, $\hat{\omega}_k > 0$ *for* $k = 1, 2, \cdots, m$; $n > 0$,
(c) R^{-1} *and* P^{-1} *exist.*

Proof. $d/dt[u(t)/v(t)] = w(t)/v^2(t) > 0$ by (B) and (D). Hence $u(t)/v(t)$ increases as t increases and so $u(t) > v(t)u(a)/v(a)$, $t \in (a, b]$, and $0 < [u_k/v_k] - [u_{k-1}/v_{k-1}] = \hat{w}_k/[v_kv_{k-1}]$, $k = 1, 2, \cdots, m$, giving the first parts of (a) and (b). One may actually compute R^{-1}, the result being

$$(4.0) \quad R^{-1} = \begin{bmatrix} \frac{u_2}{u_1\hat{w}_2} & -\frac{1}{\hat{w}_2} & 0 & & & \\ -\frac{1}{\hat{w}_2} & \frac{u_3v_1 - u_1v_3}{\hat{w}_3\hat{w}_2} & -\frac{1}{\hat{w}_3} & \ddots & & \\ 0 & -\frac{1}{\hat{w}_3} & \frac{u_4v_2 - u_2v_4}{\hat{w}_3\hat{w}_4} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & -\frac{1}{\hat{w}_m} \\ & & & & -\frac{1}{\hat{w}_m} & \frac{v_{m-1}}{\hat{w}_m v_m} \end{bmatrix}$$

We remark for later reference that $|R|$ may also be computed explicitly.

$$(4.1) \qquad |R| = u_1\hat{w}_2\hat{w}_3 \cdots \hat{\omega}_m v_m \; .$$

Completely analogous results hold for $\theta(t)$, $\hat{w}_k$, P^{-1}, and $|P|$.

LEMMA 2. *Under* (A)-(D) *of Definition* 1,
(a) $2^n\hat{w}_k, 2^n\hat{\omega}_k$ *are bounded uniformly* (in k and n) *away from* 0,
(b) *if* $w(t) \equiv \omega(t)$, *then* $\lim_{n\to\infty} |\hat{\omega}_k - \hat{w}_k| 2^{2n} = 0$ *uniformly* (in k).

Proof. A simple algebraic manipulation followed by use of the mean value the theorem for derivatives gives

$$2^n\hat{w}_k = w(t_{k-1}) + 2^{-n-1}[b-a][v(t_{k-1})u''(X_{1k}) - u(t_{k-1})v''(X_{2k})]$$

where $t_{k-1} < X_{1k}, X_{2k} < t_k$. That $2^n\hat{w}_k$ is uniformly bounded away from 0 now follows from the fact that w is (uniformly) continuous and positive on $[a, b]$ and hence bounded away from 0 and from the fact that the second term in the expression for $2^n\hat{w}_k$ becomes uniformly small as n gets large. A similar argument is pertinent for $2^n\hat{\omega}_k$ giving us part (a).

A somewhat more lengthy algebraic exercise together with application of Taylor's formula (two terms plus remainder) gives

$$\begin{aligned} &2^{2n}[\hat{\omega}_k - \hat{w}_k] \\ &\quad = 2^{2n}\{[\omega(t_{k-1}) - w(t_{k-1})][b-a]2^{-n} + [\omega'(t_{k-1}) - w'(t_{k-1})][b-a]^2 2^{-2n-1}\} \\ &\qquad + [(b-a)^2/2]\{\phi(t_{k-1})[\theta''(X_{1k}) - \theta''(t_{k-1})] - \theta(t_{k-1})[\phi''(X_{2k}) - \phi''(t_{k-1})] \\ &\qquad - v(t_{k-1})[u''(X_{3k}) - u''(t_{k-1})] - u(t_{k-1})[v''(X_{4k}) - v''(t_{k-1})]\} \; . \end{aligned}$$

Since $\omega(t) \equiv w(t)$ (and hence $\omega'(t) \equiv w(t)$) and since θ'', ϕ'', u'' and v'' are (uniformly) continuous on $[a, b]$, it follows that (b) holds.

LEMMA 3. *If Conditions* (A)-(F) *of Definition* 1 *and Theorem* 1 *hold, then*

(a) $\lim_{n\to\infty} [|R|/|P|] = C_1$,

(b) $\lim_{n\to\infty} x_1^2\left\{\frac{1}{u_1 v_1} - \frac{1}{\theta_1\phi_1}\right\} = C_2 x^2$ (a) *for almost all* x (m_r sense),

(see Theorem 1 for the definitions of C_1 and C_2).

Proof. Using formula (4.1) for $|R|$ and its analog for $|P|$ we have

$$ln[|R|/|P|] = ln[u_1/\theta_1] + ln[v_m/\phi_m] + \sum_{k=2}^{m} ln[\hat{w}_k/\hat{\omega}_k] .$$

Now

$$|ln[\hat{w}_k/\hat{\omega}_k]| = |ln\hat{w}_k - ln\hat{\omega}_k| = |\hat{w}_k - \hat{\omega}_k|/X_k$$

where $\min(\hat{\omega}_k, \hat{w}_k) \leqq X_k \leqq \max(\hat{\omega}_k, \hat{w}_k)$. But by Lemma 2, $|\hat{w}_k - \hat{\omega}_k| = o(2^{-2n})$ uniformly in k while $X_k 2^n$ is bounded away from 0 uniformly in k and n. It follows that

$$\lim_{n\to\infty} \sum_{k=2}^{m} ln[\hat{w}_k/\hat{\omega}_k] = 0 .$$

Hence

$$\lim_{n\to\infty} ln[|R|/|P|] = \lim_{n\to\infty} ln[u_1/\theta_1] + ln[v(b)/\phi(b)] .$$

This gives part (a) immediately in case $\theta(a) \neq 0$. If $\theta(a) = 0$ (and hence $u(a) = 0$) we use the fact that $w(t) \equiv \omega(t)$ to write

$$v^2(t)\frac{d}{dt}\left\{\frac{u(t)}{v(t)}\right\} \equiv \phi^2(t)\frac{d}{dt}\left\{\frac{\theta(t)}{\phi(t)}\right\}$$

so that

$$\int_a^t d[u(s)/v(s)] \equiv \int_a^t [\phi^2(s)/v^2(s)] d[\theta(s)/\phi(s)] .$$

By the mean value theorem for integrals and the fact that $u(a) = \theta(a) = 0$, we have

(4.2) $\quad u(t)/v(t) = [\phi^2(X)\theta(t)]/v^2(X)\phi(t)]$ for some X, $a < X < t$.

Rewriting and letting $t \to 0$, we see that $\lim_{t\to 0} [u(t)/\theta(t)] = \phi(a)/v(a)$, the needed result.

Part (b) is immediate in case $\theta(a)$ and $u(a)$ are not zero. If $\theta(a) = u(a) = 0$, then $x(a) = 0$ with m_r (and m_ρ) measure one. Hence it will be sufficient to show that $[1/u_1 v_1] - [1/\theta_1\phi_1]$ is bounded. But $[1/u_1 v_1] - [1/\theta_1\phi_1] = \{[\phi_1/u_1] - [v_1/\theta_1]\}/[v_1\phi_1]$ which will be bounded if $[\phi_1/u_1] - [v_1/\theta_1]$ is bounded. Now

$$\left|\frac{\phi(t)}{u(t)} - \frac{v(t)}{\theta(t)}\right| = \frac{|\phi(t)\theta(t) - u(t)v(t)|}{u(t)\theta(t)}$$

$$= \frac{|\phi(t)\theta(t) - [\phi^2(X)\theta(t)v^2(t)]/[v^2(X)\phi(t)]|}{u(t)\theta(t)}$$

where $a < X < t$, (by 4.2),

$$= \frac{|\phi^2(t)v^2(X) - \phi^2(X)v^2(t)|}{u(t)v^2(X)\phi(t)}$$

$$= \frac{\left|v^2(X)\dfrac{\phi^2(t) - \phi^2(a)}{t - a} + \phi^2(a)\dfrac{v^2(X) - v^2(t)}{t - a} + v^2(t)\dfrac{\phi^2(a) - \phi^2(X)}{t - a}\right|}{v^2(X)\phi(t)v(t)\dfrac{[u(t)/v(t)] - [u(a)/v(a)]}{t - a}}$$

$$= \frac{\left|2v^2(X)\phi'(X_1)\phi(X_1) + 2\phi^2(a)v'(X_2)v(X_2)\dfrac{t - X}{t - a} + 2v^2(t)\phi'(X_3)\phi(X_3)\dfrac{X - a}{t - a}\right|}{v^2(X)\phi(t)v(t)w(X_4)/v^2(X_4)}$$

where $a < X_1 < t$, $X < X_2 < t$, $a < X_3 < X$, $a < X_4 < t$. This last expression is clearly uniformly bounded for $t \in (a, b]$.

The first three lemmas allow us to prove the following key result.

LEMMA 4. *If* (A)-(F) *of Definition* 1 *and Theorem* 1 *hold, then*

$$\lim_{n\to\infty} \bar{x}'(R^{-1} - P^{-1})\bar{x} = C_2x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\} .$$

Proof. We may verify using formula (4.0) for R^{-1} and its analog for P^{-1} that

$$\bar{x}'(R^{-1} - P^{-1})\bar{x} = \frac{x_1^2}{u_1v_1} - \frac{x_1^2}{\theta_1\phi_1} + \sum_{k=2}^{m}\left\{\frac{(v_{k-1}x_k - v_kx_{k-1})^2}{v_{k-1}v_k\hat{w}_k} - \frac{(\phi_{k-1}x_k - \phi_kx_{k-1})^2}{\phi_{k-1}\phi_k\hat{w}_k}\right\}$$
$$= J_n(x) + K_n(x) + L_n(x)$$

where

$$J_n(x) = x_1^2\left\{\frac{1}{u_1v_1} - \frac{1}{\theta_1\phi_1}\right\} ,$$

$$K_n(x) = \sum_{k=2}^{m}[\hat{\omega}_k - \hat{w}_k][\phi_{k-1}x_k - \phi_kx_{k-1}[^2/[\phi_{k-1}\phi_k\hat{\omega}_k\hat{w}_k] ,$$

$$L_n(x) = \sum_{k=2}^{m}\hat{\omega}_k\left\{\frac{\phi_{k-1}\phi_k[v_{k-1}x_k - v_kx_{k-1}]^2 - v_{k-1}v_k[\phi_{k-1}x_k - \phi_kx_{k-1}]^2}{v_{k-1}v_k\phi_{k-1}\phi_k\hat{\omega}_k\hat{w}_k}\right\} .$$

We note first that $J_n(x) \to C_2x^2(a)$ as $n \to \infty$ by Lemma 3 part (b). We show next that $K_n(x) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given and choose N so large that $n \geqq N$ implies that $m^2|\hat{\omega}_k - \hat{w}_k| < \varepsilon$, (see Lemma 2). Let $\tau = \min_{0\leqq k\leqq m, n>0}[m^2\phi_{k-1}\phi_k\hat{\omega}_k\hat{w}_k]$. Then for $n \geqq N$,

$$| K_n | = \left| \sum_{k=2}^{m} \frac{m^2(\hat{\omega}_k - \hat{w}_k)}{m^2\phi_{k-1}\phi_k\hat{\omega}_k\hat{w}_k} \left\{ x_k \frac{\phi_{k-1} - \phi_k}{t_k - t_{k-1}} + \phi_k \frac{x_k - x_{k-1}}{t_k - t_{k-1}} \right\} (t_k - t_{k-1})^2 \right|$$
$$\leqq (\varepsilon/\tau) \sum_{k=2}^{m} \left\{ -x(t_k)\phi'(X_k) + \phi(t_k) \frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}} \right\}^2 (t_k - t_{k-1})^2$$

where $t_{k-1} < X_k < t_k$. Thus

$$(4.3) \qquad | K_n | \leqq [\varepsilon(b - a)/\tau m] \sum_{k=2}^{m} [x(t_k)\phi'(X_k)]^2 [t_k - t_{k-1}]$$
$$+ (2\varepsilon/\tau) \left| \sum_{k=2}^{m} x^2(t_k)\phi(t_k)\phi'(t_k)(t_k - t_{k-1}) \right|$$
$$+ (2\varepsilon/\tau) \left| \sum_{k=2}^{m} x(t_k)x(t_{k-1})\phi(t_k)\phi'(X_k)(t_k - t_{k-1}) \right|$$
$$+ (\varepsilon/\tau) \max_{a \leqq t \leqq b} \phi^2(t) \sum_{k=2}^{m} [x(t) - x(t_{k-1})]^2 .$$

The first three terms are small since the sums involved are Riemann sums. Futhermore the sum in the fourth term approaches a limit by Baxter's theorem. The result now follows.

Lastly, we consider $L_n(x)$.

$$L_n(x)$$
$$= \sum_{k=2}^{m} \frac{\phi_{k-1}\phi_k[v_{k-1}x_k - v_k x_{k-1}]^2 - v_{k-1}v_k[\phi_{k-1}x_k - \phi_k x_{k-1}]^2}{v_{k-1}v_k\phi_{k-1}\phi_k\hat{w}_k}$$
$$= \sum_{k=1}^{m} \left\{ \frac{\phi_k v_{k-1} - \phi_{k-1}v_k}{\hat{w}_k} \right\} \left\{ \frac{\phi_{k-1}v_{k-1}x_k^2 - x_{k-1}^2 v_k \phi_k}{v_{k-1}v_k\phi_{k-1}\phi_k} \right\}$$
$$= \sum_{k=2}^{m} \left\{ \frac{v_k(\phi_k - \phi_{k-1}) - \phi_k(v_k - v_{k-1})}{v_k(u_k - u_{k-1}) - u_k(v_k - v_{k-1})} \right\} \left\{ \frac{x_k^2}{\phi_k v_k} - \frac{x_{k-1}^2}{\phi_{k-1}v_{k-1}} \right\}$$
$$= \sum_{k=2}^{m} \left\{ \frac{v_k\phi_k' - \phi_k v_k' + [v_k\phi''(X_{1k}) - \phi_k v''(X_{2k})][(b - a)/2m]}{v_k u_k' - u_k v_k' + [v_k u''(X_{3k}) - u_k v''(X_{4k})][(b - a)/2m]} \right\} \left\{ \frac{x_k^2}{\phi_k v_k} - \frac{x_{k-1}^2}{\phi_{k-1}v_{k-1}} \right\}$$

(by Taylor's formula)

$$= \sum_{k=2}^{m} \left\{ \frac{v(t_k)\phi'(t_k) - \phi(t_k)v'(t_k)}{v(t_k)u'(t_k) - u(t_k)v'(t_k)} + B_{km}/m \right\} \left\{ \frac{x^2(t_k)}{\phi(t_k)v(t_k)} - \frac{x^2(t_{k-1})}{\phi(t_{k-1})v(t_{k-1})} \right\}$$

(where B_{km} is bounded independently of k and m)

$$\to \int_{\alpha}^{\beta} f(t) d\{x^2(t)/[\phi(t)v(t)]\} \quad \text{as} \quad n \to \infty .$$

For later re ;rence we note that the last expression for $L_n(x)$ may be rewritten using partial summation giving

$$(4.4) \quad L_n(x) = \left\{ \frac{x^2(t_m)}{\phi(t_m)v(t_m)} \right\} \{f(t_m) + O(1/m)\} - \left\{ \frac{x^2(t_1)}{\phi(t_1)v(t_1)} \right\} \{f(t_2) + O(1/m)\}$$
$$- \sum_{k=2}^{m-1} \frac{'x^2(t_k)}{\phi(t_k)v(t_k)} [f(t_{k+1}) - f(t_k)] .$$

5. Proof of Theorem 1.

Necessity. Here we assume that m_ρ is equivalent to m_r. Now by the corollary to Baxter's theorem, $f_\rho(t) \equiv f_r(t)$. An easy calculation shows that $f_r(t) \equiv v(t)u'(t) - u(t)v'(t)$ and $f_\rho(t) \equiv \phi(t)\theta'(t) - \theta(t)\phi'(t)$ so that Condition (E) holds. To see Condition (F), we note that $x(a) = 0$ with m_ρ measure 0 or 1 according as $\theta(a) \neq 0$ or $\theta(a) = 0$. Similarly $x(a) = 0$ with m_r measure 0 or 1 according as $u(a) \neq 0$ or $u(a) = 0$. But since m_ρ is equivalent to m_r, null sets with respect to the two measures must correspond. Hence F holds.

Sufficiency. We assume that Conditions (E) and (F) of the theorem are satisfied. Define two functions F_M and $F_{M,n}$ on $C[a, b]$ by

$$F_M(y) = \begin{cases} 1 & \text{if } \sup_{a \leqq t \leqq b} |y(t)| \leqq M \\ 0 & \text{otherwise} \end{cases}$$

$$F_{M,n}(y) = \begin{cases} 1 & \text{if } \sup_{1 \leqq k \leqq m} |y(t_k)| \leqq M \text{ and } \sum_{k=2}^{m} [y(t_k) - y(t_{k-1})]^2 \leqq \int_a^b w(t)dt + 1 \\ 0 & \text{otherwise} \end{cases} .$$

Let $H_{M,n}$ be a function of m real variables $\bar{u} = (u_1, u_1, \cdots, u_m)$ such that

$$H_{M,n}(\bar{u}) = \begin{cases} 1 & \text{if } \sup_{1 \leqq k \leqq m} |u_k| \leqq M \text{ and } \sum_{k=2}^{m} [u_k - u_{k-1}]^2 \leqq \int_a^b w(t)dt + 1 \\ 0 & \text{otherwise} \end{cases}$$

and note that $H_{M,n}(\bar{y}) = F_{M,n}(y)$. By Baxter's theorem and using the continuity of y and the fact that $f_\rho(t) \equiv f_r(t) \equiv w(t)$, we have that for almost all $y \in C$ (in the sense of both measures m_ρ and m_r) $\lim_{n\to\infty} F_{M,n}(y) = F_M(y)$ and $\lim_{M\to\infty} F_M(y) = 1$. Hence for any step function p on $[a, b]$,

$$\begin{aligned}
E^\rho\left\{F_M(y) \exp \int_\alpha^\beta y(t)dp(t)\right\} &= F^\rho\left\{\lim_{n\to\infty}\left[F_{M,n}(y) \exp \sum_{k=1}^{m} y(t_k)[p(t_k) - p(t_{k-1})]\right]\right\} \\
&= E^\rho\left\{\lim_{n\to\infty} [H_{M,n}(\bar{y}) \exp \bar{y}'\overline{\Delta p})]\right\} \\
&= \lim_{n\to\infty} E^\rho\{H_{M,n}(\bar{y}) \exp (\bar{y}'\overline{\Delta p})\} \qquad \text{(by bounded convergence)} \\
&= \lim_{n\to\infty} [(2\pi)^m |P|]^{-1/2} \int_{-\infty}^{\infty} H_{M,n}(\bar{u}) \exp [\bar{u}'\overline{\Delta p} - (1/2)\bar{u}'P^{-1}\bar{u}]\overline{du} \\
&= \lim_{n\to\infty} \{[|P|/|R|][(2\pi)^m |R|]\}^{-1/2} \int_a^b H_{M,n}(\bar{u}) \\
&\quad \times \exp \{\bar{u}'\overline{\Delta p} + (1/2)[\bar{u}'(R^{-1} - P^{-1})\bar{u} - \bar{u}'R^{-1}\bar{u}]\}\overline{du} \\
&= \lim_{n\to\infty} [|R|/|P|]^{1/2} \lim_{n\to\infty} E^r\{H_{M,n}(\bar{x}) \exp [\bar{x}'\overline{\Delta p} + (1/2)\bar{x}'(R^{-1} - P^{-1})\bar{x}]\} \\
&= C_1 \lim_{n\to\infty} E^r\{F_{M,n}(x) \exp [\bar{x}'\overline{\Delta p} + (1/2)\bar{x}'(R^{-1} - P^{-1})\bar{x}]\} ,
\end{aligned}$$

the last equality following from Lemma 3, part (a). Now the expectant will be bounded (independently of n and x) provided $\bar{x}'\overline{\Delta p}+(1/2)\bar{x}'(R^{-1}-P^{-1})\bar{x}$ is so bounded on the subset A_n of C where $F_{M,n}$ is different from 0. But on A_n, $\bar{x}'\overline{\Delta p} \leqq M\sum_{k=1}^{m}|p(t_k)-p(t_{k-1})|$ which is bounded (independently of n and x). Furthermore, $\bar{x}'(R^{-1}-P^{-1})\bar{x} = J_n(x)+K_n(x)+L_n(x)$. $J_n(x)$ is bounded on A_n as may be seen from the proof of Lemma 3, part (b), while $K_n(x)$ and $L_n(x)$ may been to be bounded on A_n by examination of formulas (4.3) and (4.4) respectively. This allows us to take the limit inside the expected value from which we obtain (see Lemma 4)

(4.5)
$$E^p\left\{F_M(y)\exp\int_a^b y(t)dp(t)\right\}$$
$$= C_1E^r\left\{F_M(x)\exp\left[\int_a^b x(t)dp(t) + (1/2)(C_2x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\}\right]\right\}.$$

Now letting $M\to\infty$, we obtain by monotone convergence

(4.6)
$$E^p\left\{\exp\int_a^b y(t)dp(t)\right\}$$
$$= C_1E^r\left\{\exp\left[\int_a^b x(t)dp(t) + (1/2)(C_2x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\}\right]\right\}.$$

Now consider the stochastic process $\{z(t), a \leqq t \leqq b\}$ with space of sample functions $\{C, B\}$ whose Radon-Nikodym derivative with respect to $\{x(t), a\leqq t\leqq b\}$ is $C_1\exp[1/2]\left[C_2x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\}\right]$. Then for all measurable (B) functions F,

$$(4.7)\quad E\{F(z)\} = C_1E^r\left\{F(x)\exp[1/2]\left[C_2x^2(a) + \int_a^b f(t)d\{x^2(t)/[\phi(t)v(t)]\}\right]\right\}.$$

Hence in particular, formula (4.6) holds for the process $\{z(t), a\leqq t\leqq b\}$. But this means that $\{z(t), a\leqq t\leqq b\}$ and $\{y(t), a\leqq t\leqq b\}$ have the same multidimensional moment generating functions and since they determine the measures of all the measurable subsets of C (i.e., all sets in B), the processes $z(t)$ and $y(t)$ are identical. Since (4.7) holds for the former, it also holds for the latter. This shows that m_ρ is absolutely continuous with respect to m_r. By symmetry, m_r is also absolutely continuous with respect to m_ρ. This completes the proof.

5. Examples. The best known process in the class $M[0, T]$ is the Wiener process (Brownian motion process) with probability measure m_{w_σ} determined by the covariance function

$$w_\sigma(s,t) = \sigma^2\min(s,t) = \begin{cases}\sigma^2 s & s\leqq t\\ \sigma^2 t & s\geqq t\end{cases},\quad \sigma^2>0.$$

Theorem 1 shows immediately that two such processes determined by covariance functions w_{σ_1} and w_{σ_2} are equivalent if and only if $\sigma_1 = \sigma_2$, (see [2] for a discussion of what is essentially this problem). We can actually say much more. In fact, we can easily characterize all processes in the class $M[0, T]$ which are equivalent to the Wiener process determined by $w_\sigma(s, t)$[6]. Let ϕ be any function which is positive and has a continuous second derivative on $[0, T]$ and define θ by $\theta(s) = \sigma^2\phi(s)\int_0^s [1/\phi^2(t)]dt$. Then those processes (and only those) in the class $M[0, T]$ with covariance functions of the type

$$\rho(s, t) = \begin{cases} \theta(s)\phi(t) & s \leqq t \\ \theta(t)\phi(s) & s \geqq t \end{cases}$$

are equivalent to the process in $M[0, T]$ determined by $w_\sigma(s, t)$. We give two examples.

EXAMPLE 1. Let $\{y(t), 0 \leqq t \leqq T < 1\}$ be the process belonging to $M[0, T]$ with probability measure m_ρ determined by the covariance function

$$\rho(s, t) = \begin{cases} s(1-t) & s \leqq t \\ t(1-s) & s \geqq t \end{cases}^{7} .$$

This process is equivalent to the Wiener process $\{x(t), 0 \leqq t \leqq T\}$ with covariance function $w_1(s, t) = \min(s, t)$. Moreover,

$$dm_\rho/dm_{w_1} = (1 - T)^{-1/2} \exp\{-x^2(T)/[2(1 - T)]\} .$$

EXAMPLE 2. Let $\{y(t), 0 \leqq t \leqq 1\}$ be the process belonging to $M[0,1]$ with probability measure m_{ρ_λ} determined by the covariance function

$$\rho_\lambda(s, t) = \begin{cases} \dfrac{\sin\sqrt{\lambda}s \cos\sqrt{\lambda}(1-t)}{\sqrt{\lambda}\cos\sqrt{\lambda}} & s \leqq t \\ \dfrac{\sin\sqrt{\lambda}t \cos\sqrt{\lambda}(1-s)}{\sqrt{\lambda}\cos\sqrt{\lambda}} & s \leqq t \end{cases}^{8} \qquad \lambda < (\pi^2/4) .$$

This process is equivalent to the Wiener process $\{x(t), 0 \leqq t \leqq 1\}$ with covariance function $w_1(s, t) = \min(s, t)$. Moreover,

$$dm_{\rho_\lambda}/dm_{w_1} = (\cos\sqrt{\lambda})^{1/2} \exp[(\lambda/2)\int_0^1 x^2(t)dt] .$$

[6] We reason as follows. For $r(s, t) = w_\sigma(s, t)$, $w(s) = \sigma^2$. Hence $\omega(s)$ must equal σ^2, i.e., $d/ds[\theta(s)/\phi(s)] = \sigma^2/\phi^2(s)$. This together with $\theta(0) = 0$ implies that $\theta(s) = \sigma^2\phi(s)\int_0^s [1/\phi^2(t)]dt$.

[7] This process has been studied by various authors, among them Doob [**4**].

[8] For $\lambda = 0$, $\rho_\lambda(s, t) = \lim_{\lambda\to 0} \rho_\lambda(s, t) = \min(s, t)$.

For our last example we consider the class of (Ornstien Uhlenbeck) processes in $M[a, b]$ determined by covariance functions of the type

$$\rho_{\sigma,\beta}(s, t) = \sigma^2 \exp[-\beta | s - t |] = \begin{cases} \theta(s)\phi(t) & s \leqq t \\ \theta(t)\phi(s) & s \geqq t \end{cases}$$

where $\theta(t) = \sigma^2 \exp(\beta t)$, $\phi(t) = \exp(-\beta t)$, $(\sigma^2 > 0, \beta > 0)$. If $U[a, b]$ denotes the class of such processes, we have the following result.

EXAMPLE 3. Let $\{x(t), 0 \leqq t \leqq T\}$ and $\{y(t), 0 \leqq t \leqq T\}$ be two processes belonging to $U[0, T]$ with covariance functions ρ_{σ_0,β_0} and ρ_{σ_1,β_1} respectively determining two probability measures m_{σ_0,β_0} and m_{σ_1,β_1} on $\{C, B\}$. Then m_{σ_0,β_0} is equivalent to m_{σ_1,β_1} if and only if $\sigma_0^2\beta_0 = \sigma_1^2\beta_1$. Moreover if this condition is satisfied and if we let $K = 2\sigma_0^2\beta_0 = 2\sigma_1^2\beta_1$, then

$$\frac{dm_{\sigma_1,\beta_1}}{dm_{\sigma_0,\beta_0}} = (\beta_1/\beta_0)^{1/2}$$
$$\times \exp\left\{(-1/2K)\left[(\beta_1 - \beta_0)(x^2(0) + x^2(T) - KT) + (\beta_1^2 - \beta_0^2)\int_0^T x^2(t)dt\right]\right\}.$$ [9]

6. Conjecture. Consider two general Gaussian processes determined by covariance functions $r(s, t)$ and $\rho(s, t)$ respectively. Under regularity and boundary conditions of the type (A)-(D) of Definition 1, a necessary and sufficient condition that the two processes be equivalent is that $f_r(t) \equiv f_\rho(t)$, (see § 2).

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[9] This result is due to Charlotte T. Striebel, the formula above being displayed near the top of page 566 in [**7**].