THE STRUCTURE OF CERTAIN MEASURE ALGEBRAS

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Introduction. In their paper [3], Hewitt and Zuckerman study the measure algebra $\mathscr{M}(G)$ where G is a topological semigroup of the following type: G is a linearly ordered set topologized with the order topology, is compact in this topology, and multiplication is defined by $xy = \max(x, y)$. In this study, we will suppose that G has the above properties except that compactness will be replaced by local compactness. (See § 8.5 [3]). As the reader will readily observe, we are heavily indebted to Hewitt and Zuckerman for their initial study of these measure algebras. For completeness, we have listed, without proof, a few of their results; they are stated in their paper for compact semigroups but the proofs easily carry over to locally compact semigroups.

In § 2 we study $\hat{G}$ and $\hat{G}_0$. The characterization of the Gel'fand topology on $\hat{G}$ is somewhat simpler than that of Theorem 5.5 [3]. The major result of this study is Theorem 3.4, stating that every closed ideal in $\mathscr{M}(G)$ is the intersection of maximal ideals; i.e., spectral synthesis holds for $\mathscr{M}(G)$. Malliavin [7] has recently shown that spectral synthesis fails for $\mathscr{M}(G)$ when G is a non-compact locally compact commutative group.[1] Theorem 3.4 shows that this result cannot be generalized to locally compact commutative semigroups. In § 4, a generalization of Theorem 6.7 [3] is indicated; see Theorem 4.5. This is used to obtain additional facts about $\mathscr{M}(G)$ (§ 5). In 5.8 we show that our theory is not a special case of the theory of function algebras.

1. Preliminaries.

1.1. We will be concerned with linearly ordered sets; i.e. sets ordered by transitive, irreflexive relations $<$. For elements x and y in such a set X, we define $]x, y[= \{z \in X : x < z < y\}$ and $[x, y] = \{z \in X : x \leqq z \leqq y\}$. The half-open intervals $[x, y[$ and $]x, y]$ are defined analogously. We also define $]-\infty, x[= \{z \in X : z < x\}$ and $]-\infty, x] = \{z \in X : z \leq x\}$ with analogous definitions for $[x, \infty[,]x, \infty[$, and $]-\infty, \infty[$. The symbols $-\infty$ and ∞ will never denote actual elements of X. The order topology for X is the topology having the family $\{]-\infty, x[\}_{x \in X} \cup$

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[1] Actually Malliavin shows that spectral synthesis fails for $L_1(G)$; the result for $\mathscr{M}(G)$ follows easily from this.

$\{]x, \infty[\}_{x \in X}$ for a sub-base.

For terminology not explained here in measure theory, topology, and harmonic analysis, see [1], [5], and [6], respectively. If A is a subset of B, we will write $A \subseteqq B$; $A \subset B$ will mean that A is a proper subset of B. For sets A and B, we write $A - B = \{x : x \in A, x \notin B\}$ and $A \Delta B = (A - B) \cup (B - A)$. The empty set will be denoted by 0. For any set A, χ_A will denote the characteristic function of A.

1.2. STANDING HYPOTHESES. Let G be a set linearly ordered by the relation $<$. Suppose also that G has the order topology and that under this topology G is locally compact. For $x, y \in G$, we define $xy = \max(x, y)$. With this multiplication G is a locally compact topological semigroup.

1.3. Let $\mathfrak{C}_0(G)$ denote the linear space of all complex-valued continuous functions on G that are arbitrarily small outside of compact sets. For $f \in \mathfrak{C}_0(G)$, let $\|f\| = \max_{x \in G} |f(x)|$. Let $\mathscr{M}(G)$ consist of all countably additive, complex-valued, regular, finite Borel measures on G. Let $\mathfrak{C}_0^*(G)$ be the linear space of all complex-valued bounded linear functionals L on $\mathfrak{C}_0(G)$. For each $L \in \mathfrak{C}_0^*(G)$ there is a unique $\lambda \in \mathscr{M}(G)$ such that

$$L(f) = \int_G f(x) d\lambda(x) \tag{1.3.1}$$

for all $f \in \mathfrak{C}_0(G)$. Also for each $\lambda \in \mathscr{M}(G)$, 1.3.1 defines a member of $\mathfrak{C}_0^*(G)$. Under this correspondence, $\mathscr{M}(G) \cong \mathfrak{C}_0^*(G)$. We will associate L with λ, M with μ, etc.

Let $\lambda \in \mathscr{M}(G)$. Then for Borel sets $E \subseteqq G$, we define

$$|\lambda|(E) = \sup\left\{ \sum_{k=1}^{m} |\lambda(E_k)| : \{E_k\}_{k=1}^{m} \text{ is a Borel partition of } E. \right\} \tag{1.3.2}$$

Then the set-function $|\lambda|$ belongs to $\mathscr{M}(G)$ and

$$\|\lambda\| = |\lambda|(G) = \|L\| \tag{1.3.3}$$

where $L \in \mathfrak{C}_0^*(G)$ is defined by 1.3.1. See [2].

1.4. THEOREM. *Let L and M be in $\mathfrak{C}_0^*(G)$.* For all $f \in \mathfrak{C}_0(G)$, *let*

$$L*M(f) = \int_G \int_G f(xy) d\lambda(x) d\mu(y) . \tag{1.4.1}$$

Then $L*M \in \mathfrak{C}_0^*(G)$, and

$$\|L*M\| \leqq \|L\| \cdot \|M\| . \tag{1.4.2}$$

1.5. For $\lambda, \mu \in \mathscr{M}(G)$, we define $\lambda*\mu$ to be the unique measure in

$\mathscr{M}(G)$ that corresponds to $L*M \in \mathfrak{C}_0^*(G)$.

1.6. THEOREM. *Under the convolution defined in* 1.5 *and the ordinary linear operations,* $\mathscr{M}(G)$ *is a commutative Banach algebra.*

We omit the proof; see § 2 [3].

1.7. For $a \in G$, let $\varepsilon_a \in \mathscr{M}(G)$ be defined by

$$(1.7.1) \qquad \varepsilon_a(E) = \begin{cases} 1 \text{ if } a \in E\,, \\ 0 \text{ if } a \notin E\,, \end{cases}$$

for Borel sets $E \subseteqq G$. For $\lambda \in \mathscr{M}(G)$ and $A \subseteqq G$ a Borel set, $\lambda^A \in \mathscr{M}(G)$ is defined by $\lambda^A(E) = \lambda(A \cap E)$ for all Borel sets $E \subseteqq G$.

The proofs of the following four lemmas are routine and uninteresting.

1.8. LEMMA. *Let* $E \subseteqq G$ *be a Borel set and* $\lambda \in \mathscr{M}(G)$. *Then for any* $\varepsilon > 0$, *there exist* $a, b \in E$ *such that*

$$(1.8.1) \qquad |\lambda|(E \cap]-\infty, a[) < \varepsilon \quad and \quad |\lambda|(E \cap]\, b, \infty[) < \varepsilon\,.$$

1.9. LEMMA. *Let* X *be a linearly ordered set and* $U \subseteqq X$ *be a finite union of open intervals. Then* U *is the pairwise disjoint union of open intervals*:

$$U = \bigcup_{i=1}^{m}]a_i, b_i[\,,$$

where intervals of the form $[\inf X, b_i[$, $]a_i, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible if* $\inf X$ *or* $\sup X$ *exist. Moreover,* $a_i \notin U$ *except possibly in the case where* $a_i = \inf X$, and $b_i \notin U$ *except possibly in the case that* $b_i = \sup X$.

1.10. LEMMA. *Let* X *be a compact linearly ordered set and* $U \subseteqq X$ *be an open set. Then* U *is the pairwise disjoint union of open intervals:*

$$U = \bigcup_{\alpha}]a_\alpha, b_\alpha[$$

where intervals of the form $[\inf X, b_\alpha[$, $]a_\alpha, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible. In addition,* $a_\alpha \notin U$ *except possibly in the case that* $a_\alpha = \inf X$, *and* $b_\alpha \notin U$ *except possibly in the case that* $b_\alpha = \sup X$.

1.11. LEMMA. *Let* X *be a locally compact linearly ordered set. Suppose that* $K \subseteqq X$ *is compact and that* U *is an open set such that* $K \subseteqq U \subseteqq X$. *Then there exist finitely many pairwise disjoint closed compact intervals* $\{[a_i, b_i]\}_{i=1}^m$ *such that* $U \supseteqq \bigcup_{i=1}^m [a_i, b_i] \supseteqq K$. *Also there* *exist finitely many pairwise disjoint open intervals* $\{]u_i, v_i[\}_{i=1}^n$ *such that* $U \supseteqq \bigcup_{i=1}^n]u_i, v_i[\supseteqq K$ *and each closed interval* $[u_i, v_i]$ *is compact. Intervals of the form* $[\inf X, v_i[$, $]u_i, \sup X]$, *and* $[\inf X, \sup X]$ *are also admissible whenever* $\inf X$ *or* $\sup X$ *exists.*

2. The spaces $\hat{G}$ and $\hat{G}_0$.

2.1. A Dedekind cut $\{A, B\}$ of G is a pair of subsets of G such that $A \cap B = 0$, $A \cup B = G$, and $x < y$ whenever $x \in A$ and $y \in B$. Let $\hat{G}$ denote the set of all semicharacters of G.

2.2 THEOREM. *Let* $\{A, B\}$ *be a Dedekind cut of* G *such that* $A \neq 0$. *Then the function*

$$\psi_{A,B}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in B, \end{cases} \tag{2.2.1}$$

is a semicharacter of G. *Conversely, every semicharacter on* G *has the form* 2.2.1.

2.3. THEOREM. *Let* $\{A, B\}$ *be a Dedekind cut of* G *such that* $A \neq 0$. *Then the mapping*

$$\pi_A(\lambda) = \lambda(A) = \int_G \psi_{A,B}(x)d\lambda(x) \qquad (\lambda \in \mathscr{M}(G)) \tag{2.3.1}$$

is a homomorphism of $\mathscr{M}(G)$ *onto the complex numbers. Moreover, every homomorphism of* $\mathscr{M}(G)$ *onto the complex numbers has the form* 2.3.1.

Proof. This is essentially proved in Theorems 3.2 and 3.3 [3]; however the proof in [3] that π_A is multiplicative can be simplified. Let $\lambda, \mu \in \mathscr{M}(G)$. According to Theorem 2 [8], $\lambda*\mu(E) = \lambda \times \mu\{(x, y) \in G \times G: xy \in E\}$ for Borel sets $E \subseteqq G$ where $\lambda \times \mu$ is the product measure of λ and μ. Hence if $\{A, B\}$ is a Dedekind cut of G, then

$$\begin{aligned} \pi_A(\lambda*\mu) = \lambda*\mu(A) &= \lambda \times \mu\{(x, y) \in G \times G: \max(x, y) \in A\} \\ &= \lambda \times \mu(A \times A) = \lambda(A)\mu(A) = \pi_A(\lambda)\pi_A(\mu). \end{aligned}$$

2.4. THEOREM. *The Banach algebra* $\mathscr{M}(G)$ *is semisimple.*

Proof. In virtue of 2.3 we need to prove that if $\lambda(A) = 0$ for all Dedekind cuts $\{A, B\}$, then λ is identically zero. Suppose that $\lambda(A) = 0$ for all Dedekind cuts $\{A, B\}$; evidently $\lambda(I) = 0$ for all intervals I. If

λ is not identically zero, then $\lambda(K) \neq 0$ for some compact set $K \subseteqq G$. By regularity there is an open set $U \supseteqq K$ such that $|\lambda|(U - K) < |\lambda(K)|$. For each $x \in K$, let I_x be an open interval such that $x \in I_x \subseteq U$. Let $I_1, \cdots, I_m$ be a finite subset of $\{I_x\}_{x \in K}$ covering K. Let $V = \bigcup_{i=1}^m I_i$; clearly $K \subseteqq V \subseteqq U$. By 1.9, V is the pairwise disjoint union of a finite number of open intervals. Hence $\lambda(V) = 0$. Thus

$$\begin{aligned} |\lambda(V - K)| &= |\lambda(V) - \lambda(K)| \\ &= |\lambda(K)| > |\lambda|(U - K) \geqq |\lambda|(V - K) \geqq |\lambda(V - K)| \end{aligned}$$

which is a contradiction. Hence λ is identically zero.

2.5. Theorem 2.3 identifies completely the homomorphisms of $\mathscr{M}(G)$ onto the complex numbers. Relation 2.3.1 associates each homomorphism π_A of $\mathscr{M}(G)$ with the semicharacter $\psi_{A,B}$. Hence we will usually consider $\hat{G}$ as consisting of the homomorphisms π_A. For $\lambda \in \mathscr{M}(G)$, we define $\hat{\lambda}$ on $\hat{G}$ by

$$\hat{\lambda}(\pi_A) = \pi_A(\lambda) = \lambda(A) \qquad (\pi_A \in \hat{G}); \tag{2.5.1}$$

$\hat{\lambda}$ is the Fourier transform of λ.

For $\pi_A, \pi_{A'} \in \hat{G}$, we will write $\pi_A < \pi_{A'}$ if and only if $A \subset A'$. Under this ordering, $\hat{G}$ is obviously linearly ordered. Evidently $\hat{G}$ is isomorphic to the maximal ideal space of $\mathscr{M}(G)$. The Gel'fand topology for $\hat{G}$ is the weakest topology for which all the functions $\hat{\lambda}$ are continuous.

Henceforth we will write $\pi_{a]}$ for $\pi_{]-\infty, a]}$ and $\pi_{a[}$ for $\pi_{]-\infty, a[}$ $(a \in G)$.

2.6. DEFINITION. Let $\hat{G}_0 = \hat{G} \cup \{\pi_0\}$ where $\pi_0 < \pi$ for all $\pi \in \hat{G}$.

The symbol π_0 may be taken to correspond to the zero homomorphism of $\mathscr{M}(G)$, the zero semicharacter of G, and the Dedekind cut $\{0, G\}$.

2.7. THEOREM. *The Gel'fand topology on $\hat{G}$ coincides with the order topology.*

Proof. Let $\pi_A \in \hat{G}$ where $A \neq G$, $\lambda \in \mathscr{M}(G)$, and $\varepsilon > 0$. Using 1.8, we can find $b \in A$ and $c \notin A$ such that $|\lambda|(]b, c[) < \varepsilon$. Clearly $\pi_A \in]\pi_{b[}, \pi_{c]}[$. For $\pi_B \in]\pi_{b[}, \pi_{c]}[$, we have

$$\begin{aligned} |\hat{\lambda}(\pi_A) - \hat{\lambda}(\pi_B)| &= |\lambda(A) - \lambda(B)| \\ &= |\lambda(A \mathbin{\Delta} B)| \leqq |\lambda|(A \mathbin{\Delta} B) \leqq |\lambda|(]b, c[) < \varepsilon . \end{aligned}$$

Thus $\hat{\lambda}$ is continuous at $\pi_A \in \hat{G}(A \neq G)$ in the order topology. Similarly $\hat{\lambda}$ is continuous at π_G in the order topology. Hence the Gel'fand topology is weaker than or equivalent to the order topology.

For $b, c \in G$, $b < c$, it is easy to verify that

$$\hat{\varepsilon}_b - \hat{\varepsilon}_c = \chi_{]\pi_b[,\pi_c][} \quad \text{and} \quad \hat{\varepsilon}_b = \chi_{]\pi_b[,\pi_G]} .$$

Hence sets of the form

$$]\pi_{b[}, \pi_{c]}[\quad b < c , \tag{2.7.1}$$

and

$$]\pi_{b[}, \pi_G] , \tag{2.7.2}$$

are open in the Gel'fand topology. All sets of the forms 2.7.1 and 2.7.2 comprise a basis for the order topology. It follows that the order topology on $\hat{G}$ is weaker than or equivalent to the Gel'fand topology on $\hat{G}$.

2.8. THEOREM. *The set $\hat{G}_0$ with the order topology is a totally disconnected compact Hausdorff space. For $\lambda \in \mathscr{M}(G)$, let $\hat{\lambda}$ be defined on $\hat{G}_0$ to agree with $\hat{\lambda}$ on $\hat{G}$ and such that $\hat{\lambda}(\pi_0) = \lambda(0) = 0$. Then $\hat{\lambda}$ is continuous on $\hat{G}_0$.*

Proof. Let $\mathscr{B}$ consist of all subsets of $\hat{G}_0$ of the form:

(2.8.1) $\quad]\pi_{a[}, \pi_{b]}[\quad (a < b)$,

(2.8.2) $\quad [\pi_0, \pi_{b]}[$,

(2.8.3) $\quad]\pi_{a[}, \pi_G]$.

Each set in $\mathscr{B}$ is open and closed and $\mathscr{B}$ is a base for the order topology on $\hat{G}_0$. Hence $\hat{G}_0$ is totally disconnected. The remainder of the proof is omitted.

2.9. DEFINITION. Let I be an interval of $\hat{G}_0$ and let h be a continuous function on $\hat{G}_0$. Then we define:

$$V(h; I) = \sup\left\{ \sum_{i=1}^{m-1} |h(\pi_{i+1}) - h(\pi_i)| : \pi_1 \leqq \pi_2 \leqq \cdots \leqq \pi_m,\ \pi_i \in I \right\}. \tag{2.9.1}$$

In particular, we define $V(h) = V(h ; \hat{G}_0)$ and say that h has finite variation if $V(h) < \infty$.

2.10. Let h be a continuous function on $\hat{G}_0$ and let $\pi_{A_1} \leqq \pi_{A_2} \leqq \cdots \leqq \pi_{A_k}, \pi_{A_i} \in \hat{G}_0$. Then

$$V(h ; [\pi_{A_1}, \pi_{A_k}]) = \sum_{i=2}^{k} V(h ; [\pi_{A_{i-1}}, \pi_{A_i}]) . \tag{2.10.1}$$

Let h be a continuous, real-valued function on $\hat{G}_0$ of finite variation. For $\pi_A \in \hat{G}_0$, let $h_1(\pi_A) = V(h; [\pi_0, \pi_A])$. Let $h_2 = h_1 - h$. Then h_1 and h_2 are continuous, non-decreasing functions on $\hat{G}_0$.

3. The closed ideals of $\mathscr{M}(G)$.

3.1. LEMMA. *Let $\pi_A, \pi_B \in \hat{G}_0$, where $\pi_A \leqq \pi_B$,* and let $\lambda \in \mathscr{M}(G)$.

Then

$$|\lambda|(B-A) = V(\hat{\lambda}\,;\ [\pi_A, \pi_B])\ . \tag{3.1.1}$$

In particular, $\|\lambda\| = |\lambda|(G) = V(\hat{\lambda})$.

Proof. It is easy to show that $V(\hat{\lambda}\,;\ [\pi_A,\pi_B]) \leqq |\lambda|(B-A)$.

Let $\varepsilon > 0$. Let $E_1, \cdots, E_m$ be pairwise disjoint non-void Borel sets whose union is $B-A$. For $i = 1, \cdots, m$, let $K_i \subseteqq E_i$ be a compact set for which $|\lambda|(E_i - K_i) < \varepsilon/m$. By induction (and using the second part of 1.11) we obtain pairwise disjoint open sets $U_1, \cdots, U_m$ such that

(i) $K_i \subseteqq U_i \subseteq \bar{U}_i \subseteqq G - (\bigcup_{j=i+1}^m K_j \cup \bigcup_{j=1}^{i-1} \bar{U}_j)$,

(ii) $|\lambda|(U_i - K_i) < \varepsilon/m$,

(iii) U_i is a finite union of pairwise disjoint open intervals ;

$i = 1, \cdots, m$. Now $\bigcup_{i=1}^m U_i$ is the finite union of pairwise disjoint open intervals, say $\{I'_j\}_{j=1}^r$, such that each I'_j is a subset of some U_i. For $j = 1, \cdots, r$, let $I_j = I'_j \cap (B-A)$. Evidently $\bigcup_{j=1}^r I_j = \bigcup_{i=1}^m (U_i \cap (B - A))$; we may suppose that each I_j is non-void. Let $A_{2j} = \{x \in G : x \leqq y$ for some $y \in I_j\}$ $(j = 1, \cdots, r)$. Relabelling if necessary, we may suppose that $A_2 \subset A_4 \subset \cdots \subset A_{2r}$. Let $A_{2j-1} = \{x \in G : x < y$ for all $y \in I_j\}$. Then $\pi_A \leqq \pi_{A_1} < \pi_{A_2} \leqq \pi_{A_3} < \pi_{A_4} \leqq \cdots < \pi_{A_{2r}} \leqq \pi_B$ and $I_j = A_{2j} - A_{2j-1}$ for $j = 1, \cdots, r$. Now

$$\begin{aligned} V(\hat{\lambda}\,;\ [\pi_A, \pi_B]) &\geqq \sum_{i=1}^{2r-1} |\hat{\lambda}(\pi_{A_{i+1}}) - \hat{\lambda}(\pi_{A_i})| = \sum_{i=1}^{2r-1} |\lambda(A_{i+1} - A_i)| \\ &\geqq \sum_{j=1}^{r} |\lambda(I_j)| \geqq \sum_{i=1}^{m} |\lambda(U_i \cap (B-A))| \end{aligned}$$

whereas

$$\begin{aligned} \sum_{i=1}^m |\lambda(E_i)| &= \sum_{i=1}^m |\lambda(E_i - K_i) + \lambda(U_i \cap (B-A)) \\ &- \lambda((U_i \cap (B-A)) - K_i)| \leqq 2\varepsilon + \sum_{i=1}^m |\lambda(U_i \cap (B-A))| \end{aligned}$$

so that

$$\sum_{i=1}^m |\lambda(E_i)| \leqq 2\varepsilon + V(\hat{\lambda}\,;\ [\pi_A, \pi_B]).$$

It follows that $|\lambda|(B-A) \leqq V(\hat{\lambda}\,;\ [\pi_A, \pi_B])$ since $\{E_i\}_{i=1}^m$ and ε are arbitrary.

3.2. LEMMA. *Let R be an interval of $\hat{G}_0$ of the form* 2.8.1 *or* 2.8.3. *Suppose that $\lambda \in \mathscr{M}(G)$ and that $\hat{\lambda}(\pi) \neq 0$ for all $\pi \in R$. Then there exists a $\nu \in \mathscr{M}(G)$ such that*

$$\hat{\nu}(\pi) = \begin{cases} \dfrac{1}{\hat{\lambda}(\pi)} & \text{for } \pi \in R\,, \\ 0 & \text{for } \pi \notin R\,. \end{cases} \tag{3.2.1}$$

Proof. Suppose that $R =]\pi_{x[}\,, \pi_{y]}[$ and let $X = [x, y[$. Evidently X is a locally compact subsemigroup of G. Throughout this proof, elements of $\hat{X}$ will be denoted by $\tilde{\pi}$; whenever the symbol $\tilde{\pi}_A$ occurs, it is tacitly assumed that $A \subseteqq X$ and that $\{A, X - A\}$ is a Dedekind cut of X. The functions $\hat{\lambda}$ will be considered defined on $\hat{G}$ or $\hat{X}$ rather than $\hat{G}_0$ or $\hat{X}_0$. For Borel sets $E \subseteqq X$, let $\tilde{\lambda}(E) = \lambda(E \cap X) + \lambda(]-\infty, x[)\,\varepsilon_x(E)$. We have $\tilde{\lambda} \in \mathscr{M}(X)$. We now show that

$$\hat{\tilde{\lambda}}(\tilde{\pi}_A) = \hat{\lambda}(\pi_{A \cup]-\infty, x[}) \text{ for } \tilde{\pi}_A \in \hat{X}\,. \tag{3.2.2}$$

Indeed $\hat{\tilde{\lambda}}(\tilde{\pi}_A) = \tilde{\lambda}(A) = \lambda(A \cap X) + \lambda(]-\infty, x[)\,\varepsilon_x(A) = \lambda(A) + \lambda(]-\infty, x[) = \lambda(A \cup]-\infty, x[) = \hat{\lambda}(\pi_{A \cup]-\infty, x[})$. Since $\pi_{A \cup]-\infty, x[} \in R$ whenever $\tilde{\pi}_A \in \hat{X}$, it follows from 3.2.2. that

$$\hat{\tilde{\lambda}}(\tilde{\pi}_A) \neq 0 \quad \text{for} \quad \tilde{\pi}_A \in \hat{X}\,. \tag{3.2.3}$$

By Theorem 4.15.1 (9) [4], $\tilde{\lambda} \in \mathscr{M}(X)$ has an inverse $\tilde{\nu} \in \mathscr{M}(X)$. For Borel sets $E \subseteqq G$, let

$$\nu(E) = \tilde{\nu}(E \cap X) - \tilde{\nu}(X)\varepsilon_y(E)\,.$$

Evidently $\nu \in \mathscr{M}(G)$. It is now routine to verify 3.2.1.

If $R =]\pi_{x[}\,, \pi_G]$, we let $X = [x, \infty[$ and repeat the preceding proof with the appropriate modifications.

3.3. NOTATION. For subsets A and B of G (or $\hat{G}_0$), we write $A < B$ if $x \in A$ and $y \in B$ imply $x < y$ and $A \leqq B$ if $x \in A$ and $y \in B$ imply $x \leqq y$. Note, in particular, that $0 < A$ and $A < 0$ for any set A. Let $P = \{\pi_1, \cdots, \pi_m\}$ be a finite subset of $\hat{G}_0$ where $\pi_1 < \pi_2 < \cdots < \pi_m$. We will sometimes write $\sum(\hat{\lambda}\,; P)$ for $\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)|$, $\lambda \in \mathscr{M}(G)$,.

For $\pi_A \in \hat{G}_0$, let $I_A = \{\lambda \in \mathscr{M}(G)\colon \lambda(A) = 0\}$. Note that $I_0 = \mathscr{M}(G)$. Since each $I_A(\pi_A \in \hat{G})$ is the kernel of the homomorphism π_A, the set $\{I_A\}_{\pi_A \in \hat{G}}$ is precisely the set of all regular maximal closed ideals in $\mathscr{M}(G)$.

The following theorem characterizes the closed ideals in $\mathscr{M}(G)$.

3.4. THEOREM. *Let $I \subseteqq \mathscr{M}(G)$ be a closed ideal. Let $H = \{\pi \in \hat{G}_0\colon \hat{\lambda}(\pi) = 0$ for all $\lambda \in I\}$. Then H is closed in $\hat{G}_0$ and*

$$I = \bigcap_{\pi_A \in H} I_A\,. \tag{3.4.1}$$

Proof. Obviously $H = \bigcap_{\lambda \in I} (\hat{\lambda})^{-1}(0)$ is closed and $I \subseteqq \bigcap_{\pi_A \in H} I_A$.

Let λ be a fixed element of $\bigcap_{\pi_A \in H} I_A$. Let $Z = \{\pi \in \hat{G}_0 : \hat{\lambda}(\pi) = 0\}$. Clearly Z is closed in $\hat{G}_0$, $H \subseteqq Z$, and $\pi_0 \in Z$. By Lemma 1.10, the complement Z' of Z in $\hat{G}_0$ is a pairwise disjoint union of open intervals:

$$Z' = \bigcup_\alpha]\,\pi_{A_\alpha}, \pi_{B_\alpha}[$$

where one of these intervals may be of the form $]\pi_{A_\alpha}, \pi_G]$. Moreover, $\pi_{A_\alpha} \in Z$ for all α and $\pi_{B_\alpha} \in Z$ for all α except possibly when $\pi_{B_\alpha} = \pi_G$. We assume in the following that $\pi_G \notin Z'$; elementary modifications are necessary when $\pi_G \in Z'$.

We first prove

$$V(\hat{\lambda}) = \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) \,. \tag{3.4.2}$$

Using 3.1, we have $\sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) = \sum_\alpha |\lambda|(B_\alpha - A_\alpha) \leqq |\lambda|(G) = V(\hat{\lambda})$. Let $\pi_1 < \pi_2 < \cdots < \pi_m$, $\pi_i \in \hat{G}_0$, and call this partition P'. Let $P = P' \cup \{\pi_G\}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be precisely those α such that $]\pi_{A_{\alpha_i}}, \pi_{B_{\alpha_i}}[\cap P \neq 0$. For this paragraph we write A_i for A_{α_i} and B_i for B_{α_i}. We may suppose that $]\pi_{A_i}, \pi_{B_i}[<]\pi_{A_{i+1}}, \pi_{B_{i+1}}[$ $(i = 1, \cdots, k-1)$. For $i = 1, \cdots, k$, let $P_i =]\pi_{A_i}, \pi_{B_i}[\cap P$. Let $Z_0 = [\pi_0, \pi_{A_1}] \cap P$. For $i = 1, \cdots, k-1$, let $Z_i = [\pi_{B_i}, \pi_{A_{i+1}}] \cap P$. Let $Z_k = [\pi_{B_k}, \pi_G] \cap P$. Clearly some or all of the Z_i may be void. Evidently we have:

(i) $P = Z_0 \cup P_1 \cup Z_1 \cup P_2 \cup \cdots \cup P_{k-1} \cup Z_{k-1} \cup P_k \cup Z_k$;
(ii) $Z_0 < P_1 < Z_1 < P_2 < \cdots < P_{k-1} < Z_{k-1} < P_k < Z_k$;
(iii) $Z \cap P = \bigcup_{i=0}^k Z_i$;
(iv) $P_i \subseteqq]\pi_{A_i}, \pi_{B_i}[$ $(i = 1, \cdots, k)$;
(v) the intervals given in (iv) are pairwise disjoint.

Now let $P^* = P \cup \{\pi_{A_1}, \pi_{B_1}, \pi_{A_2}, \pi_{B_2}, \cdots, \pi_{A_k}, \pi_{B_k}\}$. Clearly $Z_0 \leqq \{\pi_{A_1}\} < P_1 < \{\pi_{B_1}\} \leqq Z_1 \leqq \{\pi_{A_2}\} < P_2 < \cdots \leqq Z_{k-1} \leqq \{\pi_{A_k}\} < P_k < \{\pi_{B_k}\} \leqq Z_k$. Using the notation established in 3.3, we now get

$$\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)| = \sum(\hat{\lambda}\,; P') \leqq \sum(\hat{\lambda}\,; P^*)$$
$$= \sum_{i=1}^{k} \sum(\hat{\lambda}\,; \{\pi_{A_i}\} \cup P_i \cup \{\pi_{B_i}\}) \,.$$

By 2.9, we have $\sum(\hat{\lambda}\,; \{\pi_{A_i}\} \cup P_i \cup \{\pi_{B_i}\}) \leqq V(\hat{\lambda}\,; [\pi_{A_i}, \pi_{B_i}])$ for $i = 1, \cdots, k$. Combining these inequalities, we obtain

$$\sum_{i=1}^{m-1} |\hat{\lambda}(\pi_{i+1}) - \hat{\lambda}(\pi_i)| \leqq \sum_{i=1}^{k} V(\hat{\lambda}; [\pi_{A_i}, \pi_{B_i}]) \leqq \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}]) \,.$$

Since the partition P' was arbitrary, we have $V(\hat{\lambda}) \leqq \sum_\alpha V(\hat{\lambda}\,; [\pi_{A_\alpha}, \pi_{B_\alpha}])$ and hence 3.4.2 is proved.

Let $\varepsilon > 0$. We shall ultimately show that there is a $\mu \in I$ such that $\|\lambda - \mu\| \leqq 3\varepsilon$. Since ε is arbitrary and I is closed, this will prove that

$\lambda \in I$. It will then follow that $\bigcap_{\pi_A \in H} I_A \subseteqq I$, completing the proof. By 3.4.2, there exist $\alpha_1, \cdots, \alpha_m$ such that $\sum_{i=1}^m V(\hat{\lambda}; [\pi_{A_{\alpha_i}}, \pi_{B_{\alpha_i}}]) + \varepsilon \geqq V(\hat{\lambda})$. We shall henceforth write A_i for A_{α_i} and B_i for B_{α_i}. Then

$$V(\hat{\lambda}) - \sum_{i=1}^{m} V(\hat{\lambda}; [\pi_{A_i}, \pi_{B_i}]) \leqq \varepsilon . \tag{3.4.3}$$

We may suppose that $A_1 \subset B_1 \subseteqq A_2 \subset B_2 \subseteqq \cdots \subseteqq A_m \subset B_m$. By 1.8, there exist $x_i, y_i \in B_i - A_i$ such that

$$|\lambda| ((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m} \qquad (i = 1, \cdots, m) . \tag{3.4.4}$$

Let $U_i =]\pi_{x_i[}, \pi_{y_i]}[$; obviously U_i is open and closed. Note also that $U_i \subseteqq]\pi_{A_i}, \pi_{B_i}[\subseteqq Z'$. Let $U = \bigcup_{i=1}^m U_i$; U is open and closed (and hence compact). Also $U \subseteqq Z' \subseteqq H'$ where H' denotes the complement of H in $\hat{G}_0$. Thus for each $\pi_A \in U$, there is a $\lambda_A \in I$ such that $\lambda_A(A) = \hat{\lambda}_A(\pi_A) \neq 0$. Note that $\pi_0 \notin U$ since $\pi_0 \in H$ and $\pi_G \notin U$ since $\pi_G \notin Z'$. By the continuity of $\hat{\lambda}_A$ on $\hat{G}_0$ and Theorem 2.8, there exists an open and closed set V_A such that

(a) $\pi_A \in V_A$;

(b) $\pi \in V_A$ implies $\hat{\lambda}_A(\pi) \neq 0$;

(c) $V_A \subseteqq U$;

(d) V_A has the form 2.8.1.[2]

Since U is compact and $\bigcup_{\pi_A \in U} V_A = U$, there is a finite set $\{V_{A_i}\}_{i=1}^p$ such that $\bigcup_{i=1}^p V_{A_i} = U$.

For $V_{A_i} =]\pi_{a_i[}, \pi_{b_i]}[$, let $V_{A_i}^- = [\pi_0, \pi_{a_i]}[$ and $V_{A_i}^+ =]\pi_{b_i[}, \pi_G]$. Let $\mathscr{V}$ be the family of sets consisting of all V_{A_i}, $V_{A_i}^-$, and $V_{A_i}^+$. For $\pi \in U$, let $R_\pi = \bigcap \{V \in \mathscr{V} : \pi \in V\}$. Clearly there exist only finite many distinct R_π — say $\{R_i\}_{i=1}^k$.

The following assertions are easily shown:

(a′) $\bigcup_{i=1}^k R_i = U$;

(b′) each R_i has the form 2.8.1[3];

(c′) the family $\{R_i\}_{i=1}^k$ is pairwise disjoint;

(d′) for each i, there exists a $\lambda_i \in I$ such that $\pi \in R_i$ implies $\hat{\lambda}_i(\pi) \neq 0$.

By Lemma 3.2[3], there are $\nu_i \in \mathscr{M}(G)$ such that

$$\hat{\nu}_i(\pi) = \begin{cases} \dfrac{1}{\hat{\lambda}_i(\pi)} & \text{if } \pi \in R_i , \\ 0 & \text{if } \pi \notin R_i ; \end{cases}$$

$i = 1, \cdots, k$. Let $\mu = \sum_{i=1}^k \lambda_i * \nu_i * \lambda$; clearly $\mu \in I$. Evidently

[2] If $\pi_G \in Z'$, then V_A can be of the form 2.8.3.

[3] If $\pi_G \in Z'$, then R_i can be of the form 2.8.3.

$$\hat{\mu}(\pi) = \begin{cases} \hat{\lambda}(\pi) & \text{if } \pi \in U, \\ 0 & \text{if } \pi \notin U. \end{cases}$$

We observe that

$$(\hat{\lambda} - \hat{\mu})(\pi) = \begin{cases} 0 & \text{if } \pi \in U_i =]\pi_{x_i[}, \pi_{y_i]}[, \\ \hat{\lambda}(\pi) & \text{if } \pi = \pi_{x_i[} \text{ or } \pi = \pi_{y_i]}. \end{cases}$$

Using this, Lemma 3.1, and relation 3.4.4, we have

$$\begin{aligned} (3.4.5) \quad V(\hat{\lambda} - \hat{\mu}; [\pi_{x_i[}, \pi_{y_i]}]) &= |\hat{\lambda}(\pi_{x_i[})| + |\hat{\lambda}(\pi_{y_i]})| \\ &= |\hat{\lambda}(\pi_{x_i[}) - \hat{\lambda}(\pi_{A_i})| + |\hat{\lambda}(\pi_{B_i}) - \hat{\lambda}(\pi_{y_i]})| \\ &\leqq V(\hat{\lambda}; [\pi_{A_i}, \pi_{x_i[}]) + V(\hat{\lambda}; [\pi_{y_i]}, \pi_{B_i}]) \\ &\leqq |\lambda|(]-\infty, x_i[- A_i) + |\lambda|(B_i -]-\infty, y_i]) \\ &= |\lambda|((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m}. \end{aligned}$$

We also have from 3.1 that

$$\begin{aligned} (3.4.6) \quad & V(\hat{\lambda}; [\pi_{y_i]}, \pi_{B_i}]) + V(\hat{\lambda}; [\pi_{A_i}, \pi_{x_i[}]) \\ &= |\lambda|((B_i - A_i) - [x_i, y_i]) \leqq \frac{\varepsilon}{m}. \end{aligned}$$

Using 2.10, 3.4.5, and 3.4.6, we obtain

$$\begin{aligned} (3.4.7) \quad V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) &= V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{x_i[}]) + V(\hat{\lambda} - \hat{\mu}; [\pi_{x_i[}, \pi_{y_i]}]) \\ &+ V(\hat{\lambda} - \hat{\mu}; [\pi_{y_i]}, \pi_{B_i}]) = V(\hat{\lambda}; [\pi_{A_i}, \pi_{x_i[}]) \\ &+ V(\hat{\lambda} - \hat{\mu}; [\pi_{x_i[}, \pi_{y_i]}]) + V(\hat{\lambda}; [\pi_{y_i]}, \pi_{B_i}]) \leqq \frac{2\varepsilon}{m}. \end{aligned}$$

We used the fact that $\hat{\mu}$ is zero on $[\pi_{A_i}, \pi_{x_i[}]$ and $[\pi_{y_i]}, \pi_{B_i}]$ since these sets are disjoint from U. Finally, using 2.10, 3.1, and 3.4.7, we get

$$\begin{aligned} \|\lambda - \mu\| &= V(\hat{\lambda} - \hat{\mu}) = V(\hat{\lambda} - \hat{\mu}; [\pi_{B_m}, \pi_G]) + V(\hat{\lambda} - \hat{\mu}; [\pi_0, \pi_{A_1}]) \\ &+ \sum_{i=2}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{B_{i-1}}, \pi_{A_i}]) + \sum_{i=1}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) \\ &= V(\hat{\lambda}; [\pi_{B_m}, \pi_G]) + V(\hat{\lambda}; [\pi_0, \pi_{A_1}]) + \sum_{i=2}^{m} V(\hat{\lambda}; [\pi_{B_{i-1}}, \pi_{A_i}]) \\ &+ \sum_{i=1}^{m} V(\hat{\lambda} - \hat{\mu}; [\pi_{A_i}, \pi_{B_i}]) \leqq |\lambda|(G - B_m) + |\lambda|(A_1) + \sum_{i=2}^{m} |\lambda|(A_i - B_{i-1}) \\ &+ 2\varepsilon = |\lambda|(G) - \sum_{i=1}^{m} |\lambda|(B_i - A_i) + 2\varepsilon = V(\hat{\lambda}) \\ &- \sum_{i=1}^{m} V(\hat{\lambda}; [\pi_{A_i}, \pi_{B_i}]) + 2\varepsilon. \end{aligned}$$

Now applying 3.4.3, we obtain $\|\lambda - \mu\| \leq 3\varepsilon$. This completes the proof.

3.5. EXAMPLES. Let $G =]0, 1[$ and $\lambda \in \mathscr{M}(G)$ be ordinary Lebesgue measure. Then the ideal $I = \{\lambda * \mu + \alpha\lambda : \mu \in \mathscr{M}(G)$ and α is a complex number$\}$ is dense in $\mathscr{M}(G)$ since $\hat{\lambda}$ vanishes only at π_0; I is the ideal generated by λ. If $G = [0, 1]$ and λ is Lebesgue measure, then $I = \{\lambda * \mu : \mu \in \mathscr{M}(G)\}$ is the ideal generated by λ and I is dense in $\{\lambda \in \mathscr{M}(G): \lambda(\{0\}) = 0\}$.

4. The Herglotz-Bochner theorem for $\mathscr{M}(G)$.

This section generalizes § 6 [3].

4.1. DEFINITION. Let h be any bounded, real-valued, nondecreasing function on $\hat{G}_0$. Let Δ denote a partition $\{t_k\}_{k=0}^{m}$ of G where $t_0 < t_1 < \cdots < t_m$. For an arbitrary complex-valued function f on G, let

$$S(f, \Delta) = f(t_0)\,[h(\pi_{t_0]}) - h(\pi_{t_0[})] + \sum_{k=1}^{m} f(t_k)\,[h(\pi_{t_k]}) - h(\pi_{t_{k-1}]})]\ .$$

4.2. THEOREM. *Let $f \in \mathfrak{C}_0(G)$ and h be as in 4.1. Then there exists a unique number $L(f)$ such that for every $\varepsilon > 0$ there exists a Δ_0 as in 4.1 with the property that $|L(f) - S(f, \Delta)| \leqq \varepsilon$ for all $\Delta \supseteqq \Delta_0$. We write this relation as $L(f) = \lim_\Delta S(f, \Delta)$.*

4.3. THEOREM. *The function L defined in 4.2 for all $f \in \mathfrak{C}_0(G)$ is a bounded nonnegative linear functional on $\mathfrak{C}_0(G)$.*

4.4. DEFINITION. Let h be a continuous function on $\hat{G}_0$ and let $\pi_A, \pi_B \in \hat{G}_0$, $\pi_A < \pi_B$. Then we define

$$V_c(h; [\pi_A, \pi_B]) = \sup\Big\{\sum_{i=1}^{m} V(h; [\pi_{x_i[}, \pi_{y_i]}]): x_1 \leqq y_1 < x_2 \leqq y_2 < \cdots < x_m \leqq y_m,\ \pi_A \leqq \pi_{x_1[}, \pi_{y_m]} \leqq \pi_B,\ [x_i, y_i] \ \text{compact}\Big\}\ . \tag{4.4.1}$$

In particular, we define $V_c(h) = V_c(h; [\pi_0, \pi_G])$. We also define

$$V_c(h; [\pi_A, \pi_A]) = 0 \tag{4.4.2}$$

for $\pi_A \in \hat{G}_0$.

4.5. Let h be a real-valued continuous function on $\hat{G}_0$ having finite variation and let $\pi_{A_1} \leqq \pi_{A_2} \leqq \cdots \leqq \pi_{A_k}$. Then

$$V_c(h; [\pi_{A_1}, \pi_{A_k}]) = \sum_{i=2}^{k} V_c(h; [\pi_{A_{i-1}}, \pi_{A_i}])\ . \tag{4.5.1}$$

4.6. THEOREM. *Let h be a continuous function on $\hat{G}_0$ having finite* *variation and such that* $h(\pi_0) = 0$. *Then there exists a* $\lambda \in \mathscr{M}(G)$ *such that* $\hat{\lambda} = h$ *if and only if*

(4.6.1) $$V(h) = V_c(h)$$

The proof is a tedious lengthy extension of the proof of Theorem 6.7 [3] and uses 4.2, 4.3, 3.1, 4.5, and 1.11 in the case that h is non-decreasing. The general case is proved by applying 2.10.

4.7. EXAMPLES. Let G be the real line under the usual ordering. Then a function h on $\hat{G}_0$ is the Fourier transform of some measure $\lambda \in \mathscr{M}(G)$ if and only if h is continuous, has finite variation, and $h(\pi_0) = 0$.

Condition 4.6.1 is not always satisfied by continuous functions h on $\hat{G}_0$ having finite variation and satisfying $h(\pi_0) = 0$. Let $G = [0, 1] \times]0, 1[$ where $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. Let h on $\hat{G}_0$ be defined by

$$h(\pi_A) = \sup\{a \in [0, 1]: (a, x) \in A \text{ for some } x \in \ ]0, 1[\} .$$

The function h is continuous, $V(h) = 1$, and $V_c(h) = 0$. The linear functional L obtained from h in 4.3 turns out to be the zero functional.

5. Some consequences of the Herglotz-Bochner theorem.

Theorems 5.1 and 5.2 are routine applications of 4.6.

5.1. THEOREM. *Let* ϕ *be a continuous function from a subset* $H \supseteqq \{0\}$ *of the complex plane to the complex plane such that* $\phi(0) = 0$ *and*

(5.1.1) *for every* $M > 0$, *there exists a* $K_M > 0$ *such that*
$$|\phi(z) - \phi(w)| \leqq K_M |z - w| \text{ for } z, w \in H, |z| \leqq M, |w| \leqq M .$$

(I.e., ϕ *satisfies a Lipschitz condition for arbitrarily large disks.) Then for every* $\lambda \in \mathscr{M}(G)$ *for which (range* $\hat{\lambda}) \subseteqq H$, *there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu} = \phi \circ \hat{\lambda}$.

5.2. THEOREM. *Let* ϕ *be a continuous function from* $[0, \infty[$ to $[0, \infty[$ *that is non-decreasing, absolutely continuous on all intervals* $[0, M]$, *and such that* $\phi(0) = 0$. *Then for every nonnegative measure* $\lambda \in \mathscr{M}(G)$ *there exists a nonnegative* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu} = \phi \circ \hat{\lambda}$.

5.3. COROLLARY. *Let* $\lambda \in \mathscr{M}(G)$. *Then there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu}(\pi) = |\hat{\lambda}(\pi)|$ *for all* $\pi \in \hat{G}_0$.

5.4. COROLLARY. *Let* $\lambda \in \mathscr{M}(G)$. *Then there exists a* $\nu \in \mathscr{M}(G)$ *such that* $\hat{\nu}(\pi) = \overline{\hat{\lambda}(\pi)}$ *for all* $\pi \in \hat{G}_0$; *here* $\bar{z}$ *denotes the complex conju-*

gate of z. In other words, $\mathscr{M}(G)$ is self-adjoint (see page 88 [6]).

5.5. COROLLARY. *Let $\lambda \in \mathscr{M}(G)$ be a nonnegative measure. Then there exists a nonnegative $\nu \in \mathscr{M}(G)$ such that $\nu * \nu = \lambda$.*

5.6. It is natural to ask whether Theorem 5.2 is valid for more general measures λ; one might hope that the result would be valid at least for $\lambda \in \mathscr{M}(G)$ for which $\hat{\lambda}$ is nonnegative. If this were the case, 5.5 would also generalize. However, we will see in 5.7 that this is not the case whenever G is infinite. Theorem 5.7 also shows that the Lipschitz condition assumed for ϕ in 5.1 cannot be replaced by absolute continuity. (The function $\phi(x) = \sqrt{x}$ is absolutely continuous on all intervals $[0, M]$ but does not satisfy 5.1.1.)

5.7. THEOREM. *Suppose that G is infinite. Then there exists a $\lambda \in \mathscr{M}(G)$ such that $\hat{\lambda}$ is nonnegative on $\hat{G}_0$ and such that $\lambda \neq \nu * \nu$ for all $\nu \in \mathscr{M}(G)$.*

Proof. Suppose G has an infinite subset $\{x_i\}_{i=1}^{\infty}$ such that $x_i < x_{i+1}$ for all i. Let λ be the discrete measure defined by

$$\lambda(\{x_n\}) = \begin{cases} \dfrac{1}{n^2} & \text{if } n \text{ odd}\,, \\[2ex] -\dfrac{1}{(n-1)^2} & \text{if } n \text{ even}\,. \end{cases}$$

It can be shown that λ satisfies the conclusions of the theorem. If G does not have an infinite subset as above, then G has an infinite subset $\{x_i\}_{i=1}^{\infty}$ such that $x_i > x_{i+1}$ for all i. This case is treated in a similar manner.

5.8. It is evident from 5.7 that $\mathscr{M}(G)$ (G infinite) is not isomorphic as an algebra to the algebra $\mathfrak{C}_0(X)$ for any locally compact space X. In the contrary case, $\mathscr{M}(G)$ would be isomorphic to $\mathfrak{C}_0(\hat{G})$ and the isomorphism would be $\lambda \to \hat{\lambda}$. However, if $h \in \mathfrak{C}_0(\hat{G})$ is nonnegative, then for some $h_0 \in \mathfrak{C}_0(\hat{G})$, we have $h_0^2 = h$.

Finally, the result of 8.3 [3] holds for locally compact G. That is,

5.9. THEOREM. *A measure $\lambda \in \mathscr{M}(G)$ is idempotent if and only if λ is of the form:*

$$\lambda = \varepsilon_{c_0} - \varepsilon_{c_1} + \cdots + (-1)^k \varepsilon_{c_k} \tag{5.9.1}$$

where $c_0 < c_1 < \cdots < c_k$.

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