SOME GENERALIZATIONS OF DOEBLIN'S DECOMPOSITION

RICHARD ISAAC

Introduction. We consider a Markov chain $\{X_i\}$ $i = 0, 1, \cdots$ with stationary transition probabilities $P^k(t, E)$ defined on a measure space (Ω, Σ). All sets discussed in the following will be Σ-sets. A set N is called null if $P^1(t, N) = P(t, N) = 0$ for all $t \in \Omega$, and a set S is called invariant if $P(t, S) = 1$ for $t \in S - N$ where N is a null set. $\mathscr{I}_p$ will denote the σ-field determined by the invariant sets given the transition probability $P(t, E)$. A set S is indecomposable if it does not contain two disjoint non-null invariant subsets. The concept of a strictly separable σ-field will be employed, together with the fact that such a σ-field is atomic. S^c is the complement of the set S.

This paper considers several conditions under which we have a general decomposition $\Omega = F + \sum_\alpha A_\alpha$ where F is a transient state and the A_α are ergodic, indecomposable state, i. e., defining

$$P_1(t, E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(t, E) ,$$

then $P_1(t, \sum_\alpha A_\alpha) = 1$ for all $t \in \Omega$, $P(t, A_\alpha) = 1$ for $t \in A_\alpha$, and the A_α are minimal, up to an equivalence. This work may be considered as a further step in Doob's discussion in [3] on generalizing Doeblin's classical results. Our results are sometimes generalizations of Doob's work and other times give slightly stronger conclusions, but replace Doob's assumption of an a priori stationary measure for the process by general conditions in terms of measures.

Theorem 1 is due to Blackwell and is the basis for Theorem 2, the decomposition theorem, which is proved under the assumption of the existence of the Cesàro limit $P_1(t, E)$ for all $t \in \Omega$, $E \in \Sigma$. Theorem 3 gives Doeblin type conditions in terms of measures implying the existence of $P_1(t, E)$. Theorem 4 discusses the special case of a priori knowledge of a σ-finite stationary measure for the process. Finally, Theorem 5 gives a countable decomposition when the Cesàro limit is absolutely continuous with respect to a σ-finite measure.

THEOREM 1. (Blackwell). *Let $P(t, E)$ be an idempotent Markov*

Received June 3, 1960. This work was done while the author held a Fulbright fellowship in Paris.
The author takes this opportunity to express his thanks to the referee for several suggested modifications, including the correction of a mistake in the original proof of Theorem 2. The revised proof is presented here.

chain: $P(t, E) = \int P(y, E)\, P(t, dy)$, *and let* Σ *be a strictly separable* σ*-field. Then there is a decomposition:* $\Omega = F + \sum_\alpha A_\alpha$ *of disjoint sets, satisfying*:

$$(1)\quad P(t, E) = x_\alpha(E)\,, \qquad t \in A_\alpha$$
$$(2)\quad x_\alpha(A_\alpha) = 1$$
$$(3)\quad P(t, F) = 0\,, \qquad \textit{for all } t \in \Omega\,.$$

Moreover, the A_α are indecomposable, and indeed, are even atoms of $\mathscr{F}_p$.

Proof. See [1], Theorem 6 and 7.

THEOREM 2. *Let* Σ *be strictly separable. If* $P_1(t, E)$ (as defined in the introduction) *exists for each* $t \in \Omega$, $E \in \Sigma$, *then there is a decomposition:* $\Omega = F^* + \sum_\alpha A^*_\alpha$ *of disjoint sets such that:*

(a) $P_1(t, E) = x_\alpha(E)\,, \quad t \in A^*_\alpha$
(b) $P_1(t, A^*_\alpha) = x_\alpha(A^*_\alpha) = 1\,, \quad t \in A^*_\alpha$
(c) $P(t, A^*_\alpha) = 1\,, \quad t \in A^*_\alpha$
(d) $P_1(t, \sum_\alpha A^*_\alpha) = 1\,, \quad t \in \Omega$
(e) *Each* A^*_α *is indecomposable with respect to* P_1 *and* P.

Proof. Determine an indempotent chain by using $P_1(t, E)$ as transition probability, and apply Theorem 1. This yields a decomposition $\Omega = F + \sum_\alpha A_\alpha$. Let $F = F_0$ and define $F_i = \{t : P(t, F_{i-1}) > 0\}$ for $i = 1, 2, \cdots$ and set $F^* = \bigcup_{i=1}^\infty F_i$. By induction, we show that $P_1(t, F_i) = 0$ for all t and all i. First, $P_1(t, F_0) = 0$ for all t, by Theorem 1. If $P_1(t, F_i) = 0$ for all t, then:

$$0 = P_1(t, F_i) = \int P(x, F_i) P_1(t, dx) = \int_{F_{i+1}} P(x, F_i)\, P_1(t, dx)$$

which implies that $P_1(t, F_{i+1}) = 0$ for all t. Thus F^* is a P_1 null set. Define new sets $A^*_\alpha : A^*_\alpha = A_\alpha - A_\alpha \cdot F^*$. Then $\Omega = \Sigma\, A^*_\alpha + F^*$. (a), (b) and (d) follow immediately from Theorem 1 and the definition of the new decomposition. If $t \in A^*_\alpha$:

$$1 = P_1(t, A^*_\alpha) = \int_{A^*_\alpha} P_1(x, A^*_\alpha)\, P(t, dx) + \int_{A^{*c}_\alpha - F^*} P_1(x, A^*_\alpha)\, P(t, dx)$$
$$+ \int_{F^*} P_1(x, A^*_\alpha)\, P(t, dx) = P(t, A^*_\alpha)$$

the second integral vanishing since $P_1(x, A^*_\alpha) = 0$ for $x \in A^*_\beta$, $\beta \neq \alpha$, and the last since $P(t, F^*) = 0$ for $t \notin F^*$. This proves (c). A^*_α is indecomposable with respect to P_1 by Theorem 1, and hence also with respect to P, since a decomposition relative to P is also a decomposition relative to P_1. The proof is concluded.

THEOREM 3. *Suppose for $t_0 \in \Omega$, there exists a σ-finite measure m_{t_0} with the following Lebesgue decomposition:*

$$P^n(t_0, E) = \int_E f_n(y) m_{t_0}(dy) + K_n(t_0, E)$$

where $K_n(t_0, E)$ is the singular part of P^n with respect to m_{t_0}, and the $f_n(y) \in L_1(\Omega, \Sigma, m_{t_0})$. Assume that:

(1) $\lim_{n\to\infty} K_n(t_0, E) = 0$

(2) *There is a function $f \in L_1(\Omega, \Sigma, m_{t_0})$ with $f_n \leqq f$* a. e. (m_{t_0}).

Then: $P_1(t_0, E)$ exists for all $E \in \Sigma$ and defines a stationary probability measure. In fact, the total variation of $|1/n \sum_{k=1}^n P^k(t_0, \cdot) - P_1(t_0, \cdot)|$ tends to 0 as n tends to ∞.

Proof. Let $\mathscr{M}$ be the B-space of all finite measures on (Ω, Σ). A set K of measures in $\mathscr{M}$ is weakly sequentially compact if it is bounded and the countable additivity of μ on Σ is uniform with respect to $\mu \in K$. Consider the set $\{P^n(t_0, \cdot)\ n = 1, 2, \cdots\}$. Let $E = \sum_{i=1}^\infty E_i$. Then, for any n:

$$\begin{aligned} P^n(t_0, E) - P^n\left(t_0, \sum_{i=1}^k E_i\right) &= P^n\left(t_0, \sum_{i=k+1}^\infty E_i\right) \\ &= \int_{\sum_{i=k+1}^\infty E_i} f_n(y)\, m_{t_0}(dy) + K_n\left(t_0, \sum_{i=k+1}^\infty E_i\right) \\ &\leqq \int_{\sum_{i=k+1}^\infty E_i} f(y)\, m_{t_0}(dy) + K_n\left(t_0, \sum_{i=k+1}^\infty E_i\right) . \end{aligned}$$

Since $f(y) \in L_1(\Omega, \Sigma, m_{t_0})$, by choosing k sufficiently large, the first terms can be made as small as desired and for n large enough, the second term may be made small. Hence, the countable additivity is uniform with respect to all sufficiently large n.

Consider the operator $Tx\int P(t, E)\, x(dt)$ for $x \in \mathscr{M}$. Then $TP(t_0, \cdot) = P^2(t_0, \cdot)$. Let $1/(n-1)\,(\sum_{k=0}^{n-1} T^k) = A(n)$. Then by the mean ergodic theorem in B-space, $\{A(n)\, P(t_0, \cdot)\}$ converges in norm if the set $\{A(n)\, P(t_0, \cdot)\}$ is weakly sequentially compact. Since the set $\{P^n(t_0, \cdot)\}$ is weakly sequentially compact, so is the set $\{A(n)\, P(t_0, \cdot)\}$ and therefore $1/n \sum_{k=1}^n P^k(t_0, \cdot)$ converges to $P_1(t_0, \cdot) \in \mathscr{M}$ in the strong sense, i. e., in the norm of total variation, and hence also for each $E \in \Sigma$. That P_1 defines a stationary probability measure is clear.

COROLLARY 1. *If the set* $\{P^n(t_0, \cdot)\}$ *is weakly sequentially compact, then the conclusion of the theorem holds.*

COROLLARY 2. *The decomposition of Theorem 2 obtains if there is a class of measures* $\{m_t : t \in \Omega\}$ *where* m_t *satisfies the hypothese for Theorem 3 for* $t \in \Omega$.

REMARK. If there is one measure which satisfies the hypotheses for all $t \in \Omega$, then we obtain a countable decomposition as described in Theorem 5.

THEOREM 4. *Let* $\Phi(\cdot)$ *be a* σ*-finite stationary measure for the process and suppose that, if* $\Phi(H) = \infty$, *then for* $\varepsilon > 0$ *and each* $t \in \Omega$, *there is a set* S, $S \subset H$, $\Phi(S) < \infty$ *and* S *depends upon* H, ε, t, *such that* $P^n(t, H - S) < \varepsilon$ *uniformly in* n. *Let, in addition, the singular component,* $P(t, n, \Omega)$, *of* P^n *with respect to* Φ *tend to zero for all* $t \in \Omega$. *Then*: $P_1(t, E)$ *exists for all* $t \in \Omega$, $E \in \Sigma$, *and* $P_1(t, \cdot)$ *defines a probability measure for each* t.

Proof. Consider the space $L_1(\Omega, \Sigma, \Phi)$ and define the map $T : g(t) = \int f(x)\, P(t, dx)$ which maps L_1 into itself. We now show $|T|_1 \leqq 1$ and $|T|_\infty \leqq 1$. If $\int |f(x)|\, \Phi(dx) \leqq 1$, then $\int |g(t)|\, \Phi(dt) = \int \left| \int f(x)\, P(t, dx) \right| \Phi(dt) \leqq \int |f(x)|\, \Phi(dx) \leqq 1$ by Fubini's theorem and thus $|T|_1 \leqq 1$. If $|f(x)| \leqq C$ a. e. (Φ) where C is a constant, let N be such that $|f(x)| > 0$ on N, $\Phi(N) = 0$. Then:

$$|g(t)| \leqq \left| \int_{N^c} f(x)\, P(t, dx) \right| + \left| \int_N f(x)\, P(t, dx) \right| \leqq C + \left| \int_N f(x)\, P(t, dx) \right|$$

and $0 = \Phi(N) = \int P(t, N)\, \Phi(dt)$ implies that $P(t, N) = 0$ a. e. (Φ), say on N_1^c, $\Phi(N_1) = 0$. Then $|g(t)| \leqq C$ on N_1^c, and $|T|_\infty \leqq 1$. By the generalized point ergodic theorem $\lim_{n\to\infty}[1(n-1)]\sum_{k=0}^{n-1} (T^k f)(x)$ exists a. e. (Φ). If E is fixed and $\Phi(E) < \infty$, $\int P(x, E)\, \Phi(dx) = \Phi(E) < \infty$ and $P(x, E) \in L_1(\Omega, \Sigma, \Phi)$. Thus, if $\Phi(E) < \infty$, $1/n\sum_{k=1}^{n} P^k(x, E)$ converges a. e. (Φ). If $P(x, n, \Omega) \xrightarrow{n} 0$, the proof is the same as Doob's in [3] and the convergence holds everywhere as long as $\Phi(E) < \infty$. Let $\Phi(H) = \infty$, and hold x fixed. A simple approximation arugument yields convergence for each fixed x. Since convergence holds for all Σ-sets, by a theorem of Nikodym the limit is countably additive and therefore a probability measure.

THEOREM 5. *Let* m *be a* σ*-finite measure on* Σ *with* $P_1(t, E) =$

$\int_E h(t, y)\, m(dy)$. *If* $\mathscr{I}_{p_1}$ *is the invariant* σ*-field under* $P_1(t, E)$, *then there is a decomposition:*

$$\Omega = F + M + X_1 + X_2 + \cdots$$

where $F, M, X_i \in \mathscr{I}_{p_1}$ *and:*

(a) $0 < m(X_i) < \infty$, *and* $E \in \mathscr{I}_{p_1}$, $E \subseteq X_i$ *implies* $m(E) = m(X_i)$ *or* $m(E) = 0$

(b) $m(E) = \infty$ *and* $E \in \mathscr{I}_{p_1}$, $E \subseteq M$ *implies* $m(E) = \infty$ *or* $m(E) = 0$.

(c) F *is a null set, and* $m(F) \leqq \infty$

(d) *The* X_i *are indecomposable, but* M *may be decomposable.*

Proof. The proof is similar to Lemma 3 in [1]. Put the sets satisfying (a) into equivalence classes by defining two such sets as equivalent if their symmetric difference has measure zero. Any two sets in different classes can have only a set of measure zero in common, and each set has positive measure. If there were an uncountable number of classes, there would have to be an uncountable number of disjoint sets of positive measure. By σ-finiteness, $\Omega = \sum_{i=1}^{\infty} E_i$, $m(E_i) > \infty$, and each E_i allows the trace of only a countable number of sets of positive measure. The number of classes is therefore countable. Consider $S = \Omega - (X_1 + X_2 \cdots)$ where the X_i are sets of representatives, one from each class. Consider the totality of disjoint null sets in S of measure ∞. By the same reasoning as above, there are at most a countable number of them, whose union is also a null set. Removing this set from S, S now contains a null set of largest finite measure; remove it also. Let F be the union of all the null sets removed from S. Let $S - F = M$. If $m(M) < \infty$, the related argument of Blackwell remains unchanged, and we have that $m(M) = 0$, and hence M is empty. If $m(M) = \infty$, then for $E \in \mathscr{I}_{p_1}$, $E \subseteq M$, either $m(E) = \infty$ or $m(E) < \infty$. If $m(E)$ is finite, again $m(E) = 0$. M may be decomposable into two disjoint non-null sets of measure ∞.

The following two corollaries examine when the singular M disappears.

COROLLARY 1. *If* m *is* σ*-finite with respect to* $\mathscr{I}_{p_1}$, *then the decomposition in Theorem* 5 *lacks the term* M.

Proof. For then M may be expressed as the sum of disjoint $\mathscr{I}_{p_1}$ sets of finite measure. As shown above, if $m(E) < \infty$, $E \subseteq M$, then $m(E) = 0$, and therefore $m(M) = 0$. Thus $M \subseteq F$.

COROLLARY 2. *If there exists a sequence of increasing* $\mathscr{I}_{p_1}$ *sets* *of finite measure* $\{H_i\}$ *such that* $\lim_{i\to\infty} H_i = M$, *then the decomposition lacks the term* M.

Proof. $m(H_i) < \infty$ implies $m(H_i) = 0$

$$m(\lim H_i) = m(M) = \lim m(H_i) = 0 \ .$$

EXAMPLE 1. In [3], Doob considers the following example of a process with a stationary absolute measure, but which does not satisfy Doeblin's condition: $\Omega = (-\infty, \infty)$, $\Sigma =$ Borel subsets of Ω, and:

$$P^n(t, A) = \frac{1}{[2\pi(1-\rho^{2n})]^{1/2}} \int_A \exp - \frac{(y-\rho^n t)^2}{2(1-\rho^{2n})}\, dy$$

where ρ is constant, $0 \leqq \rho < 1$.

A stationary measure is the limit:

$$\lim_{n\to\infty} P^n(t, A) = \frac{1}{(2\pi)^{1/2}} \int_A \exp - \frac{y^2}{2}\, dy \ .$$

Without noticing, however, that there is a stationary absolute measure, we may directly apply Theorem 3 by taking m_{t_0} to be Legesgue measure on the real line for each t_0. It is easily seen that the integrand of $P^n(t, A)$ is uniformly bounded by a measurable function for large n. In this case, $\mathscr{F}p_1$ consists of the Borel subsets of $(-\infty, \infty)$. The regularity of Lebesgue measure implies that the X_i of Theorem 5 disappear, leaving only $M = \Omega = (-\infty, \infty)$ and $F =$ the empty set. Our result is a bit stronger than Doob's, yielding strong convergence.

EXAMPLE 2. If we consider the identity transition operator: $P(t, t) = 1$ on $\Omega = [0, 1]$, then we have an example of a decomposition which is not countable.

BIBLIOGRAPHY

1. D. Blackwell, *Idempotent Markov chains,* Annals of Math., **43** No. 3 (1942).
2. J. L. Doob, *Stochastic Processes,* Wiley 1953.
3. ———, *Asymptotic properties of Markoff transition probabilities.* Trans. Amer. Math. Soc., **63** (1948), 393-421.
4. Dunford-Schwartz, *Linear Operators, Part 1,* Interscience 1958.
5. M. Loève, *Probability Theory,* Van Nostrand 1955.

INSTITUTE HENRI POINCARÉ PARIS, FRANCE