THOSE ABELIAN GROUPS CHARACTERIZED BY THEIR COMPLETELY DECOMPOSABLE SUBGROUPS OF FINITE RANK

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(*To The Memory of Our Grandparents*)

I. A non-zero group[1] is said to have rank n, if any finitely generated subgroup can be generated by n or fewer elements and n is the smallest integer with this property.[2] A group G is completely decomposable if G is a direct sum of groups of rank one.[3] A subgroup H of a given group G is completely reducible in G if there exists a direct decomposition of G

$$G = \Sigma_j G_j + G' \tag{1}$$

such that all G_j's are of rank one and

$$H = \Sigma_j (H \cap G_j) . \tag{2}$$

Roughly speaking, a subgroup H is a completely reducible subgroup of G if it is a direct sum of rank one groups in a way that can be pulled up to a decomposition of G. It is clear that every completely reducible subgroup H of G is completely decomposable. But in general, the converse is not true. The main purpose of this paper is to determine those groups in which complete reducibility coincides with the complete decomposability for some classes of subgroups.

For convenience, the following terminology is adopted.

DEFINITION 1.1. A group is said to have property (α), if every subgroup of rank one is completely reducible in it.

DEFINITION 1.2. A group is said to have property (β) if every completely decomposable subgroup of finite rank is completely reducible in it.

DEFINITION 1.3. A group is said to have property (γ) if every subgroup of finite rank is completely reducible in it.

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[1] In this paper, "group" will always mean "abelian group".

[2] This definition of rank was given in [**5**].

[3] All the groups of rank one are known to us; e.g., a primary group of rank one is either cyclic or of the type $Z(p^\infty)$ and a torsion-free group of rank one is a subgroup of the additive group of all rational numbers. In torsion-free case, the above definition for completely decomposable groups coincides with the usual one; e.g., see [**4** pp. 211-216].

It follows from the definitions that property (γ) implies property (β) and property (β) implies property (α).

We begin with the following lemmas dealing with those torsion groups having the above properties.

LEMMA 1.1. *A torsion group G has property (α), (β) or (γ) if and only if each of its primary components has property (α), (β) or (γ) respectively.*

Proof. We decompose G into its p-primary components

$$G = \Sigma_p G^{(p)} \ . \tag{3}$$

Suppose G has property (β). Let $H^{(p)}$ be a completely decomposable subgroup of finite rank in the p-primary component $G^{(p)}$ of G. By our assumption that G has property (β), we have a direct decomposition of G

$$G = \Sigma_j G_j + G' \tag{4}$$

satisfying the following conditions:

(i) each G_j has rank one

(ii) $H^{(p)} = \Sigma_j(G_j \cap H^{(p)})$.

Since $H^{(p)}$ is contained in the p-primary component $G^{(p)}$ of G, we get what we need by simply considering the p-primary components of the decomposition (4).

Conversely, suppose that each $G^{(p)}$ had property (β). Let H be a given completely decomposable subgroup of finite rank n in G. It is clear that each p-primary component $H^{(p)}$ of H is either zero or also completely decomposable. Since we suppose that each $G^{(p)}$ has property (β), there exists a direct decomposition of $G^{(p)}$ for each p

$$G^{(p)} = \textstyle\sum_{j=1}^{n_p} (G_j^{(p)} \cap H^{(p)}) \tag{5}$$

satisfying the following conditions:

(i) each $G_j^{(p)}$ is at most of rank one,

(ii) $H^{(p)} = \sum_{j=1}^{n_p} (G_j^{(p)} \cap H^{(p)})$

(iii) $0 \leqq n_p \leqq n$

(iv) the maximum of n_p is n.

One may make $n_p = n$ by inserting zero groups. So, we assume $n_p = n$ for each p.

Denote $\Sigma_p G_j^{(p)}$ by G_j and $\Sigma_p G'_p$ by G'. It is easy to see that each G_j has rank one and

$$G = \textstyle\sum_{j=1}^{n} G_j + G' \ , \qquad H = \sum_{j=1}^{n} (H \cap G_j) \ . \tag{6}$$

This means that G has property (β). The other two parts of the lemma can be shown in an analogous manner.

LEMMA 1.2. *If a torsion group has property* (α), *then each of its p-primary components is either divisible or of the form*

$$\Sigma_{j_p} z_{j_p}(p^{h_p}) + \Sigma_{k_p} z_{k_p}(p^{h_p+1}) \text{ .}^{4} \tag{7}$$

Conversely, if each p-primary component of a torsion group G is either divisible or of the form (7), *then G has property* (γ).

Proof. It suffices to show that a p-primary groups having property (α) must be either divisible or reduced, since all the other cases are sonsequences of Lemma 1.1 and some known results [3, Lemma III]. If this is not the case, then we can find a group G with property (α) of the following form

$$Z(p^h) + Z(p^\infty) + G' \tag{8}$$

where $Z(p^h)$ is a cyclic group of order p^h generated by u_1 and is a p-primary divisible group of rank one. Let u_2 be an element in $Z(p^\infty)$ of order p^{h+1}. Consider the subgroup U generated by $u_1 + u_2$. Since G has property (α), there exists a direct decomposition of G

$$G = H + K \tag{9}$$

such that $U \subset H$ and H is of rank one. It is clear that the p-layer[5] of H coincides with that of U, and it is a cyclic group generated by $p^h u_2$. Since H is a direct summand of G, the heights of its elements in it are equal to the heights of those in G. It follows that $p^h u_2$ is of infinite height in H and H is divisible. Since $u_1 + u_2 \in U \subset H$, $u_1 + u_2$ is of infinite height in G. On the other hand, it follows from (8) that the height of $u_1 + u_2$ is equal to the minimum of the heights of u_1 and u_2 in G. Since the minimum is 0, we have a contradiction. This completes our proof.

II. Groups with property (α). To discuss groups with property (α), we need the following lemmas.

LEMMA 2.1. *If a group G has property* (α), *so does its torsion subgroup* $T(G)$.

It is an immediate consequence.

LEMMA 2.2. *A torsion-free group G has property* (α) *if and only if*

(a) *G and G/D are separable*[6], *where D is the maximal divisible*

[4] $Z(e)$ denotes the cyclic group of order e.

[5] The p-layer of a primary group G is defined to be the subgroup $P = \{x \mid px = 0, x \in G\}$.

[6] A torsion-free group is called separable, if every finite subset is contained in a completely decomposable direct summand (See [1]).

subgroup of G,

(b) *all the elements* $\neq 0$ *in* G/D *have the same type* [4, pp. 207–211] *in* G/D.

Proof. In fact, it is known [1, Corollary 4.5] that a torsion-free group G satisfies conditions (a) and (b) if and only if every pure subgroup of finite rank is a direct summand. Obviously, the property that every pure subgroup of finite rank is a direct summand implies our property (α) which deals with subgroup of rank one. We prove the converse by finite induction on the rank of the pure subgroups in G. It is trivially true for the pure subgroups of rank one in G. Now, let H be a pure subgroup of rank n and $\{x_1, x_2, \cdots, x_{n-1}, x_n\}$ be one of its maximal independent subsets. By induction hypothesis, G can be expressed as

$$G_1 + G_2 \tag{10}$$

where G_1 is the pure subgroup generated by $\{x_1, x_2, \cdots, x_{n-1}\}$. Then x_n can be expressed as

$$y_1 + y_2 \tag{11}$$

where $y_i \in G_i$ $(i = 1, 2)$ and $y_2 \neq 0$. Since G has property (α), the pure subgroup Y generated by y_2 is a direct summand of G_2. Consequently, $H = G_1 + Y$ is a direct summand of G.

From now on, a torsion-free group satisfying the conditions (a) and (b) in Lemma 2.2 will be called a uniformly separable group.

In fact, one can show that every reduced uniformly separable group is isomorphic to a subgroup of a strong direct sum of torsion-free groups of rank one each of which has the same type[7] as G.

LEMMA 2.3. *If G has property (α) and $T(G)$ is divisible, then $G/T(G)$ has property (α).*

Proof. Let U be a rank one subgroup if $G/T(G)$. Consider the complete inverse image $\widetilde{U}$ of U with respect to the natural projection q: $G \to G/T(G)$. Since $T(G)$ is divisible, we have $\widetilde{U} = U_1 + T(G)$ where U_1 is a rank one torsion-free subgroup in G. Applying the presumed property (α) in G and passing to the natural projection, we have our property (α) in $G/T(G)$.

THEOREM A. *A group G has property (α) if and only if*

(a) *G is a torsion group with property (α)* (see Lemma 1.2), *or*

[7] In a reduced uniformly separable group, all the non-zero elements have the same type; "the type of G" therefore has a definite meaning.

(b) $G = J + D$ *where* J *is a reduced uniformly separable group and* D *is divisible.*

Proof. Suppose G has property (α). If the torsion subgroup $T(G)$ of G is divisible, then $G = J_0 + T(G)$ where J_0 is torsion-free and J_0 has property (α) by Lemma 2.3. Now by Lemma 2.2, and hence G has form (b). We now show that if $T(G)$ is not divisible, then $T(G) = G$, so that G has form (a). If this is not the case, we can find a group G with property (α) of the following form

$$(12) \qquad R_1 + T^{(p_1)}(G) + G'^{8}$$

where R_1 is a torsion-free group of rank one and $T^{(p_1)}(G)$ is a non-zero p_1-primary component of the torsion subgroup $T(G)$ which, by assumption, is of the form (b). There are two cases according to the type $|R_1|$ of R_1.[9]

Case 1. The type $|R_1|$ has value ∞ at p_1. In this case, we consider the cyclic subgroup $\tilde{R}$ generated by $r_1 + a_{p_1}$ where and a_{p_1} are non-zero elements in R_1 and $T^{(p_1)}(G)$ respectively. Since G has property (α) and R is a torsion-free subgroup of rank one in G, we can decompose G as

$$(13) \qquad R_2 + T^{(p_1)}(G) + T''$$

such that (i) R_2 is a torsion-free direct summand of rank one in G and (ii) $\tilde{R} \subset R_2$. It follows from (13ii) that $p^{h_{p_1}+1} r_1 (= p_1^{h_{p_1}+1}(r_1 + a_{p_1}))$ whose type has value ∞ at p_1, is an element in R_2. Hence $|R_2|$ has value ∞ at p_1. On the other hand, it also follows from (13ii) that $r_1 + a_{p_1}$, which is not divisible by $p_1^{h_{p_1}+2}$, is an element of R_2 also. This means $|R_2|$ cannot have the value ∞ at p_1, hence, we have a contradiction. Thus, Case 1 cannot exist.

Case 2. The type $|R_1|$ has a finite value at p_1. In this case, we consider a torsion-free cyclic subgroup U of G satisfying the following conditions:

(i) U is generated by $p_1^s r_1 + a_{p_1}$ where $r_1 \in R_1$ and $a_{p_1} \in T^{(p_1)}(G)$

(ii) the characteristic [4, pp. 207–211] of r_1 in R_1 has zero value at p_1.

(iii) $s > h_{p_1} + 1 \geqq 1$.

Since G has property (α), we have the following decomposition

[8] Here, we use the fact that every p-primary subgroup of bounded order is a direct summand.

[9] A torsion-free group of rank one is automatically a uniformly separable group and all its elements have the same type. We denote this definite type by $|R|$. In fact, a torsion-free group of rank one is characterized by the type.

(14) $$G = R_3 + G''$$

where R_3 is a torsion-free group of rank one and contains U as a subgroup. It follows from our construction that $p_1^s r_1 + a_{p_1}$ is not divisible by p_1 in G. Consider the factor group $G/T(G)$; we have

(15) $$R_1 + (T^{(p_1)}(G) + G')/T(G) = G/T(G) = R_3 + G''/T(G)\ .$$

Since $p_1^s r_1 \in R_1$, the characteristic of $p_1^s r_1 + T(G)$ in $G/T(G)$ has value $s(>0)$ at p_1. On the other hand, since $p_1^s r_1 + a_{p_1} \in U \subset R_3$, the characteristic of $(p_1^s r_1 + a_{p_1}) + T(G)$ in $G/T(G)$ has value zero at p_1. But $(p_1^s r_1 + a_{p_1}) + T(G)$ and $p_1^s r_1 + T(G)$ represent the same element in $G/T(G)$. This means that Case 2 cannot exist either. Hence, the necessity is thereby proved.

Conversely, we prove the sufficiency; i.e., groups of form (a) or (b) have property (α). Groups of the form (a) have property (α). In the case of groups having the form (b), the torsion subgroup $T(G)$ is contained in D, which is divisible. Therefore $T(G)$ is divisible. Hence we have no trouble with those subgroups of rank one contained in $T(G)$. Now, we need only to prove that a torsion-free subgroup L of rank one in G can be enlarged to a direct summand of rank one in G. Let us consider the factor group $G/T(G)$ which satisfies the conditions (a) and (b) in Lemma 2.2. Therefore, $G/T(G)$ has property (α). Since $(L + T(G))/T(G)$ is a torsion-free subgroup of rank one in $G/T(G)$, we can enlarge $(L + T(G))/T(G)$ to a torsion-free direct summand L of rank one in $G/T(G)$. Denote the complete inverse image of L with respect to the natural projection $q\colon G \to G/T(G)$ by S. It is clear that $L \subset S$ and $L \cap T(G) = 0$. By Zorn's Lemma, there exists a maximal subgroup M of S such that (i) $L \subset M$ and (ii) $M \cap T(G) = 0$. It follows from the divisibility of $T(G)$ that M is a direct summand of rank one in S. Moreover, since $(M + T(G))/T(G)$ is a direct summand of $G/T(G)$ and $M \cap T(G) = 0$, M is in turn a direct summand of rank one in G. By our construction, $L \subset M$. We thereby complete our proof.

III. Groups with property (β). As we did in § II, we first prove the following lemma.

LEMMA 3.1. *A torsion-free group G has property (β) if and only if it can be expressed as one of the following forms*

(a) *$J_0 + D$ where J_0 is uniformly separable group of null type* [4, pp. 207–211] *and D is a divisible group.*

(b) *A torsion-free group of rank one.*

Proof. Sufficiency: It is obvious that groups of the form (b) have property (β). Let us turn to the case (a). We first recall that J_0 has the following properties:

(i) every subgroup of finite rank is free,

(ii) every pure subgroup of finite rank is a direct summand [1]. Now, let H be a given completely decomposable subgroup of finite rank in G. We first consider the factor group $H/(D \cap H)$ where D is the maximal divisible subgroup of G; by (i), we can directly decompose H as

$$H_1 + (H \cap D) \tag{16}$$

where H_1 is a free group of finite rank and $(H \cap D)$ is a completely decomposable subgroup of finite rank. Applying Zorn's lemma and divisibility of D as before, G can be directly decomposed as

$$J_0' + D \tag{17}$$

such that

(i) D has the same meeting as above,

(ii) $J_0' \cong J_0$, and

(iii) $J_0' \supset H_1$. It follows that the properties (i) and (ii) of J_0 that H_1 is completely reducible in J_0'. To sum these results, H is completely reducible in G.

Necessity. Now, we prove that a torsion-free group G having property (β) must be either (a) or (b). First, it follows from Lemma 2.2 that G must be a uniformly separable group. If G is neither of the form (a) nor of the form (b), we have a direct decomposition of G as follows

$$R_1 + R_2 + G' \tag{18}$$

such that

(i) both R_1 and R_2 are of rank one,

(ii) $|R_2|$ has finite value at some prime number p_0, and

(iii) $|R_1| \geqq |R_2| >$ the null type. By (18.ii), there exist $r_i \in R_i$ $(i = 1, 2)$ such that r_2 is not divisible by p_0 in G. Consider the subgroup $H = \widetilde{R}_1 + R_2$ where $\widetilde{R}_1$ is the cyclic subgroup generated by $p_0r_1 + r_2$ of rank two in it. There exists a direct decomposition of G as follows

$$G_1 + G_2 + G'' \tag{19}$$

such that

(i) G_1 and G_2 are of rank one, and

(ii) $H = (H \cap G_1) + (H \cap G_2)$. From (18. iii) and the fact that R_1 is of null type, we conclude that the set $\{(G_1 \cap H), (G_2 \cap H)\}$ of subgroups in H coincides with $\{\widetilde{R}_1, R_2\}$ [1]. Say, $\widetilde{R}_1 = G_1 \cap H$ and $R_2 = G_2 \cap H$. Hence, G_1 and G_2 are the pure subgroups spanned by $p_0r_1 + r_2$ and r_2

respectively. Since $(G_1 + G_2)$ is a direct summand of G and neither r_2 nor $p_0r_1 + r_2$ is divisible by p_0 it follows from (19) that $p_0r_1 = (p_0r_1 + r_2) - r_2$ is not divisible by p_0 in G either. We have a contradiction. Thus, G must satisfy either (a) or (b).

LEMMA 3.2. *Suppose that the torsion subgroup $T(G)$ of G is divisible. Then, G has property (β) if and only if $G/T(G)$ does.*

Proof. Necessity: Suppose that G has property (β). Let H be a completely decomposable subgroup of finite rank in $G/T(G)$. Denote the complete inverse image of H with respect to the natural projection $q\colon G \to G/T(G)$ by S. Since S is divisible, we have

$$S = \tilde{H} + T(G) . \tag{20}$$

Clearly, $\tilde{H}$ is isomorphic to H and hence is a completely decomposable subgroup of finite rank in G. By our assumption that G has property (β), $\tilde{H}$ is completely reducible in G. Carrying back to the factor group $G/T(G)$, therefore, H is completely reducible in $G/T(G)$.

Sufficiency: There are two cases according to whether $G/T(G)$ is of the form (b) or (a) in Lemma 3.1.

Case 1. $G/T(G)$ is a torsion-free group of rank one. Then, a completely decomposable subgroup H of finite rank in G can be expressed as

$$R_1 + (T(G) \cap H) \tag{21}$$

where R_1 is the zero group or a torsion-free group of rank one. Since $T(G) \cap H = T(H)$ is a completely decomposable subgroup of $T(G)$ and since $T(G)$ is divisible, it follows from Lemma 1.2 that $(T(G) \cap H)$ is completely reducible in $T(G)$. On the other hand by applying Zorn's lemma and the divisibility of $T(G)$ as we did before, there exists a direct decomposition of G as follows

$$M + T(G) \tag{22}$$

such that $M \supset R_1$. To sum up the above results we have proven the complete reducibility of H in G.

Case 2. Suppose that G has the form (a) in Lemma 3.1. Let H be a completely decomposable subgroup of finite rank in G. We first consider the factor group $H/(H \cap D)$ where D is the maximal divisible subgroup of G. Since $H/(H \cap D)$ is isomorphic to a free subgroup $(H + D)/D$ of finite rank in G/D, we can directly decompose H as

$$H_1 + (H \cap D) \tag{23}$$

where $H_1 = H/(H \cap D)$ is a free group of finite rank. It is clear that $(H \cap D)$ is also completely decomposable, since it is a direct summand of a completely decomposable group of finite rank. Hence, we have a refinement of (23) as follows

$$H = H_1 + D_1 + (H \cap T(G)) \tag{24}$$

where $D_1 + (H \cap T(G)) = H \cap D$. Indeed, all the three terms on the right side of (24) are completely decomposable. Successively applying Zorn's lemma and divisibility of D and $T(G)$, we can directly decompose G as

$$\tilde{J}_0 + \tilde{D} + T(G) \tag{25}$$

such that

(i) $\tilde{D} + T(G) = D$ and $\tilde{D} \supset D_1$, and

(ii) $\tilde{J}_0$ is a uniformly separable of null type and $J_0 \supset H_1$. It follows from Lemma 1.2 and Lemma 3.1 that $(H \cap T(G))$ and $(H_1 + D_1)$ are completely reducible in $T(G)$ and $(\tilde{J}_0 + \tilde{D})$ respectively. To sum up these results, we have proven that H is also completely reducible in G in this case.

As a consequence of Theorem A, Lemmas 3.1 and 3.2, we have the following theorem

THEOREM B. *A group G has property (β) if and only if it can be expressed in one of the following forms*:

(a) $J_0 + D$ *where J_0 is a uniformly separable group of null type and D is a divisible group.*

(b) $R + C$ *where R is a torsion-free group of rank one and C is a torsion divisible group.*

(c) *G is a torsion group with property (β) (see Lemma* 1.2).

REMARK. If we put (β) in a stronger form (β') every completely decomposable subgroup of G is completely reducible in G, then (a) becomes

(a′) $F + D$ where F is a free group of finite rank and D is divisible.

IV. Group with property (γ). We first find the torsion-free groups having property (γ).

LEMMA 4.1. *A torsion-free group has property (γ) if and only if it can be expressed in one of the following forms*

(a) $J_0 + A$ *where J_0 is a uniformly separable group of null type and A is either the zero group of the additive group of all rational* *numbers.*

(b) *A torsion-free group of rank one.*

Proof. Necessity: It follows from Theorem B that a torsion-free group G having property (γ) is either of rank one or of the form

$$J_0 + D \tag{26}$$

where J_0 is a uniformly separable group of null type and D is a torsion-free divisible group. If it is the latter case, we claim that the rank of D is at most one. For any torsion-free divisible group of rank greater than one contains an indecomposable subgroup of rank two, and such a subgroup is not completely reducible in G.

Sufficiency. We have no trouble with groups of the form (b). Suppose that G is a group of the form (a). Let H be a subgroup of finite rank in G. Consider the factor group $H/H \cap A$. Since $H/H \cap A$ is isomorphic to a subgroup of finite rank in J_0, we can directly decompose H as

$$F + H \cap A \tag{27}$$

where F is a free group of finite rank and $H \cap A$ is at most of rank one. This means that H is completely decomposable. Finally, it follows from Theorem B that H is completely reducible in G. The proof is thereby completed.

THEOREM C. *A group has property* (γ) *if and only if it can be expressed in one of the following forms*:

(a) $J_0 + A + \sum_{j=1}^{n} C_{p_j}$ *where* J_0 *is a reduced uniformly separable group of null type. A is either the zero group or the additive group of all rational numbers and* C_{p_j} *is a* p_j*-primary divisible group.*

(b) $R + C$ *where R is a torsion-free group of rank one, and C is a torsion divisible group. Moreover, there are only a finite number of primes p such that the p-primary component of C and the p-value of the type* $|R|$ *are simultaneously not zero.*

(c) $J_0 + C$ *where* J_0 *is a reduced uniformly separable group of null type and C is a torsion divisible group.*

(d) G *is a torsion group with property* (γ) (See Lemma 1.2).

Proof. Necessity: It follows from Theorem B and Lemma 1.2 that G is either of the form (d) or there exists a subgroup in the following form

$$R + C \tag{28}$$

where R is a torsion-free group of rank one and C is a torsion divisible group. We claim that in the latter case there are only a finite number of primes such that both the p-primary component of C and the value of $|R|$ at p are simultaneously not zero. If this is not the case, there exists an element e in R and a subgroup E of C satisfying the following conditions:

(i) E is of the form

$$\Sigma_i V(p_i) \qquad (i = 1, 2, \cdots) \tag{29}$$

where $V(p_i)$ is a cyclic group of order p_i.

(ii) the equation

$$p_i x = e \qquad (i = 1, 2, \cdots) \tag{30}$$

is always solvable in G.

Let $\tilde{e}(p_i)$ be the solution of (30) and $\tilde{f}(p_i)$ be a non-zero element of $V(p_i)$ for $i = 1, 2, \cdots$. Since C is divisible, there exists an element $f(p_i)$ for each i such that $p_i f(p_i) = \tilde{f}(p_i)$. Set $e(p_i) = \tilde{e}(p_i) + f(p_i)$. Let H be a subgroup of rank two in G generated E, $e(p_i)$'s, and e. H is not completely reducible in G. In fact H is not completely decomposable [2]. Therefore, a group having this property must be in one of the stated forms.

Sufficiency. In fact, we have already known that a group G in one of those forms has property (β). Therefore, to prove the theorem it sufficies to show that every subgroup H of finite rank is always completely decomposable. There is no trouble with the groups of the form (d). In other cases, it is known [2] that H can be directly decomposed as

$$H_1 + T(H) = H_1 + H \cap T(H) \tag{31}$$

where $T(H)$ is the torsion subgroup of H and H_1 is isomorphic to a subgroup of $J_0 + A$, R, or J depending on which of cases (a), (b) or (c) holds. In any case, it follows from Lemma 4.1 that H_1 is completely decomposable. On the other hand, it is obvious that $T(H)$ is completely decomposable. The proof is thereby completed.

REMARK. If we put (γ) in a stronger form (γ') as follows: A group G is said to have property (γ') if every subgroup of G is completely reducible in G; then Theorem C becomes Theorem C′: A group G has property (γ') if and only if it can be expressed in one of the following forms

(a′) $F + A + \sum_{j=1}^{n} C_{p_i}$ where F is a free group of finite rank and A, C_{p_i} have the same meaning as in (a).

(b′) $R + C_f$ where R has the same meaning as in (b) and C_f is a torsion divisible group of finite rank. Moreover, the p-primary components of C_f and R subjected to the same relation as in (b).

(c') $F + C_f$ where F and C_f have the same meaning as above.

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