ON PLANE CURVES WITH CURVATURE

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As a temporary abbreviation we say that a (planar) curve is an R-*curve* provided that its curvature is continuous and does not exceed $1/R$. A well-known theorem of Schwartz [1 page 63, 2, 6, 7] states that if two points on a circle C of radius R are joined by any R-curve, X, then the arc length of X does not exceed the length of the smaller arc of C unless indeed the length of X is at least as great as that of the larger arc of C determined by the two points.

In this paper we call attention to an area of largely unexplored mathematics, of which the above theorem of Schwartz can be taken as a takeoff point. Here, we only begin the exploration, raise some questions that we hope will prove stimulating, and invite others to discover the proofs of the definitive theorems, proofs that have eluded us. Roughly, the principal question is: given two points (in the Euclidean plane) what kind of R-curve can connect them? One approach towards making this question precise is as follows: Focus attention on two R-curves that connect the two given points and ask under what circumstances is it possible to gradually deform the first curve into the second, where at each stage of the deformation the curve is an R-curve connecting the two given points. Actually, our investigation is primarily concerned with a related problem in which the two given curves, and every intermediary curve, have the same tangent direction at the first of the two points, as well as at the second. In this way we become interested in the arc components of a space of curves. This leads to similarities and connections with the work of Graustein-Whitney and Smale [8, 9]. However the curvature restriction leads to new problems.

The idea for a curvature constraint arises naturally from considerations of a particle that moves at constant speed and subject to a maximum possible force. If that particle leaves a certain point heading in a certain direction and desires to arrive at another point from a certain direction what are the paths available to the particle? If it tries to take a certain available path but through errors does not quite traverse this path what is the nature of the possible neighboring paths? (Homotopy classes.) These questions represent the background for this paper.

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1. Introduction and summary. Let X be a continuously differentiable planar curve defined for $0 \leqq t \leqq 1$ with constant speed, i.e. $X'(t) = Le^{i\theta(t)}$ where $L = L(X) > 0$ is the length of X, and θ is continuous. *Let $\mathscr{C}^0$ be the set of such X that begin at some fixed point in some fixed direction, say at the origin heading east.* That is, $X(0) = 0$ and $X'(0) = (L, 0)$. With each such X is associated its terminal position $X(1)$ and winding angle $\theta(1)$. Of course $\theta(0) = 0$. *Let p: $\mathscr{C}^0 \to E$ be defined by $p(X) = (X(1), \theta(1))$, where E is the cartesian product of the plane with the real line.* The fibre F_e^0 over a point $e \in E$ is the set of all curves $X \in \mathscr{C}^0$ that terminate at a fixed point and that have a fixed winding angle. Each fibre inherits a topology from $\mathscr{C}^0$, where convergence in $\mathscr{C}^0$ means uniform convergence of the curves and their derivatives. Say that a curve $X \in \mathscr{C}^0$ is *closed* provided that $X(1) = 0$. A fibre F_e^0 is a *fibre of closed curves* provided that e is of the form $(0, \alpha)$ for some real number α. A theorem of Graustein and Whitney [9], or rather a slight modification of its proof, implies that a fibre of closed curves is arcwise connected. It is a consequence of the work of Smale [8] that every fibre of $\mathscr{C}^0$ is arcwise connected for arbitrary e. In this connection, and as a side remark only, we mention that if $e = (u, \alpha)$ and $e' = (u', \alpha')$ with u and u' non zero, then the corresponding fibres are homeomorphic. To see this, notice that there is undoubtedly a diffeomorphism h of the plane onto itself that maps $u \to u'$ and a curve of winding angle α terminating at u into one of winding angle α', and that also preserves the origin and tangent directions at the origin. It is not hard to see that such an h induces a homeomorphism of the fibre over e with the fibre over e' (a diffeomorphism is a continuously differentiable homeomorphism whose inverse is also differentiable).

Suppose now that there is a bound on curvature. That is, *let R be a fixed positive constant, and $\mathscr{C}'$ the set of $X \in \mathscr{C}^0$ that have curvature everywhere, and nowhere greater than $1/R$.*

The facts about the connectivity of the fibres F_e' of $\mathscr{C}'$ are not as simple as for the fibres F_e^0 of $\mathscr{C}'$. On the one hand, it is possible to modify the proof of Graustein and Whitney to show that if F_e' is a fibre of closed curves in $\mathscr{C}'$, that is, if e is of the form $(0, \alpha)$ for some α, then F_e' is arcwise connected just as F_e^0 is. However, in contrast to Smale's results implying the connectivity of all F_e^0, there exist e such that F_e' is not arcwise connected. Let B' be the set of e such that F_e' is not arcwise connected. We shall see that for some $e \in B'$, an arc component of F_e' contains but a single element, and that for other $e \in B'$, each of two arc components of F_e' contains two, and hence infinitely many, elements.

2. Average curvature. It is of value to introduce some curves that do not necessarily possess a curvature everywhere. This convenience arises from the fact that the collection of curves with continuous curvature bounded by $1/R$ is not closed under uniform limits. In particular a smooth curve that consists of two pieces, the first an arc of a circle, and the second, a line segment, does not have a curvature at the point of tangency. The *average curvature* of a curve $X \in \mathscr{C}^0$ in the interval $[t_1, t_2]$ is, of course,

$$\frac{\theta(t_2) - \theta(t_1)}{L\,|\,t_2 - t_1\,|} = \frac{\theta(s_2) - \theta(s_1)}{s_2 - s_1} \tag{2.1}$$

where L is the length of X, and s is the arc length parameter for X.

Let $\mathscr{C} = \mathscr{C}(R)$ *be the set of* $X \in \mathscr{C}^0$ *such that for all* t_1, t_2, *the absolute value of* (2.1) *does not exceed* $1/R$. Then $\mathscr{C}' \subset \mathscr{C} \subset \mathscr{C}^0$. The *fibre* F_e is the set of $X \in \mathscr{C}$ such that $p(X) = e$.

It was shown in [3] that for each terminal position and direction there is an element of $\mathscr{C}$ of minimal length. *Such a path is called an R-geodesic.* Of more significance, it was also established that an R-geodesic is a smooth curve that consists of at most three pieces, each of which is either an arc of a circle of radius R, or a straight line segment, (but not all such curves are of minimal length). We observe that for not every terminal position and direction is the minimal path unique. Interest in obtaining insight into this matter of non-uniqueness led us to explore homotopies between these curves, and thereby to this note.

3. If curves are close so are their derivatives.

The following is a simple consequence of Theorem 1 in [4].

THEOREM 3.1. *Let X_n be a sequence of continuously differentiable curves defined for $0 \leqq t \leqq 1$. Suppose that for some positive constant $k > 0$,*

$$\|\,X_n'(t_2) - X_n'(t_1)\,\| \leqq k\,|\,t_2 - t_1\,| \tag{3.1}$$

for all t_1 and t_2. Then the convergence of the sequence $X_n(t)$ for all t (or just for a dense set of t) implies the uniform convergence of both the sequences X_n, and of the derivatives X_n'.

COROLLARY 3.1. *Suppose that the speeds $\|\,X_n'(t)\,\|$ satisfy a uniform Lipschitz condition of order* 1, *and the average curvatures are uniformly bounded away from infinity. Then the convergence of the sequence X_n implies the uniform convergence of the sequence X_n'.*

In the event the speeds are independent of t they certainly satisfy the first part of the hypothesis of Corollary 3.1, and this is the case if t is arc length or a multiple of arc length for all the curves, where the multiple may vary with n, as is the case for $X_n \in \mathscr{C}$.

Of course Theorem 3.1 and its corollary are not restricted to planar curves.

COROLLARY 3.2. *A homotopy X_u through a family of regular curves whose first derivative satisfies a uniform Lipschitz condition of order* 1, (*or whose second derivative is uniformly bounded*) *is a regular homotopy*. (See [8] for a definition of regular.)

4. Each fibre of closed curves is connected. This section is concerned with indicating the modifications in the proof of the Graustein Whitney Theorem [9] that are necessary in order that it be applicable to curves with a curvature constraint.

LEMMA 1. *Let $Y_0 \in \mathscr{C}^0$, and $e = p(Y_0)$. Suppose that the average curvature of Y_0 does not exceed $1/R$. Then there exists a continuous mapping $u \to Y_u$ of* $[0, 1]$ *into F_e^0 and an $\varepsilon > 0$ such that* (i) *the average curvature of the Y_u is bounded away from infinity, and* (ii), *letting $Y_1'(t) = \| Y_1'(t) \| e^{i\varphi_1(t)}$ with φ_1 continuous, one has $\varphi_1(t) > 0$ for $0 < t < \varepsilon$.*

Proof. Choose an initial part Z_0 of Y_0 on which the angle mapping varies but little, say by less than $\pi/4$. Then choose any curve Z_1 with the following 6 properties:

(i) Z_1 has constant speed

(ii) Z_1 has bounded average curvature

(iii) Z_1 begins at the origin and is initially heading east

(iv) Z_1 terminates at the terminal point of Z_0 and terminates in the same direction as does Z_0

(v) the winding angle of Z_1 varies by less than $\pi/2$

(vi) If one lets $Z_1'(t) = \| Z_1' \| e^{i\theta(t)}$ with θ continuous, then for some $\delta > 0$ and all $t, 0 < t < \delta, \theta(t) > 0$.

Now let $Z_u = uZ_1 + (1 - u)Z_0$. It is easy to see that Z_u' is never zero, and that the average curvature of the Z_u is uniformly bounded. Next let W be the curve consisting of all of Z_0 except the initial part, Y_0. Finally, let Y_u be Z_u followed by W. It is easy to see that Y_u is the homotopy sought. This completes the proof.

LEMMA 2 *For any fibre F_e of $\mathscr{C}$, and any X_0 and X_1 that are elements of F_e, there exists $X_u, 0 \leqq u \leqq 1$, such that*

(i) $X_u \in F_e^0$,

(ii) *the map $u \to X_u$ is a continuous mapping of* $[0, 1]$ *into $\mathscr{C}^0$ and*

(iii) *the average curvatures of the X_u are uniformly bounded.*

Proof. Since we will use this lemma only when F_e is a fibre of closed curves we indicate the proof only for this case. For this case use the homotopy defined in the proof of the Whitney Graustein Theorem [9], utilizing Lemma 1 above and obtain an arc Z_u whose average curvatures are uniformly bounded, and such that $Z_i = X_i$, $i = 0$ and 1. Here Z_u and Z'_u vary continuously with u. Let $Z'_u(t) = \| Z'_u(t) \| e^{i\varphi(u,t)}$ with φ continuous in u and t, and $\varphi(0, 0) = 0$. Let ρ_u be the rotation through the negative of the angle $\varphi(u, 0)$. Finally, let $X_u(t) = \rho_u(Z_u(t) - Z_u(0))$. It is simple to verify that X_u has the desired properties.

It is essential to the truth of the following lemma that the fibre be a fibre of closed curves.

LEMMA 3. *Let F_e be a fibre of closed curves, and suppose that X_0 and X_1 are elements of F_e. Suppose that X_u exists, $0 \leqq u \leqq 1$, and satisfies* (i), (ii) *and* (iii) *of Lemma 2. Then X_0 and X_1 are in the same arc component of F_e.*

Proof. Since the average curvatures of X_u are uniformly bounded there is a positive constant k sufficiently large so that $Y_u = kX_u$ has its average curvatures uniformly bounded by $1/R$. It is easy to verify that Y_u is an arc in F_e. Therefore Y_0 and Y_1 are in the same arc component of F_e. Moreover there is an obvious arc in F_e connecting Y_0 with X_0, namely $Z_v = vX_0$, $1 \leqq v \leqq k$; and one connecting Y_1 with X_1. This completes the proof of Lemma 3.

Lemmas 2 and 3 immediately yield:

THEOREM 4.1. *Every fibre of closed curves in $\mathscr{C}$ is arcwise connected.*

5. Winding angle equality does not always imply the existence of a homotopy. Theorem 4.1 implies that some fibres are arcwise connected. The point of the next theorem is to show that not all fibres are connected. In fact some fibres have isolated points. Namely an arc, X, of length less than $\pi R/2$, of a circle of radius R, cannot be deformed at all via a homotopy that keeps the initial and terminal positions and directions fixed, and such that, at each stage of the homotopy, the curvature or average curvature, does not exceed $1/R$.

THEOREM 5.1. *Let $X \in \mathscr{C}$ be an arc of a circle of radius R, and suppose that $l = l(X)$, the length of X, is less than $\pi R/2$. Let $p(X) = e$. Then X is an isolated point in F_e.*

LEMMA 5.1. *The mapping l: $\mathscr{C} \to$ reals defined by $l(Y) =$ length of Y is continuous.*

Proof. Obvious.

LEMMA 5.2. *The only curve in F_e whose length is less than or equal to the length of X is X itself.*

Proof. Immediate from Proposition 6, page 504 in [3].

LEMMA 5.3. *No curve in F_e has a length strictly between $l(X)$ and πR.*

Proof. Suppose that $Y \in F_e$ and that Y has continuous curvature. Then Schwartz's theorem implies that either $l(Y) \leqq l(X)$ or else $l(Y) \geqq \pi R$. To establish the lemma for arbitrary $Y \in F_e$, it obviously suffices to generalize Schwartz's theorem so as to apply to curves whose average curvature does not exceed $1/R$. It is easy to verify that Schwartz's theorem thus generalized is indeed valid. The essential point is to verify that the corresponding extension of Schur's theorem [1, page 61, 2; 6; 7] on which Schwartz's theorem is based, is also valid.

We merely observe that the usual technique for proving Schur's theorem can be used to establish the following generalization in which the existence of curvature, and, a fortiori, its continuity, is not assumed.

SUBLEMMA 5.4. *Schur's theorem for plane curves that are not necessarily twice differentiable. Let D and D^* be continuously differentiable planar arcs with the same arc length, L, each parametrized by arc length s. Suppose that D, together with the chord joining its end points, forms a convex curve. Let $D'(s) = e^{i\theta(s)}$, $D^{*\prime}(s) = e^{i\theta^*(s)}$ with θ and θ^* continuous functions defined on $I = [0, L]$, and suppose that $|\theta(s_2) - \theta(s_1)| \geqq |\theta^*(s_2) - \theta^*(s_1)|$ for all s_1 and s_2. Let d and d^* denote the lengths of the chords joining the end points of D and D^* respectively. Then $d \leqq d^*$, and equality holds only if D and D^* are congruent.*

Proof of Theorem 5.1. Let $0 < \varepsilon < \pi R - l(X)$. By Lemma 5.1, the set of $Y \in F_e$ such that $l(X) - \varepsilon < l(Y) < l(X) + \varepsilon$ is an open subset of F_e. But Lemmas 5.2 and 5.3 imply that X is the unique element of this set. This completes the proof.

COROLLARY 5.1. *Let e be as in Theorem* 5.1. *Then F_e is not arcwise connected.*

The question arises whether the only F_e that are not arcwise connected are as above. The next theorem implies, among other things, that such is not the case. The three curves X_1, X_2, and X_3 of Figure 1 are in the same fibre F_e. Let $0 < l \leqq 4R$, and let X_2 be the unique

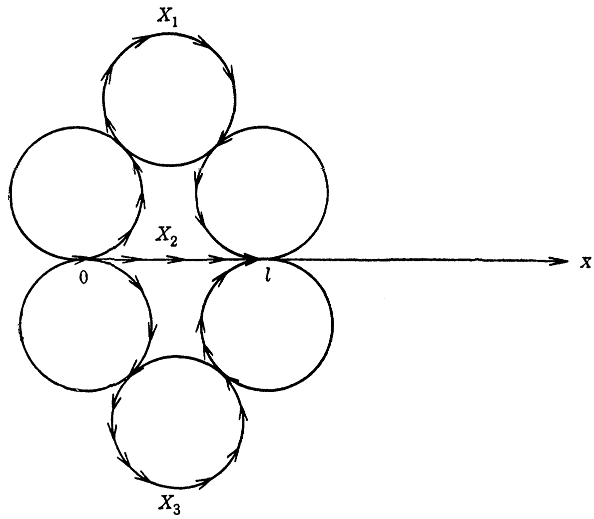

Figure 1.

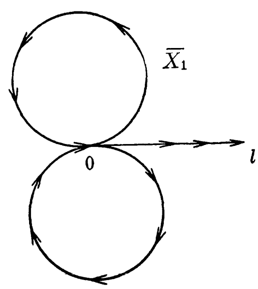

Figure 2.

straight line of length l, such that $X_2 \in \mathscr{C}$. Let $p(X_2) = e$. $X_1 \in F_e$, is determined by the condition that it consists of three pieces, each an arc of a circle of radius R, the first counterclockwise oriented. X_3 is the mirror image of X_1 where the x-axis acts as a mirror. Of course all X_i are in the same fibre F_e. It is at first not obvious whether any two of these three curves are in the same arc component of F_e, and it is at first surprising that X_1 and X_3 are in the same component, whereas X_1 and X_2 are in the same component if and only if $l = 4R$. To see that X_1 and X_3 are in the same component observe first that X_1 can be deformed into $\bar{X}_1$ where $\bar{X}_1$, depicted in Figure 2, consists in traversing first the upper circle completely, then the lower circle and finally X_2, the straight line segment of length l. And similary, X_3 can be deformed into $\bar{X}_3$ where $\bar{X}_3$ is the same as $\bar{X}_1$ except that the lower circle is traversed first.

Let $\bar{\bar{X}}_1$ be the upper circle of Figure 2 followed by the lower one, and let $\bar{\bar{X}}_3$ be the lower one followed by the upper. Of course $\bar{X}_1$ and $\bar{X}_3$ consist of closed curves $\bar{\bar{X}}_1$ and $\bar{\bar{X}}_3$ respectively followed by X_2. That $\bar{\bar{X}}_1$ and $\bar{\bar{X}}_3$ can be deformed into one another keeping their end points and directions fixed can be proven in an elementary fashion, and is also a consequence of Theorem 4.1. This easily implies that X_1 can be deformed into X_3, and hence that they are in the same arc component. Therefore we have:

THEOREM 5.2. *X_1 and X_3 are in the same component of F_e.* However X_1 and X_2 are not necessarily in the same component i.e.:

THEOREM 5.3. *X_1 and X_2 are in the same component of F_e if and only if $l = 4R$.*

Proof. If $l = 4R$ it is trivial to see that X_1 and X_2 are in the same component. Assume now that X_1 and X_2 are in the same arc component, and let $X_u, 1 \leqq u \leqq 2$ be an arc in F_e connecting X_1 with X_2. Then $X_u'(t)$ is continuous in u and t. Let $Y_u(t)$ be the unit vector in the direction $X_u'(t)$. Then Y_2 has a single point as its range, whereas Y_1 has more than a half circle as its range. There is some $u = u_0$, $1 < u_0 < 2$ such that $Y = Y_u = Y_{u_0}$ has precisely a half circle for its range. Then each point in the interior of this half circle is covered twice by Y. This is so because $Y(0) = Y(1)$, and therefore Y is topologically like a mapping of a circle into a half circle. A simple connectivity argument shows that a mapping of a circle into a half circle (or the real line) covers the interior of the range at least twice. Let v be the midpoint of the range of Y, and let $X = X_{u_0}$. Then

$$\begin{aligned} \| X(1) - X(0) \| &\geqq (X(1) - X(0), v) \\ &= \left(\int_0^1 X'(t)dt, v\right) = \int_0^1 (X'(t), v)dt \\ &= \| Y \| \int_0^1 (Y(t), v)dt = \| Y \| \int_0^1 \cos(\theta(t))dt \ , \end{aligned}$$

where $\theta(t)$ is the angle between $Y(t)$ and v, $= \| Y \| \int_{-(\pi/2)}^{\pi/2} \cos(\alpha)du(\alpha)$ where u is the measure induced on $[-(\pi/2), \pi/2] = I$ by θ. That is, $u(B)$ is the Lebesgue measure of $\theta^{-1}(B)$ for all Borel sets B. Of course $\cos(\alpha) \geqq 0$ for $\alpha \in I$. We now need a lemma to guarantee that u is a large measure, namely, not less than $2R/\| Y \|$ times $\mathscr{L}$ where $\mathscr{L}$ is Lebesgue measure. (A measure u is said to be not less than a measure v, provided that for all measurable sets B, $u(B) \geqq v(B)$). Such a lemma will permit the inequalities to continue:

$$\geqq 2R \int_{-(\pi/2)}^{\pi/2} \cos(\alpha)d\alpha = 4R \ ,$$

and the theorem will be proven.

Since the average curvature of X does not exceed $1/R$, θ satisfies the Lipschitz condition $|\theta(\alpha_2) - \theta(\alpha_1)| \leqq (\|Y\|/R)|\alpha_2 - \alpha_1|$, and, as already observed, θ covers the interior of its range at least twice. Therefore to prove that $u \geqq (2R/\|Y\|)\mathscr{L}$, and thereby complete the proof of the theorem, it suffices to establish the following general lemma:

LEMMA 5.5. *Let θ be any real valued function defined on some closed interval and satisfying the Lipschitz condition*

$$|\theta(\alpha_2) - \theta(\alpha_1)| \leqq k|\alpha_2 - \alpha_1| . \tag{5.6}$$

Let u be the distribution of θ, that is $u(B) = \mathscr{L}(\theta^{-1}(B))$ is the Lebesgue measure of $\theta^{-1}(B)$ for all Borel subsets B of the range I of θ. Suppose that every point in the interior of I is covered at least j times by θ. Then, $u \geqq (j/k)\mathscr{L}$.

Proof. Let B be any open subinterval of the interior of I. Then

$$u(B) = \mathscr{L}(\theta^{-1}(B)) = \int_{\theta^{-1}(B)} 1 \geqq \int_{\theta^{-1}(B)} \frac{1}{k}|\theta'(\alpha)|d\alpha$$
$$= \frac{1}{k}\int_{\theta^{-1}(B)} |\theta'(\alpha)|\,d\alpha = \frac{1}{k}\sum_K \int_K |\theta'(\alpha)|\,d\alpha$$

where K runs through the components of $\theta^{-1}(B)$. The equalities continue: $=(1/k)\sum_K$ (total variation of θ_K), where θ_K is θ restricted to K. Now apply a theorem of Banach [5, p. 280] which states that the total variation of any continuous function f of bounded variation defined on an interval equals $\int \bar{n}(y)dy$ where $\bar{n}(y)$ is the number of x such that $f(x)=y$, and continue the sequence of inequalities, thus, $=(1/k)\sum_K \int n_K(y)dy$, where $n_K(y)$ is the number of α such that $\theta_k(\alpha) = y$;

$$= \frac{1}{k}\int \sum_K n_K(y)dy = \frac{1}{k}\int n(y)dy$$

where $n(y)$ is the number of α such that $\theta(\alpha) = y$. But $n(y) \geqq j$ for $y \in \theta^{-1}(B)$. Therefore the inequalities continue

$\geqq (1/k)\int_{\theta^{-1}(B)} jdy = (j/k)\mathscr{L}\theta^{-1}(B)$. This completes the proof of the lemma.

Proof of Theorem continued. The average curvature being less than $1/R$ means that θ satisfies (5.6) with $k = (\|Y\|/R)$, and, taking $j = 2$, the lemma gives $u \geqq (2R/\|Y\|)\mathscr{L}$. This completes the proof of the theorem.

6. Suggestions, Conjectures, and open problems. The principal open problem of this paper is to discover necessary and sufficient conditions that two elements of F_e be in the same arc component. Let B be the set of e such that F_e is not arcwise connected. It seems likely that B is a bounded open set. In particular, if $e = (u, \alpha)$, then, as e ranges over B, we guess that u ranges over a bounded subset of the plane and, what is less intuitive, that α ranges over a bounded set of angles. Moreover, there is a reasonable chance that if $e \in B$, then F_e consists of precisely two components, $F(1, e)$ and $F(2, e)$. We conjecture that every fibre is locally arcwise connected. This would imply that these components are open, and, therefore, closed. Moreover one of them, say $F(1, e)$ is probably compact, and possesses an (unique) element X_0 of minimal length $m_0(e)$, and an (unique) element X_1 of maximal length $m_1(e)$. Moreover the other component $F(2, e)$ would not be compact, but would contain elements that wander all over the plane. Nevertheless it would contain an element X_2 of minimal length $m_2(e)$ where $m_2(e)$ is meaningful even if e is not an element of B. Examples suggest that for $e \in B, m_1(e) < m_2(e)$. This phenomenon is undoubtedly related to Schwartz's theorem [**1**, 2, 6, 7] and suggests further developments for that theorem. In this connection if $X \in F(1, e)$, it seems likely that the concatenation of a closed curve of winding angle zero with X is an element of $F(2, e)$, where the closed curve is traversed first, and, may of course be chosen to be a clockwise circle of radius R, followed by a counterclockwise circle of the same radius. Next let m_0 and m_1 be the supremum of $m_0(e)$ and $m_1(e)$ respectively for $e \in B$ and let m_2 be the infimum of $m_2(e)$ for $e \in B$, and m_3 the infimum of $m_2(e)$ for $e \in {}'B$. Of course each m_i depends upon R. Though it would be of interest to determine the m_i, we have not explored sufficiently many examples to have a firm conjecture about the values of the m_i.

As suggested earlier, a closely related problem is to find necessary and sufficient conditions that two curves with the same end points be deformable into one another by an arc of curves, each of which has its (average) curvature everywhere bounded by $1/R$, with fixed end points throughout the homotopy, but not necessarily fixed directions. Such an investigation would undoubtedly also be interrelated with Schwartz's theorem. For instance let two points be a distance $d < 2R$ apart, and let C_1 and C_2 be the two circles of radius R that pass through these two points, and let X be in the same homotopy class as the straight line joining the two points. It is a simple consequence of Schwartz's theorem that X lies in the region of intersection of the discs determined by C_1 and C_2. A particularly simple but open question is to show that the larger circular arcs of C_1 and C_2 joining the two given points are homotopic. (Added in Proof : N. H. Kuiper has kindly communicated to me his interesting discovery of a construction that shows that these circular arcs are homotopic).

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