ON THE GRAPH STRUCTURE OF CONVEX POLYHEDRA IN n-SPACE

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1. Introduction. The contents of this paper arose from work done in developing an algorithm for finding all vertices of convex polyhedral sets defined by systems of linear inequalities [1]. The following natural questions were raised: if we consider the vertices of convex polyhedral sets as the points, and the edges as the lines of a graph, does there exist a path or a cycle which goes through all points exactly once (i.e., does there exist a Hamiltonian path or cycle)? The answer to both questions is negative: there exists, in general, no Hamiltonian path or cycle. A simple example of a convex polyhedral set in 3-space whose graph contains no Hamiltonian path (and hence no Hamiltonian cycle) has recently been devised by T. A. Brown [2]. The classic example of Tutte [7] shows only that no Hamiltonian cycle exists.

In this paper, however, we show that such graphs do have the general property of being n-tuply connected. According to Whitney's Theorem [8] this implies that there exist n disjoint paths between any pair of vertices. We give a new proof of this fact based on an application of the Max-Flow Min-Cut Theorem [3], [5]. Finally, we point out that all proofs are based on the theory of linear programming, and thus on theory which itself rests on the properties of convex polyhedral sets.

2. The result. *A graph* $G(\pi, \Delta)$ is defined to be a finite collection of points π together with a collection of lines Δ. The lines consist of pairs of distinct points and Δ is thus some given subset of the collection of all possible lines formed from points in π. A line (p_1, p_2) is said to be *incident* to each of the points p_1 and p_2. A point is said to have *degree* n if n lines are incident to it. A *path* is a collection of lines $(p_1, p_2), (p_2, p_3), \cdots, (p_k, p_{k+1})$ with $p_i \neq p_j$ $j = i + 1$, and $k \geqq 1$. Paths are said to be *disjoint* if they have no points except possibly first and last points in common. A *cycle* is a path with $k \geqq 2$ whose first and last points are the same. We say a graph G is *connected* if there exists a path between any two of its points. We define an *n-tuply connected* graph G to be a graph with at least $n + 1$ points and such that it is impossible to disconnect it by dropping out $n - 1$ or fewer points.

Consider the polyhedral convex set S in n-space described by the system of linear inequalities

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$$(1) \qquad \left.\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leqq b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leqq b_m \end{array}\right\} AX \leqq b$$

where we assume that the only solution to $AX \leqq 0$ is $X = 0$ (it follows that the columns of A are linearly independent) and that there exists a solution X^0 which satisfies $AX^0 < b$. This assures us that the set S is just the convex hull of its vertices, and that it lies within no $(n-1)$-dimensional hyperplane [6]. On the other hand, every set S with these properties can be defined by a suitable system (1).

As a preliminary remark we make the obvious statement: the vertices, considered as points, and the edges, considered as lines, of the convex polyhedral set S form a graph $G(S)$ all of whose points have degree at least n.

THEOREM. *The vertices, considered as points, and the edges, considered as lines, of the convex polyhedral set S form an n-tuply connected graph $G(S)$.*

Proof. S has at least $n+1$ vertices, for otherwise it would lie within an $(n-1)$-dimensional hyperplane. Take out any $n-1$ vertices, say $v_1, v_2, \cdots, v_{n-1}$. We must show this does not disconnect $G(S)$, i.e., that from any vertex v_p to any other vertex v_q there exists a path which passes through no v_i, $i = 1, \cdots, n-1$. In the sequel we will use a two-pointed arrow to indicate the existence of a path between two vertices, $v_p \leftrightarrow v_q$.

Pass a hyperplane through $v_1, v_2, \cdots v_{n-1}$ and v_0, where v_0 is some other vertex which is a neighbor of v_1. Call this hyperplane $y_0(x_1, \cdots, x_n) = 0$. We assume that there are at least two vertices v_p and v_q of S which are not any of the vertices $v_0, v_1, \cdots, v_{n-1}$, otherwise the proof is trivial. We have a number of possibilities.

(a) $y_0 > 0$ ($y_0 < 0$) at both v_p and v_q.

(b) $y_0 > 0$ at v_p, $y_0 < 0$ at v_q.

(c) $y_0 = 0$ at v_p and v_q.

(d) $y_0 = 0$ at v_p and $y_0 > 0$ at v_q.

(a) If $y_0 > 0$ (or $y_0 < 0$) at both v_p and v_q then there exists a path v_p to v_q which goes through no v_i, $i = 1, \cdots, n-1$. Namely, if the function y_0 evaluated at v_p, $y_0(v_p)$, is not a maximum (minimum) on S then there is a neighboring vertex v_p^1 with $y_0(v_p^1) > y_0(v_p)(y_0(v_p^1) < y_0(v_p))$. Repetition of this argument defines a path $v_p \leftrightarrow v_p^r$ on $G(S)$ with $y_0(v_p^r)$ a maximum (minimum) on S. The same argument applied to v_q defines a path $v_q \leftrightarrow v_q^s$ on $G(S)$ with $y_0(v_q^s)$ a maximum (minimum) on S. Thus $y_0(v_p^r) = y_0(v_q^s)$. Either v_p^r and v_q^s are identical, and we have a path $v_p \leftrightarrow v_p^r = v_p^s \leftrightarrow v_q$ (where $v_p = v_p^r$ if $y_0(v_p)$ is optimal and $v_q = v_q^r$ if $y_0(v_q)$

is optimal) or not. If not, the intersection of S and the hyperplane $y_0 = y_0(v_p^r)$ is a convex polyhedron (a "face" of S) whose graph is clearly connected, and thus we have a path $v_p \leftrightarrow v_p^r \leftrightarrow v_q^s \leftrightarrow v_q$. In either case, there exists a path $v_p \leftrightarrow v_q$ all of whose points v satisfy $y_0(v) > 0$ ($y_0(v) < 0$).

(b) $y_0 > 0$ at v_p, and $y_0 < 0$ at v_q. Then v_0 has neighbors v_0^+ at which $y_0 > 0$, and v_0^- at which $y_0 < 0$. For suppose not, i.e., suppose that at all neighbors of v_0, $y_0 \geqq 0$ ($y_0 \leqq 0$). Then y_0 must attain its minimum (maximum) at v_0 and hence there can be no vertices v_i of S for which $y_0 < 0$ ($y_0 > 0$). This is a contradiction; so v_0 has neighbors v_0^+ and v_0^-.

By the argument given in (a) there exist paths $v_p \leftrightarrow v_0^+$ and $v_q \leftrightarrow v_0^-$, and hence a path

$$v_p \longleftrightarrow v_0^+ \longleftrightarrow v_0 \longleftrightarrow v_0^- \longleftrightarrow v_q \ ,$$

(v_p and v_0^+ or v_q and v_0^- may be identical).

(c) $y_0 = 0$ at v_p and v_q. By (b) and the fact that S lies within no $(n-1)$-dimensional hyperplane, either v_p has a neighbor v_p^+ and v_q has a neighbor v_q^+, or v_p has a neighbor v_p^- and v_q has a neighbor v_q^-. Thus, either we have a path $v_p \leftrightarrow v_p^+ \leftrightarrow v_q^+ \leftrightarrow v_q$ or a path $v_p \leftrightarrow v_p^- \leftrightarrow v_q^- \leftrightarrow v_q$, ($v_p^+$ and v_q^+ or v_p^- and v_q^- may be identical).

(d) $y_0 = 0$ at v_p and $y_0 > 0$ at v_q. By (b) we have a path $v_p \leftrightarrow v_p^+ \leftrightarrow v_q$.

This completes the proof.

Let G be a connected graph. If every point and line of G has a nonnegative number associated with it, G is a *network*. We distinguish two points of G, p_s and p_k, the source and the sink, respectively. A *path flow* from p_s to p_k in the network G is a couple (C, t) composed of a path C and a nonnegative number t representing the flow from p_s to p_k along C. A *flow* in the network G is a collection of path flows such that the sum of the numbers of all path flows through any one point or line of G is not greater than the capacity of that point or line. The value of the flow is the sum of the numbers of the collection of path flows which compose it. A *disconnecting* set is a collection of points and lines which disconnect p_s and p_k. The value of a disconnecting set is the sum of the capacities of the points and lines which make up that set.

THE MAX-FLOW MIN-CUT THEOREM [3], [5]. *Given a network G with source p_s and sink p_k, the maximum of the values of all flows from p_s to p_k, is equal to the minimum of the values of all disconnecting sets.*

We remark that the theorem can be proved by using the methods of linear programming [3]. The problem of finding a maximal flow is formulated as a linear programming problem, and the theorem deduced from the basic existence and duality theorems of programming theory. Moreover, it can be shown that if the capacities of points and lines are integers then there exists a maximum flow with all its path flows also in integers.

WHITNEY'S THEOREM. *A graph G is n-tuply connected if and only if there exists n disjoint paths between any pair of points p_s and p_k.*

Proof. That the condition is sufficient is obvious. To prove necessity we use the Max-Flow Min-Cut Theorem. Assign a capacity of 1 to each point of G, except p_s and p_k, which we consider as source and sink, respectively; and a capacity of $n + 1$ to each line of G, except for the line joining p_s and p_k, if such a line exists, which is assigned a capacity of 1. Then G is a network. Assume that the max-flow $< n$. Then the min-cut $< n$. But this contradicts the n-tuple connectedness of G and thus the max-flow $\geqq n$. Since no two unit path flows can go through one point, due to the capacity restrictions, there must be at least n disjoint paths from p_s to p_k.

COROLLARY. *There exist at least n disjoint paths between any pair of vertices of the polyhedral convex set S.*

In conclusion, it is perhaps worth while to point out that Dirac [4] proves that every connected graph in which the degree of every point is at least $n(n > 1)$ and which contains not more than $2n$ points has a cycle which goes through all points exactly once.

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