THE ANALYTIC-FUNCTIONAL CALCULUS IN COMMUTATIVE TOPOLOGICAL ALGEBRAS

RICHARD ARENS

1. Introduction. The idea of an analytic-functional calculus involving holomorphic functions f of several variables seems to have originated with Shilov [6]. Shilov uses Weil's integral formula [8] to construct, for each f holomorphic on the joint spectrum of elements $a_1, \cdots, a_n$ of a commutative Banach algebra A, an element b of that algebra, deserving the name $f(a_1, \cdots, a_n)$ because of the function b yields on the space of maximal ideals. Shilov's requirement that $a_1, \cdots, a_n$ generate the algebra was removed in [1]. Waelbrock [8], perhaps idependently of [6], treated the general case and indeed that of more general algebras. Waelbrock uses the ordinary form of Cauchy's integral, but also deeper ideal-theoretic results of K. Oka and H. Cartan. He shows moreover that one can arrange the mapping $f \rightarrow f(a_1, \cdots, a_n)$ so as to be an algebra-homomorphism, which is not obvious for the method of Shilov-Arens-Calderón [6, 1]. One purpose of the present paper is to show that this results from that method also. Another is to give a careful exposition of the Weil integral, or rather a weaker but more effective form involving integration on affine rather that analytic polyhedra. Although we have elsewhere sketched a proof of such a result, we dealt only with $n = 2$, as Weil did, and there was some question about the combinatorial procedure in the general case.

We desired to establish also a covariance property of the functional calculus (see 4.2 below) which enables us (see § 5) to extend the functional calculus to certain inverse limits of Banach algebras.

However, the most interesting discovery is that one can just as well deal with holomorphic A-valued functions f, rather than merely complex-valued functions. (For a trivial example, if $f(\lambda) = \lambda a$ on the spectrum of b, then $f(b) = ab$.) The attractive thing is that by extending the technique in this way, the distinction between the case in which $a_1, \cdots, a_n$ generate A, and that in which they do not, simply does not arise, nor does the matter of polynomial-convexity which was the great discovery of, but at the same time the indispensable tool for, Shilov.

The actual integral representation for functions holomorphic in the usual sense, on a suitably convex, compact subset of C^n is then *derived* from the theorem (4.1, 4.4 below) concerning the case of A-valued functions.

2. Holomorphic differential forms with values in a topological

Received June 7, 1960.

algebra.

By a *topological algebra* A we shall mean a linear algebra over the complex numbers $\boldsymbol{C}$ which is a locally convex topological linear space with the property that each compact set lies in some compact convex set, and in which the multiplication

$$A \times A \to A\,, \qquad (x, y) \to xy$$

is continuous. *Banach algebra* are the outstanding examples.

The condition involving the compactness is included in the definition so that the existence of the Riemann integral of a continuous function will be assured.

Let A be a commutative topological algebra with unit (written 1). The case in which $A = \boldsymbol{C}$ is a special, but not trivial case from the point of view of this section.

Let V be an open subset of $\boldsymbol{C}^n$, and f a continuous A-valued function defined on V. We shall say f is *holomorphic* on V, in symbols $f \in \mathrm{Hol}\,(V, A)$, if $\xi \circ f$ is holomorphic in the usual sense for every linear continuous functional ξ of A [4, 92].

Now let $w_1, \cdots, w_n$ be n elements of A. We shall say that an open set V of $\boldsymbol{C}^n$ is an *elementary resolvent set* for $w_1, \cdots, w_n$ if there exist functions $q_1, \cdots, q_n \in \mathrm{Hol}\,(V, A)$,

2.1 $$\Sigma q_i(\lambda)(\lambda_i - w_i) = 1 \qquad \lambda \in V\,.$$

The union of all elementary resolvent sets is an open set which we call the resolvent set, and denote by $\rho(w; A)$. Here 'w' is, as it often shall be, an abbreviation for '$(w_1, \cdots, w_n)$'.

The complement of $\rho(w; A)$ denoted by

2.2 $$\tau(w; A)$$

we call the *analytic joint spectrum* of $(w_1, \cdots, w_n)$. It is a closed set. The *joint spectrum*

2.21 $$\sigma(w; A)$$

may be defined as the set of all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \boldsymbol{C}^n$ for which there do not exist $b_1, \cdots, b_n \in A$ such that

2.22 $$\Sigma(w_i - \lambda_i)b_i = 1$$

(See [6, 2].) Thus $\sigma(w; A) \subset \tau(w; A)$.

2.23 *For A a Banach algebra $\sigma(w; A) = \tau(w; A)$.* Indeed, starting with 2.22, one sees that

$$\Sigma(w_i - \mu_i)b_i$$

has an inverse $f(\mu)$, μ in a neighborhood V of λ and $f \in \text{Hol}(V, A)$. Thus $q_i(\mu) = -b_i f(\mu)$ yields the q_i for which 2.1 holds. *These* q_i *are evidently rational functions.*

An open set $V \subset \boldsymbol{C}^n$, together with $q_1, \cdots, q_n \in \text{Hol}(V, A)$, such that 2.1 holds will be called an *elementary resolvent system.*

Now suppose we have N elementary resolvent systems for the n-tuple $w_1, \cdots, w_n$:

2.3 $$\{q_{ai}, V_a : a = 1, 2, \cdots, N;\ i = 1, \cdots, n\} .$$

For an ordered subset $\alpha = (a_1, \cdots, a_m)$ of these indices, $1 \leqq \alpha_j \leqq N$, we denote $V_{a_1} \cap \cdots \cap V_{a_m}$ by V_α. If $\alpha = (a_1, \cdots, a_n)$ we can define on V_α a function

2.31 $$Q_\alpha = \det (q_{a_i j})_{i\ j}$$

(this is an n rowed determinant), and this $Q_\alpha \in \text{Hol}(V_\alpha, A)$.

We want to study the symmetry properties of the system of functions Q_α.

2.32 PROPOSITION. *Let* $a_0, \cdots, a_n$ *be* $n + 1$ *integers,* $1 \leqq a_i \leqq N$. *Let* α_i *be the n-tuple obtained by deleting 'a_i' from* $(a_0, \cdots, a_n)$. *Then*

2.33 $$Q_{\alpha_0} - Q_{\alpha_1} + Q_{\alpha_2} - \cdots + (-)^n Q_{\alpha_n} = 0$$

on $V_{a_0 a_1 \cdots a_n}$.

The proof is as follows. The $(n + 1)$-rowed determinant

$$\begin{vmatrix} 1 & q_{a_0 1} & q_{a_0 2} \cdots q_{a_0 n} \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 1 & q_{a_n 2} & q_{a_n 2} \cdots q_{a_n n} \end{vmatrix}$$

is surely 0 on $V_{a_0 \cdots a_n}$, by 2.1. Expanding by minors of the first column gives 2.33.

A more compact notation for 2.33 is convenient. Consider the abstract complex $\mathscr{N}$ whose vertices are the numbers $1, 2, \cdots, N$, with m-simplices

2.34 $$(a_0, \cdots, a_m) ,$$

repetitions being allowed, but

2.35 $$(a_0, \cdots, a_m) = \pm(b_0, \cdots, b_m)$$

if the a's are an even (resp., odd) permutation of the b's. For an n-chain

$$\beta = \lambda_1 \alpha_1 + \cdots + \lambda_p \alpha_p$$

where the λ_i are complex numbers (or even elements of A!) and the

α_j are m-simplices, we define

2.36 $$Q_\beta = \Sigma\lambda_k Q_{\alpha_k} \qquad (on\ V_\beta)$$

which clearly depends only on β.

Recall that $\partial(a_0, \cdots, a_p)$ is defined by $(a_1a_2\cdots a_p) - (a_0a_2\cdots a) + \cdots$. Using this notation, 2.32 can be expressed as follows:

2.37 $$if \quad \gamma = (a_0a_1\cdots a_n) \quad then \quad Q_{\partial\gamma} = 0 \quad on \quad V_\gamma .$$

If f is a continuous A-valued function defined on $V_1 \cup \cdots \cup V_N$, we can define a system of differential forms

2.4 $$\Omega_\alpha(f, q_{ai}) = Q_\alpha f dz_1 \wedge \cdots \wedge dz_n , \quad \text{on} \quad V_\alpha$$

where $\alpha = (a_1\cdots a_n)$. As in 2.37 we have

2.41 $$if \quad \gamma = (a_0a_1\cdots a_n) \quad then \quad \Omega_{\partial\gamma} = 0 \quad on \quad V_\gamma .$$

Now suppose we have a polyhedral complex K (cf. [10, 357]) contained in C^n. Suppose there are subcomplexes

2.5 $$K_a \qquad (a = 1, 2, \cdots, N)$$

of K such that K_a is contained in V_a (refer to 2.3). Let K_α be used to denote the subcomplex

$$K_{a_1} \cap \cdots \cap K_{a_m} \qquad (\alpha = (a_1, \cdots, a_m)) .$$

Then for each n-cell in $K_{a_1\cdots a_n}$ we can define (see 2.4, and [10, 82])

2.51 $$\omega_\alpha(k) = \int_k \Omega_\alpha(f, q_{ai})$$

where $\alpha = (a_1, \cdots, a_n)$. This gives an n-cochain in K_α. From 2.4, and in the same notation

2.52 $$\omega_{\partial\gamma} = 0 \quad in \quad K_\gamma \qquad (\gamma = (a_0, \cdots, a_n)) .$$

However, the holomorphism of the forms Ω_α has another consequence.

2.6 *The* ω_α *is a cocycle in* K_α, i.e., $\omega_\alpha(\partial h) = 0$ *for each* $(n+1)$*-chain* h *in* K_α.

This is the "A-valued" analogue of the Cauchy-Green-Stokes theorem, and can be reduced to the complex-valued case by the use of linear functions (and there it is proved by use of the Cauchy-Riemann relations.)

A final remark on the *topology* of Hol(U, A). It is that of uniform convergence on each compact subset of the open set U.

3. **Partitioning of boundaries.** Let K be a finite complex. Let $K_1, \cdots, K_N$ be subcomplexes of K. For any *cell* k of K we define the

type of k to be the least integer j, where $1 \leqq j \leqq N$, such that k lies in K_j. If none exists, we say k has no type. For a *chain* g of K

$$g = \sum_{i=1}^{m} \lambda_i k_i$$

where the k_i are cells, and the λ_i are arbitrary coefficients, we define

3.1 $$\pi_j(g) = \sum_{j=1}^{m} \lambda_{ij} k_i$$

were $\lambda_{ij} = \lambda_i$ if k_i is of type j, and 0 otherwise. Clearly

3.2 $$\textit{if} \quad g \subset K_1 \cup \cdots \cup K_N \quad \textit{then} \quad g = \Sigma_j \pi_j(g) .$$

The main object of this section is as follows.

3.3 THEOREM. *Suppose* $\partial g \subset K_1 \cup \cdots \cup K_N$. *For* $1 \leqq a \leqq N$ *let* $g_a = \pi_a(\partial g)$. *For* $1 \leqq a_1 < a_2 < \cdots < a_m \leqq N$ *let*

3.31 $$g_{a_1 a_2 \cdots a_m} = \pi_{a_1}(\partial g_{a_2 \cdots a_m}) .$$

For any permutation t of $(1, 2, \cdots, m)$ *let*

3.32 $$g_{a_{t(1)} a_{t(2)} \cdots a_{t(m)}} = \operatorname{sgn}(t) g_{a_1 a_2 \cdots a_m} ;$$

and if there are repetitions among the $a_1, \cdots, a_m$, *let* $g_{a_1 a_2 \cdots a_m} = 0$. *Then*

3.33 $$g_{a_1 \cdots a_m} \quad \textit{lies in } K_{a_1} \cap \cdots \cap K_{a_m}$$

3.34 $$g_{a_1 \cdots a_m} \quad \textit{is alternating in its indices}$$

3.35 $$\partial g = \sum_{a=1}^{N} g_a$$

3.36 $$\partial g_{a_1 \cdots a_m} = \sum_{a=1}^{N} g_{a a_1 \cdots a_m} .$$

In the compact notation of § 2, 3.36 says that

3.36′ $$\partial g_\alpha = g_{\delta\alpha}$$

where δ is the coboundary operator (cf. [10, p. 362]).

The *proof* of 3.3 will ensue from a number of propositions, in which $k, g, h, \cdots$ are chains of K.

3.37 $$\partial k = 0 \quad \textit{and} \quad k \subset K_1 \cup \cdots \cup K_N \quad \textit{then}$$

3.371 $$\sum_{b=a}^{N} \pi_a(\partial \pi_b k) = 0 .$$

To show 3.371 we begin by calling the term in 3.371 by the name k_{ab}. Since $k = \sum_1^N \pi_b k$, and $\partial k = 0$, we have

3.372 $$\Sigma k_{ab} = 0$$

where the summation is over all a, b. It may be limited to pairs such that $a \leqq b$ because all terms in $\partial\pi_b k$ have type at most b. Now let $1 \leqq c \leqq N$ and consider

3.373 $$\sum_{c \leqq a \leqq b} k_{ab} .$$

The terms here are of type c at least. The sum S of the remaining k_{ab} contains only terms of type less than c. But $S +$ (3.373) is 0 by 3.372. It follows that (3.373) is 0. 3.371 follows at once. We have the following corollary.

3.374 $$\partial\pi_b k = \sum_{a=1}^{b-1} \pi_a \partial\pi_b k - \sum_{c=b+1}^{N} \pi_b \partial\pi_c k .$$

For any h with $\partial h \subset K_1 \cup \cdots \cup K_N$ set ${}_a h = \pi_a \partial h$. For such h we have, by 3.374

3.375 $$\partial_b h = \sum_{a=1}^{b-1} {}_{ab}h - \sum_{c=b+1}^{N} {}_{bc}h ,$$

and by 3.371

3.376 $$\sum_{b=a}^{N} {}_{ab}h = 0 ,$$

whence

3.377 $${}_{aa}h = - \sum_{b=a+1}^{N} {}_{ab}h .$$

Now we can prove, for $a_1 < a_2 < \cdots < a_m$,

3.378 $$\partial_{a_1 \cdots a_m} g = \sum_{a<a_1} {}_{aa_1 \cdots a_m} g - \sum_{a_1<a<a_2} {}_{a_1 a a_2 \cdots a_m} g + \sum_{a_2<a<a_3} {}_{a_1 a_2 a a_3 \cdots} g - \cdots$$

We let $h = {}_{a_2 \cdots a_m} g$ in 3.375, and take $b = a_1$. Thus

$$\partial_{a_1 \cdots a_m} g = \sum_{a<a_1} {}_{aa_1 \cdots a_m} g - \sum_{c>a_1} {}_{a_1 c a_2 \cdots} g .$$

This second sum may be terminated with $c = a_2$, since each term in the boundary of ${}_{a_2 \cdots} g$ has type at most a_2, so that ${}_{ca_2 \cdots} g = 0$ for $c > a_2$. To the summand in which $c = a_2$ we apply 3.377, with $a = a_2$, $h = {}_{a_3 \cdots} g$, whence

$${}_{a_1 a_2 a_2 \cdots} g = - \sum_{c>a_2} {}_{a_1 a_2 c \cdots} g .$$

This establishes 3.378.

If we use the definition 3.32 for $g_{b_1 \cdots b_m}$ with distinct indices not arranged in order of magnitude, then 3.378 takes the desired form 3.36.

The three other assertions of 3.3 are pretty obviously true. This completes our proof of 3.3.

Let g be as in 3.3 and suppose $G = \{g_\alpha\}$ and $H = \{h_\alpha\}$ are two systems satisfying 3.33–3.36 (*mutatis mutandis*, for the case of h). Here α represents an m-tuple $a_1a_2\cdots a_m$, $m = 0, 1, \cdots$, and $g_\alpha = g$ when $m = 0$. We call G and H *immediately equivalent* if there is an e, $1 \leqq e \leqq N$ such that

$$g_\alpha - h_\alpha \subset K_e \qquad \textit{for all } \alpha. \tag{3.4}$$

We call G and H *equivalent* if we can find systems $G^{(0)}, G^{(1)}, \cdots, G^{(p)}$ each satisfying 3.33–3.36 where $G^{(0)} = G^{(p)}$, $G^{(p)} = H$, and each system is *immediately equivalent* to its successor in this sequence.

3.5 LEMMA. *Let ∂g, ∂h and $g - h \subset K_1 \cup \cdots \cup K_N$ and suppose $\{g_\alpha\}, \{h_\alpha\}$ satisfy* 3.33–3.36. *Then these systems are equivalent.*

Proof. Let us linearly order the indices α, placing the shorter ones before the longer, and ordering lexicographically those of each given length. Let us also order the elements of K. If $G = \{g_\alpha\} \neq H = \{h_\alpha\}$ then there is a first index α such that $g_\alpha \neq h_\alpha$. We treat first the case where α has length 0, i.e., $g \neq h$. Then there must be a first cell k (in the ordering of K) that occurs in $g - h$, with a non zero coefficient λ. Now k must lie in some K_e. We make a new system G' as follows. Let $\{k_\alpha\}$ be formed by an application of 3.4. Let $g'_\alpha = g_\alpha - \lambda k_\alpha$. This system is immediately equivalent to G, and g' agrees with h as far as k and its predecessors is concerned. By a repetition of this process we reach a system $G^{(p)}$ in which $g^{(p)} = h$, and which is equivalent to G.

Now consider the case in which $\alpha = a_0a_1\cdots a_m$ has positive length. Let k be the first cell of K that occurs with non-zero coefficient λ in $g_\alpha - h_\alpha$. As before, we construct an auxiliary system to be added to G. We shall call it $\{l_\gamma\}$. For γ of length less than $m + 1$ we set $l_\gamma = 0$. For γ of length $m + 1$ we set

$$l_\gamma = \lambda_\gamma k \quad \textit{where} \quad \lambda_\gamma = (h_\gamma - g_\gamma)\cdot k\,; \tag{3.51}$$

that is, in the notation of [10, p. 361], λ_γ is the coefficient of k in $h_\gamma - g_\gamma$. We remark that $\sum_c \lambda_{cc_1\cdots c_m} = 0$. Indeed, $\Sigma\lambda_{cc_1\cdots c_m} = \Sigma(h_{cc_1\cdots c_m} - g_{cc_1\cdots c_m})\cdot k = (\partial h_{c_1\cdots c_m} - \partial g_{c_1\cdots c_m})\cdot k = 0$ because $g_{c_1\cdots c_m} = h_{c_1\cdots c_m}$ according to minimal property of m. This says that the function $(c_0\cdots c_m) \to \lambda_{c_1\cdots c_m}$, which is an $(m+1)$-chain of $\mathscr{N}$ vanishes on all coboundaries (cf. 3.36 and [10, p. 362]) and hence is an $(m + 1)$-cycle. Because $\mathscr{N}$ is homologically trivial, λ is the boundary of some $(m + 2)$-chain $\mu : \lambda = \partial\mu$. For $\sigma = (b_0\cdots b_m)$ we obtain $\lambda_{b_0\cdots b_m} = \lambda\cdot\sigma = \partial\mu\cdot\sigma = \mu\cdot\delta\sigma = \sum_c \mu_{cb_0\cdots b_m}$. We now define, for $\gamma = cc_0\cdots c_m$, of length $m + 2$, 3.52 $$l_\gamma = \mu_\gamma \partial k\ .$$

For γ of length greater than $m+2$, we set $l_\gamma = 0$. This system satisfies 3.33–3.36, which we shall now show. We may confine our discussion to 3.36. For γ of the form $c_0 \cdots c_i$ with $i < m$ we have $l_\gamma = 0$. For such i we also have $\Sigma l_{a\gamma} = 0$ because $l_{a\gamma} = 0$ for $i < m-1$ while for $i = m-1$, $\Sigma l_{a\gamma} = 0$ follows from $\Sigma \lambda_{cc_1 \cdots c_m} = 0$. For γ of the form $c_c \cdots c_m$ we have $\partial l_\gamma = \lambda_\gamma \partial k = \lambda_{c_0 \cdots c_m} \partial k = \Sigma_c \mu_{cc_0 \cdots c_m} \partial k = \Sigma \lambda_{cc_0 \cdots c_m}$, by 3.52. For γ longer than $m+1$, $\partial l_\gamma = 0$ again, and so is $l_{a\gamma}$. Thus 3.36 holds for $\{l_\gamma\}$. The discussion of the index $ba_1 \cdots a_m$ is similar, while for those not a permutation of these, everything is 0.

Besides 3.33, we have $l_\gamma \subset K_{a_0}$ (and indeed, also K_b). This shows that $\{l_\gamma\}$ is immediately equivalent to 0, and that

$$\{g_\gamma + l_\gamma\} = G'$$

is immediately equivalent to G. Moreover, it agrees with H for all indices preceding α and in α as far as not only the predecessors of k, *but also k itself*, is concerned.

The reader will surely appreciate that these combinatorial devices can be installed in an inductive argument serving to establish 3.5.

The intent of our definition of 'equivalence' is to be shown in the following theorem in which K is a finite complex, $K_1, \cdots, K_N$, subcomplexes of K, just as they always have been in this section, but for the coefficients in the homology theory we presuppose some vector space A over the rational numbers (e.g., a Banach algebra!).

3.6 THEOREM. *Suppose that for each $\beta = (b_1, \cdots, b_n)$, $1 \leqq b_i \leqq N$, there is an n-cocycle ω_β in $K_\beta = K_{b_1} \cap \cdots \cap K_{b_n}$ such that*

3.61 $$\omega_\beta \text{ is alternating in } (b_1, \cdots, b_n)\ ;$$

3.62 $$\text{if} \quad 1 \leqq b_0, b_1, \cdots, b_n \leqq N \quad \text{then}$$

$$\omega_{\beta_0} - \omega_{\beta_1} + \cdots = 0 \quad \text{in} \quad \bigcap_{i=0}^{n} K_{b_i}$$

where $\beta_k = (b_0 b_1 \cdots b_n)$ with 'b_k' omitted.

Let g be a $2n$-chain in K with $\partial g \subset K_1 \cup \cdots \cup K_N$.

Let $G = \{g_\alpha\}$ be a system of chains satisfying 3.33–3.36 (*such systems exist, by* 3.3.)

Then the value of

3.63 $$\frac{1}{n!} \Sigma_\beta \omega_\beta(g_\beta) = \omega(g)$$

depends only on g, and in fact only on g outside of $K_1 \cup \cdots \cup K_N$. That is, if $g - g' \subset K_1 \cup \cdots \cup K_N$ then $\omega(g) = \omega(g')$. Finally, $\omega(g + g'') = \omega(g) + \omega(g'')$ when $\partial g, \partial g'' \subset K_1 \cup \cdots \cup K_N$.

(If K is $2n$-dimensional, then this says that 3.63 defines a $2n$-cocycle of K mod $(K_1 \cup \cdots \cup K_N)$.)

Proof. In an obvious sense, the sum 3.63 depends additively on the system $G = \{g_\alpha\}$. Any two such systems G and G' are equivalent if $g - g' \subset K_1 \cup \cdots \cup K_N$. Therefore it suffices to show that 3.63, or $\omega(g)$ as we denote that sum, is 0 when G is immediately equivalent to 0. *Then* it will be clear that $\omega(g)$ depends merely on g, etc.

Accordingly, we suppose that for some e, $1 \leqq e \leqq N$, each g_α for α of length $n-1$ lies in K_e.

We shall abbreviate 3.62 in the same spirit as 2.5.

Let $\gamma = e\beta$ where β has length n. Then $\partial\gamma = \beta - e\partial\beta$, and since $\omega_{\partial\gamma} = 0$ we obtain

3.64 $$\omega_\beta - \omega_{e\partial\beta} = ((n-1)!)^{-1}\Sigma_\alpha(\alpha : \beta)\omega_{e\alpha}$$

where we have used the incidence numbers defined by

3.65 $$(n-1)!\partial\beta = \Sigma_\alpha(\alpha : \beta)\alpha ,$$

the factorial compensating for the fact that some α's are permutations of others included in the summation. Inserting 3.64 into 3.63 yields

$$(n-1)!(n!)\omega(g) = \Sigma_\beta\Sigma_\alpha(\alpha : \beta)\omega_{e\alpha}(g_\beta) .$$

Now, from the dual of 3.65 [10, 362(5)]

$$\Sigma_\beta(\alpha : \beta)g_\beta = n!\Sigma_b g_{b\alpha} ,$$

so that

3.66 $$(n-1)!\omega(g) = \Sigma_{b,\alpha}\omega_{e\alpha}(g_{b\alpha}) .$$

But by 3.36

$$\Sigma_b g_{b\alpha} = \partial g_\alpha$$

and g_α lies in K_e as well as in K_α. Thus g_α lies in $K_e \cap K_\alpha$ on which $\omega_{e\alpha}$ is a cocycle. Accordingly

$$(n-1)!\omega(g) = \omega_{e\alpha}(\partial g_\alpha) = 0 .$$

This establishes 3.6.

4. The operational calculus. Let A be a commutative topological algebra over $\boldsymbol{C}$, with unit. Let $w_1, \cdots, w_n$ be n elements of A.

Let K be a finite polyhedral-cell complex in $\boldsymbol{C}^n$, and $K_1, \cdots, K_N$ subcomplexes. Let g be a sum of non-overlapping $2n$-cells of K (each oriented so as to agree with the natural orientation of $\boldsymbol{C}^n = \boldsymbol{R}^{2n}$). Let

$\{g_a\}$ be a system satisfying 3.33–3.36, thus related to g *via* 3.35. Now let $\Delta \subset U$ be subsets of $\boldsymbol{C}^n$ such that g 'covers' Δ but is 'included' in U. Then we call $\{g_a\}$ a *contour system in* U *surrounding* Δ. For $n = 1$, $g_1 + \cdots + g_N$ would be a polygonal contour in U winding once around Δ, suitable for the path of integration of Cauchy's integral.

Let $\{g_\alpha\}$ be such a contour system. In terms of the same N and n, let $\{q_{ai} : a = 1, \cdots, N;\ i = 1, \cdots, n\}$ be some system of continuous functions defined on various open subsets of $\boldsymbol{C}^n$, but such that

4.01 $$q_{ai} \text{ is defined and continuous on } g_a, \qquad a = 1, \cdots, N;\ i = 1, \cdots, n\,.$$

Then we say that $\{g_\alpha\}$ and $\{q_{ai}\}$ are *compatible*. The point of this is that if a system 2.3 is compatible with a contour system $\{g_\alpha\}$, then the forms 2.4, for $f \in \mathrm{Hol}\,(U, A)$ give rise to cocycles in K_α, and in particular

$$\int_{g_\alpha} \Omega_\alpha(f, q_{ai})$$

exists.

Now let Δ be a compact, and U an open, subset of $\boldsymbol{C}^n$, with $\Delta \subset U$ and

4.02 $$U - \Delta \subset \rho(w; A)$$

(the $w_1, \cdots, w_n$ being the elements of A). Then

4.03 PROPOSITION. *There is a contour system* $\{g_\alpha\}$ *surrounding* Δ *in* U, *and a system* 2.3 *compatible with this contour system.*

Proof. Select a neighborhood V of Δ in U whose frontier F is compact and contained in U. Because of the definition of $\rho(w; A)$ (but readers interested in Banach algebras should remember 2.23) we can find a system 2.3 such that $F \subset V_1 \cup \cdots \cup V_N$. For convenience, we display it here:

4.04 $$\{q_{ai}, V_a : a = 1, \cdots, N;\ i = 1, \cdots, n\}\,.$$

Now we dissect $\boldsymbol{C}^n$ into $2n$-cubes each of side d, and make d so small that if a cube touch F, then it lies in some V_a. Let K be the complex generated by the cubes that touch V^-, and K_a, by those that lie in V_a. Let g be the sum of the generators of K. Then $\partial g \subset K_1 \cup \cdots \cup K_N$ and 3.3 can be applied to give a contour sysem

4.05 $$\{g_\alpha\} \text{ surrounding } \Delta \text{ in } U,$$

compatible with 4.04 because $g_\alpha \subset V_\alpha$.

We now introduce the main object (J_Δ) of our study, in a lemma whose proof involves combinatorial results of § 3.

4.1 LEMMA. *Let A be a commutative topological algebra over $\boldsymbol{C}$, with unit, and let $w_1, \cdots, W_n$ be n elements of A. Let Δ be a compact subset of $\boldsymbol{C}^n$, and U a neighborhood of Δ for which*

4.11 $$U - \Delta \subset \rho(w; A) .$$

Then there exists a linear continuous mapping $J(\Delta, U, w, A)$ or more briefly

4.12 $$J_\Delta\colon \operatorname{Hol}(U, A) \to A$$

which may be evaluated as follows. Select a contour system 4.05 *and a family* 4.04 *of elementary resolvent systems compatible* (4.01) *with it, with $V_a \subset U$. Let $f \in \operatorname{Hol}(U, A)$. Then*

4.13 $$J_\Delta(f) = (2\pi i)^{-n}(n!)^{-1} \textstyle\sum_\alpha \displaystyle\int_{g_\alpha} \Omega_\alpha(f, q_{ai}) .$$

For the Ω_α, see 2.4, 2.31.

Proof. We have already shown that such compatible pairs 4.04, 4.05 exist, so at least one such integral can be formed. We shall now show that all such integrals (with a given f) have the same value in A. Suppose we have on the one hand

4.14 $$\{p_{ai}\colon a = 1, \cdots, N;\ i = 1, \cdots, n\}$$

compatible with

4.15 $$\{g_\alpha\} .$$

Suppose

4.16 $$\{h_\alpha\colon \alpha = (a_1, \cdots, a_m),\ m = 1, 2, \cdots,\ 1 \leqq a_j \leqq N'\}$$

and

4.17 $$\{q_{ai}\colon a = 1, \cdots, N';\ i = 1, \cdots, n\}$$

is another compatible pair of systems, giving rise to an integral (we denote the numerical factor by c)

4.18 $$c\Sigma\int_{h_\alpha} \Omega_\alpha(f, q_{ai}) , \qquad \alpha = (a_1, \cdots, a_n) .$$

We don't need to suppose that the q_{ai} are constant. Suppose 4.16 are chains in a cellular complex L, with $L_1, \cdots, L_{N'}$ playing the role analogous to $K_1, \cdots, K_N$. We construct a complex M of which certain refinements of K, L are subcomplexes. Since refinement of chains does not alter 4.13 nor 4.18, we may simply suppose that K, L are subcomplexes of M.

We now define g_α for $\alpha = (a_1, \cdots, a_m)$, $1 \leqq a_j \leqq N + N' = N''$. When all $a_j \leqq N$ we use 4.15. When this does not apply, we set $g_\alpha = 0$.

We define a system of k_α for this set of indices. When $\alpha = (\alpha_1, \cdots, \alpha_m)$ and each $a_j > N$ we let $k_{a_1 \cdots a_m} = h_{a_1 - N, \cdots, a_m - N}$, and when some $a_i \leqq N$, we let $k_{a_1 \cdots a_m} = 0$. We define $p_{ai} = 0$ for $N < a \leqq N''$ (see 4.14).

Let M_a ($a = 1, \cdots, N''$) be defined as K_a when $a \leqq N$, and L_{a-N} when $N < a \leqq N''$. It is not hard to see that the systems $\{g_\alpha\}$, $\{p_{ai}\}$ (enlarged index family) are compatible, and give rise to an integral with the same value as 4.13. Also, the systems $\{k_\alpha\}$, $\{p_{ai}\}$ are compatible, and they give rise to an integral with the same value as 4.18. Consider next the cocyles (see 2.6) in M:

4.19
$$\omega_\alpha = \int \Omega_\alpha(f, p_{ai})$$

(formed with the enlarged index system). Notice that $g - k \subset M_1 \cup \cdots \cup M_{N''}$. Hence by 3.6, $\omega(g) = \omega(k)$. Hence the integrals 4.13 and 4.18 are equal. We may thus unambiguously define J_Δ by 4.13. It clearly has all the properties required.

In 4.4 we shall show that this J_Δ preserves *products* as well.

We shall turn to a covariance property, whose setting we now describe.

Let A and B be two topological algebras, commutative and with unit 1. Let

4.20
$$T\colon A \to B$$

be a continuous linear-algebra homomorphism. Let L be a linear transformation of $\boldsymbol{C}^n$ *onto* $\boldsymbol{C}^n$. Let U and V be open sets such that $L(V) \subset U$, and let Δ, Γ be compact sets such that $L(\Gamma) \supset \Delta$. Then the following diagram of mappings is "commutative"

4.21
$$\begin{array}{ccc} \mathrm{Hol}\,(U, A) & \xrightarrow{T} & \mathrm{Hol}\,(U, B) \\ \circ L \downarrow & & \downarrow \circ L \\ \mathrm{Hol}\,(V, A) & \xrightarrow{T} & \mathrm{Hol}\,(V, B) \end{array}$$

and sends f onto $T \circ f \circ L$.

The linear transformation L can be lifted up to A^n, B^n, and T induces a $T^{(n)}$ in such a way that there arises the commutative diagram

4.22
$$\begin{array}{ccc} A^n & \xrightarrow{T^{(n)}} & B^n \\ L \downarrow & & \downarrow L \\ A^n & \xrightarrow{T^{(n)}} & B^n \end{array},$$

where $T^n(a_1, \cdots, a_n) = (T(a_1), \cdots, T(a_n))$.

Let $a = (a_1, \cdots, a_n)$ be an element of A^n and $b = (b_1, \cdots, b_n)$ an element of B^n of such a sort that (see 4.22)

4.23 $$T^{(n)}(a) \equiv (T(a_1), \cdots, T(a_n)) = L(b) .$$

On these hypotheses we have

4.3 LEMMA. *If* $U - \Delta$ *lies in* $\rho(a; A)$ $V - \Gamma$ *lies in* $\rho(b; B)$, *and for* $f \in \mathrm{Hol}\,(U, A)$ *we have*

$$T[J(\Delta, U, a, A)(f)] = J(\Gamma, V, b, B)(T \circ f \circ L)$$

Proof. If $\mu \in V - \Gamma$ then $\lambda = L(\mu) \in U - \Delta$ so that p_i holomorphic near λ can be found such that $\Sigma p_i(z_i - a_i) = 1$ (z_i is the ith coordinate function) near λ. Applying T, and using 4.23, we obtain $\Sigma q_i(z_i - L(b)_i) = 1$ near λ_i, or $\Sigma q'_i(z_i - b_j) = 1$ near μ, where $(q'_1, \cdots, q'_n) = L'(q_1, \cdots, q_n)$, L' being the transpose of L. These q_i are holomorphic near μ, so that $\mu \in \rho(b, B)$. We now set up an integral for $J_\Delta(f) = J(\Delta, U, a, A)(f)$:,

$$J_\Delta(f) = c \sum \int_{g_\alpha} \Omega_\alpha(f, p_{ai}) .$$

(the index 'a' on p_{ai} is not to be confused with the a $\in A^n$!). We apply T to this and then change the variable of integration by $z = L(w)$. This changes f to TfL, and p_{ai} to $L'Tp_{ai}$. The new chains $L^{-1}g_\alpha$ do not have density 1, but the factor needed to bring this about is exactly provided by the jacobian $\partial z/\partial w$. The reader not acquainted with the transformation of such integrals [10, 88] is invited to consider only the case $L = 1$ which is really enough for our purposes. On the other hand, a much more complex situation is also conceivable: the case of L being a non-singular analytic mapping. But this would require more familiarity with integration in complex manifolds than we wish to require. This completes our sketch of the proof of 4.3.

Several corollaries are deducible from 4.3, all involving $L = 1$. We always suppose Δ a non-void compact, and U a neighborhood of Δ, in $\boldsymbol{C}^n$.

4.31 COROLLARY. *Suppose* $U - \Delta \subset \rho(a_1, \cdots, a_n; A)$ *and let* τ *be that part of* $\tau(a; A)$ *which lies in* Δ. *Suppose* τ *is not void, and let* V *be a neighborhood of* τ *in* U. *Then*

$$J(\Delta, U, a, A)(f) = J(\tau, V, a, A)(f) .$$

4.32 COROLLARY. *If* A *is a subalgebra of* B, *and* $a_1, \cdots, a_n \in A$ *and* $U - \Delta \subset \rho(a; A)$, *then* $U - \Delta \subset \rho(a; B)$ *and*

$$J(\Delta, U, a_1, \cdots, a_n, A)(f) = J(\Delta, U, a_1, \cdots, a_n; B)(f)$$

for $f \in \mathrm{Hol}(U, A)$.

4.33 COROLLARY. *Let* ξ *be a continuous linear functional of* A, *which is multiplicative. Let* λ *be the point* $(\xi(a_1), \cdots, \xi(a_n))$, *where* $U - \Delta \subset \rho(a_1, \cdots, a_n; A)$. *Let* $f \in \mathrm{Hol}(U, A)$. *Then*

$$\xi(J_\Delta(f)) = \begin{cases} \xi(f(\lambda)) & \text{if } \lambda \in \Delta \\ 0 & \text{if } \lambda \notin \Delta . \end{cases}$$

Proof. In the first plane, $\xi(a) = \lambda$ must fall into the joint spectrum $\sigma(a; A)$, so if $\lambda \notin \Delta$ then $\lambda \in U$. Select a point $\mu \in \Delta$, taking $\mu = \lambda$ if $\lambda \in \Delta$, and arbitrarity otherwise.

We have ξ: $A \to \boldsymbol{C}$, a situation to which 4.3 can be applied, with the result that

4.34 $$\xi[J(\Delta, U, a, A)(f)] = J(\Delta, U, \lambda, \boldsymbol{C})(\phi)$$

where $\phi = \xi \circ f \in \mathrm{Hol}(U, C)$. Using 4.31 twice, we obtain for the right hand side R of 4.34

$$R = J(\mu, U, \lambda, \boldsymbol{C})(\phi) .$$

Consider the system of elementary resolvent systems $q_{ai} = \delta_{ai}(z_i - a_i)^{-1}$ where δ_{ai} is the Kronecker symbol. As g take an $2n$-cube with μ as center and so small that g lies in U, and let $g_{12\cdots n}$ be the product of their boundaries in the several coordinate planes. This obviously compatible pair gives rise to the classical representation

4.341 $$R = (2\pi i)^{-n}\int \cdots \int \phi(z)(z_1 - \lambda_1)^{-1} \cdots (z_n - \lambda_n)^{-1}$$
$$dz_1 \cdots dz_n = \phi(\lambda) \text{ or } 0 \text{ according to } \lambda = \mu \text{ or } \lambda \neq \mu .$$

This establishes 4.33.

The Cauchy-Weil integral formula, in the weaker, but more intelligible form, in which one integrates not over a subset of the boundary of a *polyèdre* [8] Δ, but over an ordinary polyhedron in a neighborhood U of Δ, can be deduced from 4.1 and 4.33. We consider the matter.

A compact subset Δ of an open subset U of $\boldsymbol{C}^n$ is called $\mathrm{Hol}(U, \boldsymbol{C})$-*convex* if for $z \in U - \Delta$ there exists and $p \in \mathrm{Hol}(U, \boldsymbol{C})$ such that $p(z) = 1$, but $|p(\lambda)| < 1$ for $\lambda \in \Delta$.

Suppose $p \in \mathrm{Hol}(U, \boldsymbol{C})$. Suppose that U is convex, or that p is a polynomial, or that U is a *domain of holomorphy* [3]. Then

4.35 *there exist* $p_1, \cdots, p_n \in \mathrm{Hol}(U \times U, \boldsymbol{C})$ *such that for* $\lambda, \mu \in U$, $p(\lambda) - p(\mu) = \Sigma p_i(\lambda, \mu)(\lambda_i - \mu_i)$. (See [3].)

Suppose U is a neighborhood of the compact subset Δ of $\boldsymbol{C}^n$. Let $A(\Delta, U)$ be the closure of $\mathrm{Hol}(U, C)$ in $C(\Delta, C)$. Then $A(\Delta, U)$ is a Banach algebra with the maximum modulus norm. Let $z = (z_1, \cdots, z_n)$ be the coordinate functions in $\boldsymbol{C}^n$.

We now relate these concepts.

4.36 *If* $U - \Delta \subset \rho(z_1, \cdots, z_n; A(\Delta, U))$ *then* Δ *is* $\mathrm{Hol}(U, \boldsymbol{C})$*-convex.*

Proof. Let $\lambda \in U - \Delta$. Then there are $p_1, \cdots, p_n \in A(\Delta, U)$ such that $\Sigma p_i(z_i - \lambda_i) = 1$. Because $\mathrm{Hol}(U, \boldsymbol{C})$ is dense in $A(\Delta, U)$ there are $f_1, \cdots, f_n \in \mathrm{Hol}(U, \boldsymbol{C})$ such that $\|f\| < 1$ where $f = 1 - \Sigma f_i(z_i - \lambda_i)$. But $f(\lambda) = 1$.

4.37 *If for each* $p \in \mathrm{Hol}(U, \boldsymbol{C})$ *one has* 4.35, *then* $U - \Delta \subset \rho(z_1, \cdots, z_n; A(\Delta, U))$.

Proof. Let $\lambda \in U - \Delta$. Select $p \in \mathrm{Hol}(U, \boldsymbol{C})$ such that $p(\lambda) = 1$, $\|p\| < 1$. Using 4.35 we obtain

$$-1 + p(z) = \Sigma p_i(\lambda, z)(z_i - \lambda_i) .$$

Since $\|p(z)\| < 1$, the right side has an inverse in $A(\Delta, U)$, so that $\lambda \in \rho(z; A(\Delta, U))$.

The Cauchy-Weil integral formula, in the restricted sense already described, results from the following.

4.38 THEOREM. *Let U be an open subset of $\boldsymbol{C}^n$ and let $p_1, \cdots, p_N \in$* $\mathrm{Hol}(U, \boldsymbol{C})$. *Let $F_1, \cdots, F_N$ be closed sets in $\boldsymbol{C}$ such that*

4.381 $$\Delta = p_1^{-1}(F_1) \cap \cdots \cap p_N^{-1}(F_N)$$

is compact. Replace U by a smaller neighborhood V for which $p_a(\lambda) - p_a(\mu) = \Sigma p_{ai}(\lambda, \mu)(\lambda_i - \mu_i)$ *with* $p_{ai} \in \mathrm{Hol}(V \times V, \boldsymbol{C})$. *For each* $\lambda \in \Delta$ *define*

$$q_{ai}(\lambda)(\mu) = p_{ai}(\lambda, \mu)[p_a(\mu) - p_a(\lambda)]^{-1}$$

for all μ in the set $V_a(\lambda)$ for which it makes sense. Then there exists one contour system $\{g_\alpha\}$ surrounding Δ in V which is compatible with the $\{q_{ai}(\lambda)\}$ for every $\lambda \in \Delta$, and (for c, see 4.18)

4.382 $$c \sum \int_{g_\alpha} \Omega(f, q_{ai}(\lambda)) = f(\lambda) \qquad \lambda \in \Delta ,$$

for every $f \in \mathrm{Hol}(U, \boldsymbol{C})$.

Proof. Choose neighborhoods U_a of F_a such $V = p_1^{-1}(U_1) \cap \cdots \cap p_N^{-1}(U_N)$

is a domain of holomorphy and lies in U. Then, by 4.35, the p_{ai} can be found. For a given μ not in Δ, an index a can be found such that $q_{ai}(\lambda)(\nu)$ makes sense for all $\lambda \in \Delta$, and all ν in some neighborhood V_a of μ, namely some a such that $p_a(\mu) \notin F_a$. Thus $V_a \subset V_a(\lambda)$ for all λ. A contour system $\{g_\alpha\}$ such that $g_a \subset V_a$ can now be found by the *method* of 4.03, which after all, uses only the fact that the V_a cover the frontier of some neighborhood of Δ. We now define

$$\tilde{q}_{ai}(\mu)(\lambda) = q_{ai}(\lambda)(\mu) .$$

We have used z_i as the ith coordinate function in U, V, or even $\boldsymbol{C}^n$. It is important to use another name for its restriction to Δ. We call that w_i.

Now we ask ourselves, what is $\tilde{q}_{ai}$ itself. It is a function on Δ whose values are functions on V_a, that is $\tilde{q}_{ai} \in \mathrm{Hol}(V_a, A(\Delta, V))$ (where the holomorphy is recognized.) Moreover, $\Sigma q_{ai}(\lambda)(\mu)(\lambda_i - \mu_i) = -1$, $\lambda \in \Delta$, $\mu \in V_a$. Thus $\Sigma \tilde{q}_{ai}(\mu)(\mu_i - w_i) = 1$, $\mu \in V_a$, whence $\{\tilde{q}_{ai}, V_a\}$ is an elementary resolvent system for the elements $w_1, \cdots, w_n$ of $A(\Delta, V)$. Therefore

4.383 $$c\Sigma \int_{g_\alpha} \Omega(\phi, \tilde{q}_{ai}) = J(\Delta, V, w, A(\Delta, V))(\phi) ,$$

where $\phi(\lambda) = f(\lambda) \cdot E$, where E is the unit of $A(\Delta, V)$, and both sides of 4.382 give some element of $A(\Delta, V)$, which we call ψ. We shall show that $\psi = f$ restricted to Δ. Let $\lambda \in \Delta$. Define the linear multiplicative functional ξ on $A(\Delta, V)$ as evaluation at λ. Then $\xi(\psi) = \psi(\lambda)$. But 4.33 tells us that $\xi(\psi) = \xi(\phi(\lambda))$. Now $\phi(\lambda) = f(\lambda)E$ and $\xi(E) = E(\lambda) = 1$. Thus $\psi(\lambda) = f(\lambda)$. But if we put λ into the free place of the integral in 4.383, we find ourselves integrating something like $\tilde{q}_{ai}(z)(\lambda) \cdots dz$, which is something like $q_{ai}(\lambda)(z) \cdots dz$, which is what is obviously intended in 4.382.

In this theorem, the $q_{ai}(\lambda)$ contain the parameter in an analytic way. If we ask for an integral representation for $f \in \mathrm{Hol}(U, \boldsymbol{C})$ on Δ in terms of values on $U - \Delta$, without requiring that $q_{ai}(\lambda)$ depend analytically on λ, we can do without the special form of Δ (4.381).

4.39 THEOREM. *Let $\Delta \subset U \subset \boldsymbol{C}^n$, Δ compact, U open. Then there exists a contour system $\{g_\alpha\}$ surrounding Δ in U and a system*

$$\{q_{ia}(\lambda)\colon a = 1, \cdots, N;\ i = 1, \cdots, n\}$$

of holomorphic A-valued functions, depending continuously on a parameter λ in a neighborhood of Δ, compatible with $\{g_\alpha\}$ for every such λ, such that 4.382 *holds for every* $f \in \mathrm{Hol}(U, \boldsymbol{C})$.

Proof. Let $A = \mathscr{C}(\Delta, \boldsymbol{C})$. Then $w_1, \cdots, w_n \in A$ where w_i is the restriction of z_i to Δ, and $\sigma(w; A) = \Delta$ as is well known. Hence, when

$\mu \in \Delta$ there exist $p_1, \cdots, p_n \in A$ such that

$$\Sigma p_i(w_i - \nu_i) = 1 \in A \, .$$

Therefore there is a neighborhood $V(\mu)$ such that

$$[\Sigma p_i(w_i - \nu_i)]^{-1} \text{ exists in } A, \quad \nu \in V(\mu) \, .$$

Select $\mu_1, \cdots, \mu_N$ such that the $V_a = V(\mu_a)$ cover the frontier of some neighborhood of Δ in U.

We define

$$q_{ai}(\lambda)(\mu) = p_{ai}(\lambda)[\Sigma p_{aj}(\lambda)(\mu_i - \lambda_j)]^{-1}$$

which is to say

$$q_{ai}(\lambda) = p_{ai}(\lambda)[\Sigma p_{aj}(z_j - \lambda_j)]^{-1} \, .$$

This is holomorphic on V_a indeed rational for each λ. Thus the form $\Omega_\alpha(f, q_{ai}(\lambda))$ is holomorphic on V_α. (On the other hand the q_{ai} depend continuously on λ). We now continue as in the proof of 4.38, beginning with the words "a contour system $\{g_\alpha\}$ can be found", and the result is 4.382.

It is remarkable that although the parameter λ does not appear holomorphically in the forms $\Omega_\alpha(f, q_{ai}(\lambda))$, the result of the integration yields something which does depend holomorphically on λ.

We digress at this point to deduce from our results those propositions on which Waelbrock bases his symbolic calculus. This digression ends with 4.395.

4.394. *Let $p_1, \cdots, p_n$ be polynomials in $z_1, \cdots, z_n$ where, in fact $p_1 = z_1, \cdots, p_n = z_n$ $(n \leqq N)$. Suppose $\Delta = p_1^{-1}(D) \cap \cdots \cap P_N^{-1}(D)$ is compact where $D = \{z: |z| \leqq 1, z \in \boldsymbol{C}\}$. Then each $f \in$ Hol (Δ, A) is of the form $\varphi \circ P$ where $\varphi \in$ Hol (D^N, A) and $P(\lambda) = p_1(\lambda), \cdots, p_N(\lambda))$. Thus $P: \Delta \to D^N$. A need not be an algebra here. But Hol $(\ , A)$ is still a modul over Hol $(\ , \boldsymbol{C})$.*

Proof. Examine the integral representation 4.382:

$$f(\lambda) = c \sum_\alpha \int_{g_\alpha} f(z) \det \left(\frac{p_{aj}(\lambda, z)}{p_a(\lambda) - p_a(z)} \right) dz_1 \cdots dz_n$$

$$a = a_1, \cdots, a_n \, ; \qquad j = 1, \cdots, n \, .$$

On g_α we have $|p_a(z)| \geqq t > 1$. Thus, for $|\lambda_a| < t$ $(a = 1, \cdots, N)$ we can define $\varphi(\lambda_1, \cdots, \lambda_{n+1}, \cdots \lambda_N)$

$$= c \sum_\alpha \int g_\alpha f(z) \det \left(\frac{p_{aj}(\lambda, z)}{\lambda_a - p_a(z)} \right) dz_1 \cdots dz_n \, .$$

This is clearly holomorphic on D^N. (Moreover, by writing

$$\frac{1}{\lambda_a - p_a(z)} = -\frac{1}{p_a(z)}\cdot\Big(1 + \frac{\lambda_a}{p_a(z)} + \frac{\lambda_a^2}{p_a(z)^2} + \cdots\Big)$$

we see that $\varphi(\lambda_1\cdots\lambda_N)$ can be uniformly approximated by polynomials on D^N.)

Obviously $\varphi\circ P = f$.

4.395 *The set $P(\Delta)$ is the intersection of D^N with the algebraic variety*

$$\{\lambda : \lambda_1 = p_1(\lambda), \lambda_2 = p_2(\lambda), \cdots, \lambda_N = p_N(\lambda)\}\ ,$$

i.e.,

$$\{\lambda_1, \cdots, \lambda_N : \lambda_{n+1} = p_{n+1}(\lambda), \cdots, \lambda_N = p_N(\lambda)\}\ .$$

The homomorphism $\mathrm{Hol}\,(D^N, A) \to \mathrm{Hol}\,(\Delta, A)$ *given by* $\varphi \to \varphi\circ f$ (shown *into* in 4.394) *has as its kernel precisely the ideal generated by the polynomials* $p_{n+1}(z) - z_{n+1}, \cdots, p_n(z) - z_n$.

Let $\varphi \in \mathrm{Hol}\,(D^n, A)$. Then (looking at the Taylor series about $O \in \boldsymbol{C}^n$) $\varphi(\lambda_1,\cdots,\lambda_n,\lambda_{n+1},\cdots) - \varphi(\lambda_1,\cdots,\lambda_n,\mu_{n+1},\cdots) = \sum_{j=n+1}^{N}(\lambda_j - \mu_j)\varphi_j(\lambda\cdots\mu\cdots)$ where φ_j are holomorphic on D^k ($k = 2N - n$, to be exact).

Now let $\mu_{n+k} = p_{n+k}(\lambda_1, \cdots, \lambda_n)$. Then $|\mu_{n+k}| \leqq 1$. Thus $\varphi(\lambda_1\cdots) - \varphi(\lambda_1, \cdots, p_{n+1}(\lambda), \cdots) = \sum_{j>n}[\lambda_j - p_j(\lambda)]\psi_j(\lambda)$ where $\psi_j \in \mathrm{Hol}\,(D^N, A)$. If $\varphi\circ P = 0$ then $\varphi = \sum_{j>n}[z_j - p_j]\psi_j$. This suffices to show 4.395.

We now consider the homomorphism property of the operator J_Δ.

4.4 Lemma. *Assume the hypothesis of* 4.1 (*which ends with* 4.11). *Then*

4.41 $$J_\Delta(f_1 f_2) = J_\Delta(f_1)J_\Delta(f_2)$$

for $f_1, f_2 \in \mathrm{Hol}\,(U, A)$, *and* $J_\Delta = J(\Delta, U, w, A)$.

Proof. We select an integral representation 4.15 for J_Δ. Our meaning should be clear if we write

4.42 $$J_\Delta(f_1) = c\sum_\alpha \int_{g_\alpha} f_1(z)P_\alpha(z)dz\ .$$

We write also

4.43 $$J_\Delta(f_2) = c\sum_\alpha \int_{g_{\alpha'}} f_2(\zeta)P_{\alpha'}(\zeta)d\zeta\ .$$

Then, denoting the right side of 4.41 by R, we have

4.44 $$R = c^2\sum_{\alpha\ \beta}\int_{g_\alpha}\int_{g_\beta} f_1(z)f_2(\zeta)P_\alpha(z)P_\beta(\zeta)dz d\zeta\ .$$

We adopt the notation $(z, \zeta) = (z_1, \cdots, z_n; \zeta_1, \cdots, \zeta_n)$ for the natural coordinate system in C^{2n}.

The key to the situation is to recognize that the integral on the right side of 4.44 is a representation for

4.45 $$J(\Delta \times \Delta, U \times U, a, A)(\phi)$$

where $$a = (w_1, w_2, \cdots, w_n, w_1, w_2, \cdots, w_n) \in A^{2n}$$

and for $(\lambda, \mu) = (\lambda_1, \cdots \lambda_n, \mu_1, \cdots, \mu_n)$,

4.46 $$\phi(\lambda, \mu) = f_1(\lambda) f_2(\mu) .$$

The contour-system in 4.45 is $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} = g_\alpha \times g_\beta$ (3.34–3.36 have to be verified, and this can be done, starting with [10, 365, (1)]).

We now shrink the $\Delta \times \Delta$ in 4.45 to the smaller set $\Delta_1 = \{(\lambda, \lambda) : \lambda \in \Delta\}$. This change is justified by the fact that there is no point of the spectrum $\sigma(a; A)$ in $\Delta \times \Delta - \Delta_1$, as we shall now check. Let $(\lambda, \mu) \in \Delta \times \Delta - \Delta_1$. Then either $\lambda = \mu \notin \Delta$ or $\lambda \neq \mu$. In the first case there exist $q_1, \cdots, q_n \in$ Hol (V, A) such that

$$\sum_{i=1}^{n} q_i(z_i - a_i) = 1$$

on the neighborhood V of $\lambda \in C^n$. These q_i can be extended in a trivial way to be Holomorphic on $V \times C^n$ which is a neighborhood of (λ, λ). Thus $(\lambda, \lambda) \in \rho(a; A)$. In the latter case, there is an i such that $\lambda_i \neq \mu_i$. Then the relation

$$(z_i - \zeta_i)^{-1}(z_i - a_i) + (\zeta_i - z_i)^{-1}(\zeta_i - a_{n+i}) = 1 ,$$

valid in a neighborhood of (λ, μ), show that $(\lambda, \mu) \in \rho(a, A)$.

We now introduce the linear mapping L: $C^{2n} \longleftrightarrow C^{2n}$ for which $L(\lambda, \mu) = (\lambda + \mu, \lambda - \mu)$. Then $L(\Gamma) = \Delta_1$ where $\Gamma = \Delta \times \{O_n\}$, O_n being the origin in the second factor space C^n of C^{2n}. Moreover $a = L(b)$ where $b = (w_1, \cdots, w_n, 0, \cdots, 0) \in A^{2n}$. We pick T: $A \rightarrow A$ as the identity, and are thus in a position to apply 4.3:

4.47 $$J(\Delta_1, U_1, a, A)(\phi) = J(\Gamma, V_i, b, A)(\phi \circ L) .$$

The left side here is equal to R (4.44). The U_1, V_1 are some neighborhoods of Δ_1, Γ. To evaluate the right side of 4.47, we choose a contour-system $g_\alpha \times h_\beta$ where g_α are those we began with, and h_β is a classical box-like arrangement as in 4.341, the μ in which is be replaced by O_n. This is a contour-system surrounding $\Gamma = \Delta \times \{O_n\}$. The product of Cauchy-kernels that goes with this classical integral representation we may denote by $Q_\beta(\zeta)$. Thus we obtain

$$R = c^2 \sum_{\alpha,\beta} \int_{g_\alpha} \int_{h_\beta} f_1(z + \zeta) f_2(z - \zeta) P_\alpha(z) Q_\beta(\zeta) dz d\zeta .$$

Here let us integrate first with respect to ζ. By Cauchy's integral theorem, we obtain

$$R = c \sum_{\alpha} \int_{g_\alpha} f_1(z + 0) f_2(z - 0) P_\alpha(z) dz$$

which is $J_\Delta(f_1 f_2)$, as we intended to show.

Two remarks are in order. First, $J_\Delta(1)$ is an idempotent element which is a relative unit for all elements $J(\Delta, U, w, A)(f)$, $f \in \mathrm{Hol}\,(U, A)$, $U - \Delta \subset \rho(a; A)$. Second, the homomorphism property for algebras A which are semi-simple in the sense that for $a \neq O \in A$ there is a continuous complex linear-algebra-homomorphism ξ such that $\xi(a) \neq 0$, follows already from 4.33.

The covariance result 4.3 can be generalized to include linear mappings

4.48 $$L\colon \mathbf{C}^{n+m} \to \mathbf{C}^n \qquad \text{(onto)}$$

where $m > 0$. We prefer to isolate just the one feature of L in 4.48, namely the many-valuedness of L^{-1}, and assume that

4.49 $$L(\lambda_1, \cdots, \lambda_n, \lambda_{n+1}, \cdots, \lambda_{n+m}) = (\lambda_1, \cdots, \lambda_n) ,$$

while taking $B = A$, and $T = l$.

4.5 PROPOSITION. *Let L be as in 4.49. Let $(a_1, \cdots, {}_{m+n}) \in A^{m+n}$. Let $\Delta \subset U \subset C^n$ where U is open, Δ, compact, with $U - \Delta \subset \rho(a_1, \cdots, a_n; A)$. Let $\Gamma \subset C^{n+m}$ be compact with $\Delta \subset L(\Gamma) \subset U$, such that $U \times \mathbf{C}^m - \Gamma \subset \rho(a_1, \cdots, a_{n+m}; A)$. Then*

4.51 $$J(\Delta, U, a_1, \cdots, a_n, A)(f) = J(\Gamma, U \times \mathbf{C}^m, a_1, \cdots, a_{n+m}, A)$$

$(f \circ L)$, *for $f \in \mathrm{Hol}\,(U, A)$.*

This can be provided by selecting a contour-system $\{g_\alpha\}$ surrounding $L(\Gamma)$ in U with a compatible family $\{p_{\alpha i}\}$; and then selecting a classical-type contour-system for $(1 - L)(\Gamma)$ in $\mathbf{C}^m$. We combine these by the product method sketched below equation 4.47. This provides a representation for the right side R of 4.51 wherein the integrand is $f(z_1, \cdots, z_n)$. The integral with respect to $dz_{n+1} \cdots dz_{n+m}$ can be carried out first, and Cauchy's integral formula for constant functions on $(1 - L)(\Gamma)$ in $\mathbf{C}^m$ yields an integral representation for the left side of 4.51.

This proposition shows that the element 'a' constructed in 1, 3.3 is indeed $J(S_A, W, a_1, \cdots, a_n, A)(\mathscr{F})$, in the notation of [1], because the method there is to adjoin further elements $a_{n+1}, \cdots, a_p$ and apply Shilov's adaptation of Weil's formula to $\mathscr{F} \circ L$. However, there is no logical necessity for this observtion about the relation of [1, 3.3] to the present work because the combination of 4.31 and 4.33 in the present paper yields all that is promised by [1, 3.3], and more (e.g., 4.41, 4.32, etc.).

For an important case including Banach algebras in which Δ contains the entire topological joint spectrum $\tau(w_1, \cdots, w_n; A)$, we sum up the major results obtainable from 4.1, 4.33 and 4.44. We denote the constant functions whose value on U is a also by 'a', for each $a \in A$.

4.6 THEOREM. *Let A be a commutative topological algebra with unit. Let $w_1, \cdots, w_n$ be such elements of A for which $(z - w_i)^{-1} \to 0$ uniformly as $|z| \to \infty$. Let Δ be a compact subset of C^n such that*

$$\tau(w_1, \cdots, w_n; A) \subset \Delta .$$

Let U be open, $U \supset \Delta$. Then the mapping (*see* 4.12)

$$J_\Delta\colon \mathrm{Hol}\,(U, A) \to A$$

4.61 *is a continuous liner algebra homomorphism J such that*

4.62 $$J(a) = a \text{ for each constant } a \in A$$

and

4.63 $$J(z_i) = w_i ,$$

where z_i is the ith *coordinate function*:

$$z_i(\lambda_1, \cdots, \lambda_n) = \lambda_i .$$

Proof. Let us show 4.62. It is enough to treat the case $a = 1$. We can, by 4.3, choose U arbitrarily, so we take $U = C^n$. For each entire function f we have then

$$J_\Delta(f) = (2\pi i)^{-n}\int\cdots\int\frac{f(z_1, \cdots, z_n)dz_1\cdots dz_n}{(z_1 - w_1)\cdots(z_n - w_n)}$$

provided z_i runs around a large square of center 0, and side $2R$ in C. We take $f = 1$. Then

$$J_\Delta(1) = (2\pi i)^{-n}\int(z_i - w_i)^{-1}dz_1\cdots\int(z_n - w_n)^{-1}dz_n .$$

Let w be any one of the w_i and define $b = \int(z - w)^{-1}dz - 2\pi i1$. Thus $b = wc$ where $c = \int z^{-1}(z - w)^{-1}dz$. Let F be a linear continuous functional on the topological linear space A, and let $\phi(z) = F((z - w)^{-1})$. Then $|\phi(z)| \to 0$ uniformly for $z \to \infty$. Now $F(c) = \int z^{-1}\phi(z)dz$, so $F(c) = 0$. Thus $c = 0$, and 4.61 is proved. Now consider f where $f(z) = z_1$. We can write $J_\Delta(f)$ as a product of integrals, each of which is a scalar, except for one, which has form

$$\int z(z - w_1)^{-1}dz = \int(z - w_1)(z - w_1)^{-1}dz + w_1\int(z - w_1)dz = 0 + 2\pi i w_1 .$$

Inserting this into the product, we obtain $J_\Delta(f) = w_1$, thus proving 4.62. Remark: Waelbrock [7, 147] notices the relevance of the condition

$$(z - w)^{-1} \to 0 \quad \textit{uniformly as} \quad z \to 0$$

to the operational calculus, and points out that it follows from

$(z - w)^{-1}$ is bouned for z in some neighborhood of infinity.

Still weaker growth conditions have the same effect.

It is natural to ask [8, 174] whether in 4.6 the mapping J_Δ is *characterized* by the properties 4.61–4.63. We do not know, but it seems unlikely. Sufficient conditions may be obtained as follows. Let B be the algebra Hol (Δ, A) and make *some* hypothesis about Δ and $w_1, \cdots, w_n$ such that $\tau(z_1, \cdots, z_n; B) \subset \Delta$. Then for $f \in B$ we have an integral representation for $J(\Delta, z, B)(f)$, approximable by rational functions in z (coefficients in A). By 4.33, the element thus represented (and approximated) is f itself. Hence $J_\Delta(f)$ is determined by the conditions 4.61–4.63.

For a compact subset Δ of $\boldsymbol{C}^n$ we define Hol (Δ, A) as the direct limit of the Hol (U, A), for those open $U \supset \Delta$, and topologize Hol (Δ, A) accordingly [7, 8]. Following the pattern of [8], we can, on the hypothesis of 4.31, construct a linear continuous homomorphic mapping

4.7 $$J(\Delta, a, A)\colon \text{Hol}\,(\Delta, A) \to A\ .$$

In case Δ is precisely $\tau(a_1, \cdots, a_n; A)$, assumed to be non-void, the Δ may be dropped and we have

4.8 $$J(a, A)\colon \text{Hol}\,(\tau(a; A), A) \to A\ .$$

For $f \in$ Hol $(\tau(a; A), A)$, the element $J(a, B)(f)\ A$ is independent of the superalgebra $B \supset A$. We may denote it by $f(a)$. In order that $f(a)$ make sense, one needs to know that f is holomorphic on $\tau(a_1, \cdots, a_n; A_0)$ for some algebra A_0 containing these elements.

We haven't made any search through the literature to see where the idea of making holomorphic A-valued functions into operators may have been used before, but an example has come to our attention, namely G. Lumer and M. Rosenbloom, *Linear operator equations*, Proc. Amer. Math. Soc., 10, (1959), 32–41; see the top line of page 36.

5. Banach algebras, and their inverse limits. Let A be a commutative Banach algebra over $\boldsymbol{C}$, with unit. A' denotes the dual linear space: we consider it under the weak topology $\sigma(A', A)$. The class of homomorphisms, 0 excepted, in A' we denote by $A' \cap$ Hom. This is compact. The kernels of the $\xi \in A' \cap$ Hom are the maximal ideals of A. The joint spectrum $\sigma(a_1, \cdots, a_n; A)$ of an ordered set $a = (a_1, \cdots, a_n) \in A^n$ of elements of A may be defined either as

$$\sigma(a; A) = \{\xi(a_1), \cdots, \xi(a_n)) : \xi \in A' \cap \text{Hom}\}$$

or as we have already done in 2.21 [6]. To remove any confusion about the application of the previous section to Banach algebras, we state the following.

5.1 THEOREM. *Let A be a commutative Banach algebra with unit and $a_1, \cdots, a_n$ elements of A. Then the operator $J(a_1, \cdots, a_n; A)$*(see 4.8) *is a continuous linear homomorphism of* Hol $(\sigma(a_1, \cdots, a_n; A), A)$ *into A, sending constants in A onto themselves, sending the coordinate function z_i onto a_i, and having the covariance properties* 4.33, 4.32.

Our purpose here is to extend this theorem to a wider class of algebras, those studied in [2, 5] and for which we shall use Michael's term: *commutative F-algebras*, rather than our earlier terminology. Each such algebra A can be obtained as follows. Let

5.2 $$B_1, B_2, \cdots, B_m, \cdots$$

be a sequence of commutative Banach algebras, related by continuous homomorphisms (mapping 1 on 1)

5.21 $$B_1 \xleftarrow{\pi} B_2 \xleftarrow{\pi} B_3 \cdots \xleftarrow{\pi} B_m \xleftarrow{\pi} \cdots$$

where each $\pi(B_{m+1})$ is *dense* in B_m, $m = 1, 2, \cdots$. Let A be the inverse limit, that is the set of sequences

5.23 $$(b_1, b_2, \cdots) \quad where \quad \pi(b_{m+1}) = (b_m)$$

with the topological algebra structure derive from the topological product $B_1 \times B_2 \times \cdots$. Then A is a commutative F-algebra. It is metrizable and complete. It is *topological* in the sense of § 2.

The reader may wonder for a moment that we say we want to extend 5.1 to F-algebras. Cannot the theory of § 4 be applied to F-Algebras? Of course it can, but the results are not often interesting because $\tau(a_1, \cdots, a_n; \mathrm{A})$ is usually *unbounded*, as is the joint spectrum $\sigma(a_1, \cdots, a_n; A)$. However, because of the known relation of the joint spectrum $\sigma(a_1, \cdots, an; A)$ to the various $\sigma(\pi_m(a_1), \cdots, \pi_m(a_n); A)$, 5.1 can be extended to F-algebras as it stands—but we have first to explain this relation, and the notation. An element $a \in A$ is a sequence as in 5.23, and we shall use $\pi_m(a)$ to denote the element $b_m \in B_m$. Each π_m is a continuous homomorphism of A onto a dense subalgebra of B_m, and [2, 5.4]

5.24 $$\sigma(a_1, \cdots, a_n; A) = \bigcup_{m=1,2,\cdots} \sigma(\pi_m(a_1), \cdots, \pi_m(a_n); B_m) .$$

We consider it impractical to make all the definitions which would make 5.1 literally true for 'Banach algebra' replaced by 'F-algebra'.

Consider Hol (Δ, A) where Δ is a subset of $\boldsymbol{C}^n$. We define it simply as the class of equivalence-classes of functions each holomorphic on a neighborhood of Δ, "identifying" two functions which agree on a neighborhood of Δ. We avoid the task of topologizing Hol (Δ, A). Thus the continuity of J (or rather, its analogue) will not be discussed here.

5.3 THEOREM. *Let A be a commutative F-algebra with unit. Let $a_1, \cdots, a_n$ be elements of A. Let Δ be the joint spectrum* 5.24. *Then there is a linear algebra-homomorphism $J(a_1, \cdots, a_n; A)$ of* Hol (Δ, A) *into A which sends constants in A into themselves, and sends the coordinate function z_i into a_i.*

Proof. Let f be a function representing an element of Hol (Δ, A). For each m, f is holomorphic on a neighborhood V of the compact set $\Delta_m = \sigma(\pi_m(a_1), \cdots, \pi_m(a_n); B_m)$. We may thus apply $J(\Delta_m, V, \pi_m(a), B_m)$ to $\pi_m \circ f$, obtaining an element b_m of B_m. With $T = \pi$: $B_{m+1} \rightarrow B_m$, $L =$ the identity, $\Delta = \Gamma = \Delta_{m+1}$, $U = V$, the n-tuple 'a' of 4.3 replaced by '$\pi_{m+1}(a)$', and $b = \pi_m(a) = T(\pi_{m+1}(a))$, we apply 4.3 and obtain $\pi(b_{m+1}) = b_m$, $m = 1, 2, \cdots$. Thus we obtain an element of A, which we call $J(a_1, \cdots, a_n; A)(f)$. It is not hard to see that this J is a homomorphism. For a constant function c, we take $V = \boldsymbol{C}^n$, and we obtain $b_m = \pi_m(c)$, so that $J(a_1, \cdots, a_n; A)(c) = c$. For $f = z_i$ (and in this case we may take $V = \boldsymbol{C}^n$ again), $\pi_m \circ z_i = z_i$ where the second z_i is the scalar-valued function with values in B_m. Therefore $J(\Delta_m, V, \pi_m(a), B_m)(z_i) = \pi_m(a_i)$ for each m. Thus $J(a_1, \cdots, a_n; A) = a_i$.

This completes our proof of 5.3.

We conclude with a remark. This Theorem 5.3 was our original objective in this research. Could we have derived it form [1], at least for scalar-valued f which, candidly, from the most important case? The difficulty in such an attempt lay precisely in trying to make sure that $b_m = \pi(b_{m+1})$. If it were assumed that each B_m was semi-simple, $\pi(b_{m+1})$ would have to be b_m because of the behavior of b_m on $B'_m \cap$ Hom (the behavior in question is that $\xi(b_m) = f(\xi(\pi_m(a_1)), \cdots, \xi_m(\pi_m(a_n))$, which follows from 4.33, and which was, in [1], the only hold one had of b_m). We should be perfectly willing to assume that A were semi-simple because its radical could first be divided out. However, it can really happen that the B_m are not semi-simple even if A is semi-simple. Thus a more careful analysis leading to 4.3 was forced upon us.

Note. I wish to express my thanks to the referee for discovering an error in my previous demonstration of 3.5.

BIBLIOGRAPHY

1. R. Arens and A. P. Calderón, *Analytic functions of several Banach algebra elements,*

Amer. Math., **62**, (1955), 204-216.
2. R. Arens, *Dense inverse limit rings*, Mich. Math. Jour., **5** (1958), 169-182.
3. H. Hefer, *Zur Funktionentheorie mehrerer Veranderlichen*, Math. Annalen **122**, (1950), 276-278.
4. E. Hille and R. S. Phillips, *Functional Analysis and semigroups*, Amer. Math. Soc. Coll. Publ., **31** (1957).
5. E. Michael, *Locally multiplicatively convex topological algebras*, Memoirs of the Amer. Math. Soc., No. **11** (1952).
6. G. E. Shilov, *On the decomposition of a commutative neormed ring into a direct sum of ideals*, Mat. Shornik N.S., **32** (1953), 353-364 in Russian.
7. L. van Hove, *Topologie des espaces fonctionels analytiques*···, Bull. Acad. Belgique, **38** (1952).
8. L. Waelbrock, *Le calcule symbolique dans les algebres commutatives*, J. Math. Pures Appli., **33** (1954), 147-186.
9. A. Weil, *L'integrale de Cauchy et les functions de plusiers variables*, Math. Ann., **111** (1955).
10. H. Whitney, *Geometric integration theory*, Princeton University Press, 1957.

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA