# CONVERGENT SEQUENCES OF FINITELY ADDITIVE MEASURES

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**1. Introduction and Main theorem.** Though sequences of countably additive measures have been investigated by many authors, comparatively little attention has been paid to finitely additive measures in general. The main purpose of this paper is to give a generalization of a classical convergence theorem to the case of finitely additive measures and its improvement.

Let $\mathscr{B}$ be a $\sigma$-complete (infinite) Boolean algebra with the unit $I$. $\mathfrak{M}$ is the class of finitely additive measures on $\mathscr{B}$ with bounded variations, that is, real valued functions $\mu$ on $\mathscr{B}$ with the following properties:

$$\sup_{E\in\mathscr{B}} |\mu(E)| < \infty\,, \qquad \mu(O) = 0$$

and

$$\mu(E\cup F) + \mu(E\cap F) = \mu(E) + \mu(F)$$

for every $E, F \in \mathscr{B}$. We shall call elements of $\mathfrak{M}$ simply *measures*. Under the ordinary addition and scalar multiplication, $\mathfrak{M}$ is a linear space. Moreover it is a universally continuous semi-ordered linear space [6] (=a conditionally complete vector lattice [2]) under the order relation: $\nu \geqq \mu$ means $\nu(E) \geqq \mu(E)$ for every $E \in \mathscr{B}$. The symbols $\vee$ and $\wedge$ will denote supremum and infimum in $\mathfrak{M}$ respectively. We shall write $\mu^+ = \mu \vee 0$, $\mu^- = (-\mu)\vee 0$ and $|\mu| = \mu \vee (-\mu)$, then $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. For each subset $\mathfrak{S}$ of $\mathfrak{M}$ its orthogonal complement, i.e., $\{\mu \,|\, |\mu| \wedge |\nu| = 0$ for every $\nu\in\mathfrak{S}\}$ will be denoted by $\mathfrak{S}^\perp$. We quote some results from the theory of vector lattices (see [6] chap. I). $\mathfrak{S}$ is called *normal*, if $\mathfrak{S} = (\mathfrak{S}^\perp)^\perp$. Any orthogonal complement is normal. Every normal subset is a direct summand, that is, $\mathfrak{M} = \mathfrak{S}\oplus\mathfrak{S}^\perp$ (in linear order sense). Thus each normal subset $\mathfrak{S}$ determines a linear lattice homomorphism of $\mathfrak{M}$ onto $\mathfrak{S}$ which makes $\mathfrak{S}$ invariant. Following [6] § 5 this homomorphism will be denoted by $[\mathfrak{S}]$, that is,

$$[\mathfrak{S}](\alpha\mu + \beta\nu) = \alpha[\mathfrak{S}]\mu + \beta[\mathfrak{S}]\nu \qquad (\alpha, \beta \text{ real})\,,$$
$$[\mathfrak{S}](\mu\vee\nu) = [\mathfrak{S}]\mu \vee [\mathfrak{S}]\nu\,,$$

and $[\mathfrak{S}]\mu = \mu$ is equivalent to $\mu\in\mathfrak{S}$. When $\mathfrak{S} = (\{\nu\}^\perp)^\perp$ where $\{\nu\}$ consists of a single measure $\nu$, the linear operator $[\mathfrak{S}]$ will be denoted simply by $[\nu]$. $\mu$ is said to be *absolutely continuous with respect to* $\nu$,

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if $|\nu|(E)\to 0$ implies $\mu(E)\to 0$. It is known that this is equivalent to the relation $[\nu]\mu=\mu$. A measure $\mu$ is called *countably additive*, if $|\mu|(\bigcup_{k=1}^{\infty}E_k)\leqq\sum_{k=1}^{\infty}|\mu|(E_k)$ for every sequence $\{E_k\}$. The set of countably additive measures is normal. It will be denoted by $\mathfrak{L}$. Following [8] we shall call measures in $\mathfrak{L}^{\perp}$ *purely finitely additive*.

A classical convergence theorem of countably additive measures can be formulated as follows (see [7]):

*Let $\{\mu_k\}$ be a sequence of measures such that $\lim_{k\to\infty}\mu_k(E)$ exists and is finite for every $E\in\mathscr{B}$. If every $\mu_k$ is absolutely continuous with respect to a fixed countably additive measure $\nu$, then the function $\mu(E)\overset{\text{def}}{=}\lim_{k\to\infty}\mu_k(E)$ is also a measure absolutely continuous with respect to $\nu$, and the sequence has the following uniform absolute continuity:*

$$|\nu|(E)\to 0 \quad \textit{implies} \quad \sup_k|\mu_k|(E)\to 0\ .$$

We shall prove the theorem without assumption of countable additivity. Since, as stated before, absolute continuity can be expressed in terms of $[\mathfrak{S}]$ the following theorem will give a more complete answer.

MAIN THEOREM. *If $\lim_{k\to\infty}\mu_k(E)$ exists and is finite for every $E\in\mathscr{B}$, then the function $\mu(E)\overset{\text{def}}{=}\lim_{k\to\infty}\mu_k(E)$ is a measure, and for each normal $\mathfrak{S}$*

$$\lim_{k\to\infty}[\mathfrak{S}]\mu_k(E)=[\mathfrak{S}]\mu(E) \qquad \textit{for every } E\in\mathscr{B}\ .$$

*Moreover the sequence $\{[\mathfrak{S}]\mu_k\}$ has the following uniform absolute continuity:*

$$\nu(E)\to 0\ (\textit{for every } \nu\in\mathfrak{S}) \quad \textit{implies} \quad \sup_k|[\mathfrak{S}]\mu_k|(E)\to 0\ .$$

**2. Proof of Main theorem.** In connection with uniform absolute continuity we begin with some lemmas.

LEMMA 1. *Let $\{\mu_k\}$ be a sequence of measures with the property:*

$$(*) \qquad \lim_{k,j\to\infty}\rho(E_k-E_j)=0$$

*for every monotone sequence $\{E_k\}$ where*

$$\rho(E)=\sup_k|\mu_k|(E)\ .$$

*Then for each sequence $\{F_k\}$ and $\varepsilon>0$ there exist two sequences $\{G_k\}$ and $\{H_k\}$ such that*

$$(1) \qquad G_k=F_k\cup F_{k+1}\cup\cdots\cup F_{j(k)} \qquad \textit{for some } j(k)\ ,$$

$$(2) \qquad G_k \supset H_k \supset H_{k+1} ,$$

$$(3) \qquad \rho(G_k - H_k) \leqq \varepsilon \qquad k = 1, 2, \cdots$$

*Proof.* Since the sequence $F_{kj} \overset{\text{def}}{=} \bigcup_{i=k}^{j} F_i$ $(j \geqq k)$ is monotone (for each fixed $k$), by property $(*)$ there exists a sequence $\{j(k)\}$ of positive integers such that $j(k) \leqq j(k+1)$ and

$$\rho(F_{ki} - F_{k,j(k)}) \leqq \varepsilon/2^k \quad \text{for} \quad i \geqq j(k) .$$

We define the desired sequences by

$$G_k = F_{k,j(k)} \quad \text{and} \quad H_k = \bigcap_{i=1}^{k} G_i \qquad k = 1, 2, \cdots .$$

Then (1) and (2) are trivially satisfied. As to (3)

$$\begin{aligned} \rho(G_k - H_k) &\leqq \rho\Big(\bigcup_{i=1}^{k-1} (G_{i+1} - G_i)\Big) \\ &\leqq \sum_{i=1}^{\infty} \rho(G_{i+1} - G_i) = \sum_{i=1}^{\infty} \rho(F_{i+1,j(i+1)} - F_{i,j(i)}) \\ &\leqq \sum_{i=1}^{\infty} \rho(F_{i,j(i+1)} - F_{i,j(i)}) \leqq \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon. \end{aligned}$$

This completes the proof.

REMARK. Likewise we can choose $\{G_k\}$ and $\{H_k\}$ as follows:

$$(1') \qquad G_k = F_k \cap F_{k+1} \cap \cdots \cap F_{j(k)} \quad \text{for some } j(k) ,$$

$$(2') \qquad G_k \subset H_k \subset H_{k+1} ,$$

$$(3') \qquad \rho(H_k - G_k) \leqq \varepsilon \qquad k = 1, 2, \cdots .$$

LEMMA 2. *For any non negative measures $\nu$ and $\mu$*

$$(4) \qquad [\nu]\mu(E) = \inf \{\sup_k \mu(E_k)\}$$

*where the infimum is taken over all the sequences $\{E_k\}$ such that $\bigcup_k E_k \subset E$ and $\lim_{k\to\infty} \nu(E - E_k) = 0$.*

*Proof.* First remark that $\lim_{k\to\infty}[\nu]\mu(E_k) = [\nu]\mu(E)$ for every such sequence, because $[\nu]\mu$ is absolutely continuous with respect to $\nu$. Since

$$[\nu]\mu(E) = \inf \Big\{\sup_k [\nu]\mu(E_k)\Big\} \leqq \inf \Big\{\sup_k \mu(E_k)\Big\} \leqq \mu(E) ,$$

(because $0 \leqq [\nu]\mu \leqq \mu$), the function $\mu_1(E)$ defined by the right side of (4), is a measure with the property: $[\nu]\mu \leqq \mu_1 \leqq \mu$. If it is proved that $\mu_1$ itself is absolutely continuous with respect to $\nu$ (i.e. $[\nu]\mu_1 = \mu_1$), by the order-preserving property of $[\nu]$ we have

$$[\nu]\mu = [\nu]([\nu]\mu) \leqq [\nu]\mu_1 = \mu_1 \leqq [\nu]\mu ,$$

that is, $[\nu]\mu = \mu_1$. Now suppose that $\mu_1$ is not absolutely continuous with respect to $\nu$, then there exist a sequence $\{F_k\}$ and $\varepsilon > 0$ such that

$$(5) \qquad \nu(F_k) \leqq 1/2^k \quad \text{and} \quad \mu_1(F_k) > 3\varepsilon \qquad k = 1, 2, \cdots .$$

Since condition $(*)$ is evidently satisfied for single $\mu_1$, Lemma 1 guarantees the existence of $\{G_k\}$ and $\{H_k\}$ with the properties (1), (2) and (3) (with $\mu_1$ instead of $\rho$). From (1), (2), (3) and (5) it follows

$$(6) \qquad \begin{gathered} \mu_1(H_k) = \mu_1(G_k) - \mu_1(G_k - H_k) \geqq \mu_1(F_k) - \mu_1(G_k - H_k) \geqq 2\varepsilon \quad \text{i.e.} \\ \inf_k \mu_1(H_k) \geqq 2\varepsilon . \end{gathered}$$

By the definition of $\mu_1$ there can be chosen a double sequence $\{E_{kj}\}$ such that

$$(7) \qquad \begin{gathered} \bigcup_j E_{kj} \subset H_k - H_{k+1} , \qquad \lim_{j\to\infty} \nu(H_k - H_{k+1} - E_{kj}) = 0 , \\ \sup_j \mu(E_{kj}) \leqq \mu_1(H_k - H_{k+1}) + \varepsilon/2^k \qquad k = 1, 2, \cdots . \end{gathered}$$

Writing $D_j = \bigcup_{k=1}^j E_{kj}$, it follows

$$\nu(H_1 - D_j) \leqq \sum_{k=1}^{i-1} \nu(H_k - H_{k+1} - E_{kj}) + \nu(H_i) \xrightarrow{j\to\infty} \nu(H_i) \qquad i = 1, 2, \cdots$$

consequently we have

$$(8) \qquad \lim_{j\to\infty} \nu(H_1 - D_j) = 0 ,$$

because by (5) and (1)

$$\nu(H_i) \leqq \nu(G_i) \leqq \sum_{j=i}^{\infty} \nu(F_j) \leqq \sum_{j=i}^{\infty} 1/2_j = 1/2^{i-1} .$$

On the other hand, on account of (2), (6) and (7)

$$\begin{aligned} \mu(D_j) = \sum_{k=1}^{j} \mu(E_{kj}) &\leqq \sum_{k=1}^{j} \mu_1(H_k - H_{k+1}) + \sum_{k=1}^{\infty} \varepsilon/2^k \\ &= \mu_1(H_1) - \mu_1(H_j) + \varepsilon \leqq \mu_1(H_1) - \varepsilon , \end{aligned}$$

that is,

$$\sup_j \mu(D_j) \leqq \mu_1(H_1) - \varepsilon .$$

Taking (8) into consideration, by the definition of $\mu_1$.

$$\mu_1(H_1) \leqq \sup_j \mu(D_j) \leqq \mu_1(H_1) - \varepsilon .$$

This contradiction establishes the assertion.

We shall reduce a proof of Main theorem to the case of a concrete Boolean algebra. The simplest $\sigma$-complete (infinite) Boolean algebra is the class $\mathscr{N}$ of all subsets of natural numbers. Phillips proved a special case of our Main theorem when $\mathscr{B} = \mathscr{N}$. The following Lemma due to him is essential.

LEMMA 3 (Phillips). *Let* $\{\varphi_k\}$ *be a sequence of measures on* $\mathscr{N}$. *If* $\lim_{k\to\infty}\varphi_k(A)$ *exists and is finite for every* $A \in \mathscr{N}$, *then*

$$\lim_{k\to\infty} \varphi_k(\{k\}) = 0$$

*where* $\{k\}$ *is the set consisting of single* $k$.

This is a slight modification of [1] p. 32 Lemma.

Up to this point the $\sigma$-completeness of $\mathscr{B}$ has not been used, however in the following Lemma it plays a decisive rôle.

LEMMA 4. *If* $\lim_{k\to\infty}\mu_k(E)$ *exists and is finite for every* $E \in \mathscr{B}$, *then* $\sup_k |\mu_k|(I) < \infty$ *and* $(*)$ *is satisfied.*

*Proof.* As proofs of two assertions are similar, we confine ourselves to the proof of $(*)$. Supposing that $(*)$ is not satisfied, we can choose a sequence $\{E_k\}$ and $\varepsilon > 0$ such that $E_1 \subset E_2 \subset \cdots$ and

$$|\mu_k|(E_{k+1} - E_k) > 2\varepsilon \qquad k = 1, 2, \cdots$$

(taking a subsequence of $\{\mu_k\}$ if necessary). Since in general (see [8])

$$|\mu|(E) = \sup_{F\subset E} |\mu(F) - \mu(E - F)|,$$

there exists a sequence $\{F_k\}$ such that

$$F_k \subset E_{k+1} - E_k \quad \text{and} \quad |\mu_k(F_k)| > \varepsilon \qquad k = 1, 2, \cdots .$$

Writing $\varphi_k(A) = \mu_k(\bigcup_{i\in A}F_i)$ for $A \in \mathscr{N}$ (here the $\sigma$-completeness of $\mathscr{B}$ is necessary), we obtain a sequence of measures on $\mathscr{N}$ with the property: $\lim_{k\to\infty}\varphi_k(A)$ exists and is finite for every $A \in \mathscr{N}$. Then Lemma 3 shows that $\mu_k(F_k) = \varphi_k(\{k\}) \xrightarrow{k\to\infty} 0$. This contradiction establishes the assertion.

With these preparations we are now in position to prove Main theorem.

*Proof of Main theorem.* Since $\sup_k |\mu_k|(I) < \infty$ by Lemma 4, the function $\mu(E) \overset{\text{def}}{=} \lim_{k\to\infty}\mu_k(E)$ is a measure. Considering the sequence $\{\mu_k - \mu\}$ instead, we may assume $\mu = 0$. Define

$$\nu(E) = \sum_{k=1}^{\infty} \frac{|[\mathfrak{S}]\mu_k|(E)}{2^k |\mu_k|(I)} \qquad E \in \mathscr{B} .$$

It is not difficult to see that $\nu$ is a measure and $[\mathfrak{S}]\mu_k = [\nu]\mu_k$ $k = 1, 2, \cdots$ (see [6] § 5), hence we may also assume $[\mathfrak{S}] = [\nu]$, and we shall prove

$$\lim_{k\to\infty} [\nu]\mu_k(E) = 0 \qquad \text{for every } E \in \mathscr{B} .$$

By Lemma 2 for each $E$ there exist sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ such that

$$\bigcup_j C_{kj} \subset E , \qquad \bigcup_j D_{kj} \subset E ,$$

$$\lim_{j\to\infty} \nu(E - C_{kj}) = \lim_{j\to\infty} \nu(E - D_{kj}) = 0 ,$$

$$\lim_{j\to\infty} \mu_k^+(C_{kj}) = [\nu]\mu_k^+(E) ,$$

and

$$\lim_{j\to\infty} \mu_k^-(D_{kj}) = [\nu]\mu_k^-(E) \qquad k = 1, 2, \cdots .$$

Since

$$\nu(E - C_{kj} \cap D_{kj}) \leqq \nu(E - C_{kj}) + \nu(E - D_{kj}) \xrightarrow{j\to\infty} 0$$

by Lemma 2 we obtain

$$[\nu]\mu_k^+(E) \leqq \overline{\lim_{j\to\infty}}\, \mu_k^+(E_{kj}) \leqq \lim_{j\to\infty} \mu_k^+(C_{kj}) = [\nu]\mu_k^+(E) ,$$

where $E_{kj} = C_{kj} \cap D_{kj}$, similarly

$$\lim_{j\to\infty} \mu_k^-(E_{kj}) = [\nu]\mu_k^-(E) \qquad k = 1, 2, \cdots .$$

Writing $F_{kj} = \bigcap_{i=1}^k E_{ij}$, the similar arguments show

$$\lim_{j\to\infty} \nu(E - F_{kj}) = 0$$

and

$$\lim_{j\to\infty} \mu_i^+(F_{kj}) = [\nu]\mu_i^+(E) \qquad i = 1, 2, \cdots, k ,$$

and similarly for $\{\mu_k^-\}$. By a diagonal method we can find a subsequence $\{F_i\}$ of $\{F_{kj}\}$ such that

$$\nu(E - F_i) < 1/2^i \qquad i = 1, 2, \cdots$$

and

$$\lim_{i\to\infty} \mu_k^+(F_i) = [\nu]\mu_k^+(E) \qquad k = 1, 2, \cdots$$

and similarly for $\{\mu_k^-\}$. Since condition (*) is satisfied by Lemma 4, there exist sequences $\{G_k\}$ and $\{H_k\}$ with the properies (1′), (2′) and (3′).

On account of (1′), by the similar way as above, it is not difficult to see that

$$\lim_{j\to\infty} \mu_k^+(G_j) = [\nu]\mu_k^+(E) \qquad k = 1, 2, \cdots$$

and similarly for $\{\mu_k^-\}$, hence by subtraction

$$\lim_{j\to\infty} \mu_k(G_j) = [\nu]\mu_k(E) \qquad k = 1, 2, \cdots . \tag{9}$$

On the other hand, for each $\varepsilon > 0$ (*) and (2′) guarantee the existence of $n = n(\varepsilon)$ such that

$$\rho(H_j - H_n) \leqq \varepsilon \qquad \text{for } j \geqq n ,$$

consequently by (3′)

$$| \mu_k(H_n) - \mu_k(G_j) | \leqq \rho(H_j - H_n) + \rho(H_j - G_j) \leqq 2\varepsilon \quad \text{for } j \geqq n . \tag{10}$$

Then from (9) and (10) it follows

$$| \mu_k(H_n) - [\nu]\mu_k(E) | = \lim_{j\to\infty} | \mu_k(H_n) - \mu_k(G_j) | \leqq 2\varepsilon \quad k = 1, 2, \cdots .$$

Since $\lim_{k\to\infty}\mu_k(H_n) = 0$ by hypothesis, combining this with the above, we obtain $\overline{\lim}_{k\to\infty}[\nu]\mu_k(E) \leqq 2\varepsilon$. The arbitrariness of $\varepsilon$ implies $\lim_{k\to\infty}[\nu]\mu_k(E) = 0$.

Next we shall turn to a proof of the second assertion. Here we may again assume $[\mathfrak{S}] = [\nu]$. First remark that the sequence $\{[\nu]\mu_k\}$ satisfies (*). Given a sequence $\{F_k\}$ with the property: $\nu(F_k) < 1/2^k$ $k = 1, 2, \cdots$, applying Lemma 1 to this sequence and $\{[\nu]\mu_k\}$, for any $\varepsilon > 0$ we can find sequences $\{G_k\}$ and $\{H_k\}$ with properties (1), (2) and (3) (with $\rho(E) = \sup_k | [\nu]\mu_k | (E)$). Since (*) and the absolute continuity of every $[\nu]\mu_k$ (with respect to $\nu$) imply $\rho(H_k) \xrightarrow{k\to\infty} 0$, we obtain $\overline{\lim}_{k\to\infty}\rho(F_k) \leqq 2\varepsilon$, because

$$\rho(F_k) \leqq \rho(G_k) \leqq \rho(H_k) + \rho(G_k - H_k) \leqq \rho(H_k) + 2\varepsilon \quad k = 1, 2, \cdots .$$

The arbitrariness of $\varepsilon$ establishes the assertion.

REMARK. When $\mathscr{B}$ is moreover *complete*, our Main theorem can be deduced from a result of Grothendieck [3] by the following way. By means of the theory of Boolean algebra, $\mathscr{B}$ can be represented by the class of open-closed subsets of a compact *Stonian* space $\Omega$. Then by the natural way $\mathfrak{M}$ may be considered as the dual of the Banach space $C(\Omega)$ (the space of continuous functions on $\Omega$ with the supremum norm). Grothendieck proved that if $\mu_k$ is $\sigma(\mathfrak{M}, C(\Omega))$-convergent, then it is also $\sigma(\mathfrak{M}, \mathfrak{M}')$-convergent, where $\mathfrak{M}'$ is the dual of $\mathfrak{M}$ and $\sigma(\cdots)$ denotes the weak topology. Since every operator $[\mathfrak{S}]$ is $\sigma(\mathfrak{M}, \mathfrak{M}')$-continuous, the first part of our Main theorem follows immediately. The second part is also stated there.

**3. Corollaries.** An immediate corollary (which is a direct generalization of the classical theorem in question) is as follows.

COROLLARY 1. *Every normal set of $\mathfrak{M}$ is sequentially complete under the weak topology defined by $\mathscr{B}$.*

Let $\{E_\lambda\}$ be the set of all atoms in $\mathscr{B}$ and

$$\mathfrak{P} = \{\mu \in \mathfrak{L} \mid |\mu|(E_\lambda) = 0 \text{ for every } E_\lambda\}\ .$$

Then $\mathfrak{P}$ is normal. A modification of a recent result of Kaplan [5] § 9 shows that the closure of $\mathfrak{L}^\perp$ with respect to the topology in question coincides with $\mathfrak{P} \oplus \mathfrak{L}^\perp$. In this regards, the following special case of Corollary 1 is of some interest.

COROLLARY 2. *The set of purely finitely additive measures is sequentially complete under the weak topology defined by $\mathscr{B}$.*

**4. Sequences of purely finitely additive measures.** In connection with Main theorem a question arises whether $\mu_k \xrightarrow{k\to\infty} 0$ implies $|\mu_k| \xrightarrow{k\to\infty} 0$. When $\sup_{\mu\in\mathfrak{L}\mathfrak{B}}\mu(E) > 0$, for every $E \in \mathscr{B}$, Halperin and Nakano [4] proved that the property "$\mu_k \in \mathfrak{L}$, $\mu_k \xrightarrow{k\to\infty} 0$ implies $|\mu_k| \xrightarrow{k\to\infty} 0$" is equivalent to the atomicity of $\mathscr{B}$. We shall treat this problem in $\mathfrak{L}^\perp$, namely, does $\mu_k \xrightarrow{k\to\infty} 0$ (all $\mu_k$ being purely finitely additive) imply $|\mu_k| \xrightarrow{k\to\infty} 0$? The answer is negative.

THEOREM. *There exists a sequence $\{\mu_k\}$ of purely finitely additive measures such that*

$$\lim_{k\to\infty} \mu_k(E) = 0 \qquad \textit{for every } E \in \mathscr{B}$$

*but*

$$|\mu_k|(I) = 1 \qquad k = 1, 2, \cdots .$$

*Proof.* As in § 2 we shall reduce the proof to the case $\mathscr{B} = \mathscr{N}$. Banach (see [1] p. 83) proved that there exists a positive measure $\varphi_0$ on $\mathscr{N}$ invariant under translation, that is, $\varphi_0(I_0) = 1$ and

$$0 \leqq \varphi_0(A) = \varphi_0(\tau A) \qquad \text{for every } A \in \mathscr{N}$$

where $I_0$ is the unit of $\mathscr{N}$ and $\tau A = \{j + 1 \mid j \in A\}$. We define a sequence $\{\varphi_k\}$ recurrently by the formula

$$\varphi_{k+1}(A) = \varphi_k\Big(A \cap \bigcup_{j=1}^{2^k} B_{k+1,j}\Big) - \varphi_k\Big(A \cap \bigcup_{j=2^k+1}^{2^{k+1}} B_{k+1,j}\Big)$$

where

$$B_{kj} = \{i \mid i \equiv j \mod 2^k\} .$$

From the arguments in [4] it results

$$\lim_{k\to\infty} \varphi_k(A) = 0 \qquad \text{for every } A \in \mathscr{N} \tag{11}$$

but

$$|\varphi_k| = \varphi_0 \qquad k = 1, 2, \cdots$$

Let $\{F_k\}$ be a sequence in $\mathscr{B}$ with the property

$$\bigcup_k F_k = I \quad \text{and} \quad F_k \cap F_j = 0 \qquad (k \neq j) . \tag{12}$$

On account of the representation theorem of Boolean algebra (see [6] § 8; [2] Chap. X) there exists a sequence $\{\nu_k\}$ of two-valued (say 1 and 0) measures on $\mathscr{B}$ such that

$$\nu_k(F_j) = \delta_{kj} \qquad k, j = 1, 2, \cdots .$$

We construct the desired sequence $\{\mu_k\}$ from $\{\varphi_k\}$ and $\{\nu_k\}$ by the formula:

$$\mu_k(E) = \varphi_k(A)$$

where $A = \{j \mid \nu_j(E) = 1\}$. From (11) and (12) it results

$$\lim_{k\to\infty} \mu_k(E) = 0 \qquad \text{for every } E \in \mathscr{B} .$$

but $|\mu_k| = \mu_0$ $k = 1, 2, \cdots$. There remains to prove pure finite additivity of $\mu_k$. For this purpose it is enough to prove it for $\mu_0$ only. Invariance of $\varphi_0$ under translation shows

$$\mu_0(F_j) = \varphi_0(\{j\}) = \varphi_0(\{i\}) = \mu_0(F_i) \qquad i, j = 1, 2, \cdots$$

hence

$$k\mu_0(F_j) = \sum_{i=1}^{k} \mu_0(F_i) \leqq \mu_0(I) = 1 \qquad j, k = 1, 2, \cdots$$

finally

$$\mu_0(F_j) = 0 \qquad j = 1, 2, \cdots$$

Since $\mu_0(I_j) = 1$, this implies pure finite additivity (cf. [8] § 4).

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