AUTOMORPHISMS OF SEPARABLE ALGEBRAS

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1. Introduction. In this note we begin by noticing that for any commutative ring C, the isomorphism classes of finitely generated, projective C-modules of rank one (for the definition, see § 2) form an abelian group $\mathscr{J}(C)$ which reduces to the ordinary ideal class group if C is a Dedekind domain. In [2], Auslander and Goldman proved that if $\mathscr{J}(C)$ contains only one element then every automorphism of every central separable C-algebra is inner. Using similar techniques, we prove that for general C and for any central separable C-algebra A, $\mathscr{J}(C)$ contains a subgroup isomorphic to the group of automorphisms of A modulo inner ones. We characterize both this subgroup and the factor group. For example, in the case of an integral domain or a noetherian ring, the subgroup is the set of classes of projective ideals in C which become principal in A (i.e., Ker β in Theorem 7). If C is a Dedekind ring and A is the (split) algebra of endomorphisms of a projective C-module of rank n, the subgroup is the set of classes of ideals whose nth powers are principal.

2. Generalization of the ideal class group. Let C be a commutative ring[1] and let J be a projective C-module. Then for every maximal ideal M in C, the module[2] $J \otimes C_M$ is a projective, hence free, C_M-module. Following [7, § 3] we say J has rank one if for all M, $J \otimes C_M$ is free on one generator,[3] i.e. $J \otimes C_M \cong C_M$ as C_M-modules.

DEFINITION. $\mathscr{J}(C)$ will denote the set of isomorphism classes of finitely generated, projective, rank one C-modules. If J is a finitely generated, projective, rank one C-module, $\{J\}$ will denote the isomorphism class of J.

We note that if $\{J\} \in \mathscr{J}(C)$ then J is faithful, for if an ideal I annihilates J then $0 = I(J \otimes C_M) \cong IC_M \cong I \otimes C_M$ for every M, and so $I = 0$ [4, Chap. VII, Ex. 11].

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[1] All rings will be assumed to have units, all modules will be unitary, and if R is a subring of S then R will contain the unit element of S. A homomorphism of rings will preserve unit elements.

[2] The unadorned $\otimes$ always means tensor product over C. C_M denotes the ring of quotients of C with respect to the maximal ideal M.

[3] $J \otimes C_M \cong C_M$ for all M does not imply that J is either finitely generated or projective. For example, let C be the ring of integers and $J = \bigcup_n C p_1^{-1} \cdots p_n^{-1}$ where p_i is the ith prime.

LEMMA 1. *$\mathscr{J}(C)$ is an abelian group under the operation $\otimes$. The identity element is C.*

Proof. The only nontrivial item is the existence of inverses. If $\{J\} \in \mathscr{J}(C)$, let $J^* = \operatorname{Hom}_C(J, C)$. Since Hom distributes over direct sums and since J is a direct summand in a finite direct sum of C's we see that J^* has the same property. Furthermore $J^* \otimes C_M \cong \operatorname{Hom}_{C_M}(J \otimes C_M, C_M) \cong C_M$ [4, Chap. VI, Ex. 11] so that $J^* \in \mathscr{J}(C)$. Since J is faithful, the mapping $J \otimes J^* \to C$ defined by $x \otimes f \to f(x)$ is known to be an epimorphism [1, Prop. A. 3]. If its kernel is K then $K \otimes C_M = \operatorname{Ker}(J \otimes J^* \otimes C_M \to C \otimes C_M) = 0$ for each M. Thus $K = 0$, $J \otimes J^* \cong C$ and J^* is the inverse of J.

If C is semisimple (with minimum condition) then $\mathscr{J}(C) = 1$. Using [6, Lemma 3.14] it is easily seen that if N is a radical ideal in a ring C and $\mathscr{J}(C/N) = 1$ then $\mathscr{J}(C) = 1$; therefore $\mathscr{J}(C) = 1$ whenever C is semilocal (i.e. C has only finitely many maximal ideals, but is not necessarily noetherian). This fact also follows from Serre's theorem on the structure of projective modules over semilocal rings [7, Prop. 6 and 6, Lemma 3.15].

When C is an integral domain, $\mathscr{J}(C)$ is the ordinary group of (projective) ideal classes. We proceed to (prove and) generalize this statement by considering the functorial properties of $\mathscr{J}(C)$.

If C and D are commutative rings and $C \to D$ is a ring homomorphism, there is a corresponding homomorphism $\mathscr{J}(C) \to \mathscr{J}(D)$ given by $\{J\} \to \{J \otimes D\}$: Clearly $J \otimes D$ is a finitely generated projective D-module. To prove that it is of rank one, let N be any maximal ideal of D and let M be any maximal ideal of C containing the kernel of the composite homomorphism $C \to D \to D/N$. Since every element of C not in M maps into a unit of D_N, we have a homomorphism $C_M \to D_N$. Thus $(J \otimes D) \otimes_D D_N \cong J \otimes D_N \cong (J \otimes C_M) \otimes_{C_M} D_N \cong D_N$.

If S is a multiplicatively closed subset of C containing no zero-divisors we define an analog, $\mathscr{I}(C, S)$ of the ideal class group of an integral domain as follows:

Two ideals I and I' of C are equivalent if $I' = uI$ for some unit u in the ring of quotients C_S. Then $\mathscr{I}(C, S)$ is the set of equivalence classes of projective ideals of C which meet S.[4] Multiplication of ideals induces a product in $\mathscr{I}(C, S)$. Among other things, the following lemma shows that $\mathscr{I}(C, S)$ is a group.

LEMMA 2. *$\mathscr{I}(C, S) \cong \operatorname{Ker}(\mathscr{J}(C) \longrightarrow \mathscr{J}(C_S))$.*

Proof. Let the class of I belong to $\mathscr{I}(C, S)$. Then some element

[4] The same proof as in [4, Chap. VII, Prop. 3.3] shows that such an ideal is finitely generated.

of I is not a zero divisor, and consequently $I \otimes C_M \neq 0$ for each M. Hence $I \otimes C_M$ is a nonzero projective (hence free) ideal of C_M and so $I \otimes C_M \cong C_M$. Moreover, if I' is in the same class as I then I and I' are isomorphic C-modules. Conversely, if δ is an isomorphism of I with I', then $\delta \otimes 1$ gives an isomorphism of $C_S = I \otimes C_S$ with $C_S = I' \otimes C_S$. Thus $\delta \otimes 1$ is simply multiplication by a unit u of C_S, and so $I' = Iu$. Finally, by [4, Chap. VI, Ex. 19] $I \otimes I' \cong II'$, and therefore mapping the class of I in $\mathscr{I}(C, S)$ to $\{I\}$ in $\mathscr{J}(C)$ yields a monomorphism of $\mathscr{I}(C, S)$ into $\mathscr{J}(C)$.

As we already noted $I \otimes C_S = IC_S = C_S$ so that the image of I lies in $\operatorname{Ker}(\mathscr{J}(C) \to \mathscr{J}(C_S))$. On the other hand, if $\{J\} \in \mathscr{J}(C)$ lies in this kernel, $J \otimes C_S \cong C_S$ and so J is isomorphic to a C-submodule of C_S. Since J is finitely generated, it is isomorphic to an ideal I of C and $I \otimes C_S = IC_S = C_S$. Hence $I \cap S \neq \phi$ and the class of I lies in $\mathscr{I}(C, S)$.

COROLLARY 3. *If C is an integral domain or a noetherian ring and S is the complement of the set of zero divisors then $\mathscr{J}(C_S) = 1$, and hence $\mathscr{I}(C, S) \cong \mathscr{J}(C)$.*

Proof. If C is an integral domain this is now clear. If C is noetherian, S is the complement of the union of the primes of zero. Since there are only finitely many of these, the standard theorems concerning the relation of ideals in C and C_S show that C_S is a semilocal ring and so by the remarks following Lemma 1, $\mathscr{J}(C_S) = 1$.

3. Separable algebras. If A is an algebra over the commutative ring C, A is said to be separable over C if the left A^e-module[5] A is projective. Central separable C-algebras are a natural generalization of central simple algebras, and their basic theory has been given in [2] and [3]. In particular, we single out the following results which we use several times:

PROPOSITION 4. *Let A be a central separable C-algebra and X a left A^e-module. Then $X \cong A \otimes Y$ as A^e-modules where the C-module $Y = \{x \in X \mid ax = xa \text{ for all } a \text{ in } A\} \cong \operatorname{Hom}_{A^e}(A, X)$. The C-module Y is unique: If $X \cong A \otimes Y'$ as A^e-modules then $Y' \cong Y$ as C-modules. The following three statements are equivalent:*

(a) *X is a finitely generated projective C-module*
(b) *Y is a finitely generated projective C-module*
(c) *X is a finitely generated projective A^e-module.*

Proof. The first assertion is [2, Theorem 3.1]. As for the uniqueness: $Y \cong \operatorname{Hom}_{A^e}(A, A \otimes Y') \cong \operatorname{Hom}_{A^e}(A, A) \otimes Y' \cong C \otimes Y' \cong Y'$ where

[5] The algebra A^e is the tensor product over C of A and its opposite.

all the isomorphisms are C-isomorphisms; the second isomorphism follows from the statement "φ_3 is an isomorphism" on p. 210 of [4] if Y' is identified with $\mathrm{Hom}_C(C, Y')$; the third isomorphism follows from $\mathrm{Hom}_{A^e}(A, A) \cong C$ which is the condition that A is central.

For the rest, we prove the implications a $\Rightarrow$ b $\Rightarrow$ c $\Rightarrow$ a. Since A is a finitely generated projective A^e-module, the C-module $Y \cong \mathrm{Hom}_{A^e}(A, X)$ is a direct summand in a finite direct sum of copies of X. Thus if X is a finitely generated projective C-module, Y is also. If Y is finitely generated and C-projective then $X \cong A \otimes Y$ is an A^e-direct summand in a finite direct sum of copies of A's and thus a finitely generated projective A^e-module. Finally, since A is a finitely generated projective C-module [2, Theorem 2.1], A^e is also, and so if X is finitely generated and projective as an A^e-module it has the same properties as C-module.

As is usual in the study of simple algebras, for any central separable C-algebra A and a pair of C-algebra automorphisms σ, τ of A we make A into a new A^e-module, ${}_\sigma A_\tau$ by defining

$$(x \otimes y)(a) = \sigma(x)a\tau(y) \qquad \text{for } x \otimes y \in A^e,\ a \in A.$$

Of course as a C-module, ${}_\sigma A_\tau \cong A$ and so is finitely generated and projective.

Since ${}_\sigma A_\tau$ is A^e-isomorphic to ${}_{\rho\sigma}A_{\rho\tau}$ by the mapping $a \to \rho(a)$, we need only be concerned about ${}_\sigma A_\tau$ with $\tau = 1$. Proposition 4 shows that ${}_\sigma A_1$ is isomorphic to $A \otimes J_\sigma$ with $J_\sigma = \{a \in A \mid \sigma(x)a = ax \text{ for all } x \text{ in } A\}$, a finitely generated projective C-module. Moreover the chain of C_M-module isomorphisms

$$A \otimes C_M \cong {}_\sigma A_1 \otimes C_M \cong (A \otimes C_M) \otimes_{C_M} (J_\sigma \otimes C_M)\ ,$$

together with the fact that $A \otimes C_M$ is a finitely generated free C_M-module, shows that $J_\sigma \otimes C_M \cong C_M$ and so $\{J_\sigma\} \in \mathscr{J}(C)$.

LEMMA 5. *$J_\sigma \cong C$ if and only if σ is an inner automorphism. Also $J_\sigma \otimes J_\tau \cong J_{\tau\sigma}$.*

Proof. The first part of the Lemma is essentially [2, Theorem 3.6] (cf. also [5, p. 143]): If $\sigma(x) = uxu^{-1}$ then $J_\sigma = \{a \in A \mid uxu^{-1}a = ax$ for all x in $A\} = \{a \in A \mid x(u^{-1}a) = (u^{-1}a)x$ for all x in $A\} = uC \cong C$. Conversely, suppose J_σ is a free C-module on one generator, u. Since the isomorphism $A \otimes J_\sigma \cong {}_\sigma A_1$ is defined by $a \otimes j \to ja$, we have $A = J_\sigma A = uA = \sigma(A)u = Au$. Thus u is a unit in A lying in J_σ. The definition of J_σ then shows that $\sigma(x) = uxu^{-1}$ for all x in A.

By Proposition 4 and the remark following the definition of ${}_\sigma A_\tau$ we have the following chain of A^e-isomorphisms:

$$A \otimes J_\tau \otimes J_\sigma \cong ({}_\sigma A_1) \otimes_A ({}_\tau A_1) \cong ({}_\tau A_1) \otimes_A ({}_1 A_{\tau^{-1}}) \cong {}_\sigma A_{\tau^{-1}} \cong {}_{\tau\sigma}A_1 \cong A \otimes J_{\tau\sigma}\ .$$

The uniqueness statement in Proposition 4 then asserts $J_\sigma \otimes J_\tau \cong J_{\tau\sigma}$.

DEFINITION. If A is a central separable C-algebra, $\mathscr{O}(A)$ denotes the group of automorphisms of A modulo inner ones.

By Lemma 5 the mapping $\sigma \to J_\sigma$ induces a group monomorphism α: $\mathscr{O}(A) \to \mathscr{J}(C)$.

COROLLARY 6. *$\mathscr{O}(A)$ is an abelian group.*

We next obtain a description of $Im\,\alpha$.

DEFINITION. $\mathscr{J}(A)$ is the set of left A-isomorphism classes of left A^e-modules P with the properties

(i) P is C-projective and finitely generated

(ii) $P \otimes C_M \cong A \otimes C_M$ as C_M-modules for all M.

For the same reasons that ${}_\sigma A_1 \cong A \otimes J_\sigma$ with $\{J_\sigma\} \in \mathscr{J}(C)$, we have that if $\{P\} \in \mathscr{J}(A)$ then $P \cong A \otimes J$ as A^e-modules with $J \in \mathscr{J}(C)$. Since, conversely, $\{J \otimes A\} \in \mathscr{J}(A)$ whenever $\{J\} \in \mathscr{J}(C)$, we see that $\mathscr{J}(A)$ is just the set of left A-isomorphism classes of A^e-modules $A \otimes J$ with $\{J\} \in \mathscr{J}(C)$.

Note that if we had defined $\mathscr{J}(A)$ to be the set of A^e-isomorphism classes instead of left A-isomorphism classes, we would have had a set in one-to-one correspondence with $\mathscr{J}(C)$, according to Proposition 4. See also the remark after Theorem 7.

There is a natural multiplication in $\mathscr{J}(A)$: $(P_1, P_2) \to P_1 \otimes_A P_2$. If $P_1 = A \otimes J_1$ and $P_2 = A \otimes J_2$ then $P_1 \otimes_A P_2 \cong A \otimes (J_1 \otimes J_2)$. Thus the mapping β: $\mathscr{J}(C) \to \mathscr{J}(A)$ defined by $\{J\} \to \{A \otimes J\}$ is an epimorphism and so $\mathscr{J}(A)$ is a group.

THEOREM 7. *The sequence*

$$1 \longrightarrow \mathscr{O}(A) \xrightarrow{\alpha} \mathscr{J}(C) \xrightarrow{\beta} \mathscr{J}(A) \longrightarrow 1$$

is exact.

Proof. The only thing that still needs to be shown is that $Im\,\alpha = \operatorname{Ker}\beta$. If $\{J\} \in Im\,\alpha$ then $J = J_\sigma$ and $\beta\{J\} = \{A \otimes J_\sigma\} = \{{}_\sigma A_1\} = \{{}_1 A_{\sigma^{-1}}\} = \{A\}$ which is the unit element of $\mathscr{J}(A)$. Thus $Im\,\alpha \subset \operatorname{Ker}\beta$. Conversely, if $\{J\} \in \operatorname{Ker}\beta$ then $P = A \otimes J$ is left A-isomorphic to A. That is, each element of P is of the form $p = aw$ for some fixed w in P and for suitable a in A, uniquely determined by p. Since P is an A^e-module, $wa \in P$ for every a in A. Hence $wa = \sigma(a)w$ where σ is a well defined mapping of A to A. It is trivial to verify that σ is a C-algebra endomorphism. Now by [2, Theorem 3.5] σ is an automorphism and so

$a \to \sigma(a)w$ is an A^e-isomorphism of ${}_{\sigma^{-1}}A_1$ to P. Thus $P \cong A \otimes J_{\sigma^{-1}}$. By the uniqueness in Proposition 4, $J \cong J_{\sigma-1}$ and $\{J\} \in Im\,\alpha$.

REMARK. The proof of Theorem 7 shows that $A \otimes J \cong A \otimes J'$ as left A-modules if and only if $J \cong J' \otimes J''$ as C-modules with $\{J''\} \in Im\,\alpha$. Clearly the same proof will show $A \otimes J \cong A \otimes J'$ as *right* A-modules if and only if $J \cong J' \otimes J''$ with $\{J''\} \in Im\,\alpha$. Thus, given two A^e-modules P and P', satisfying conditions (i) and (ii) in the definition of $\mathscr{J}(A)$, the following conditions are equivalent: $P \cong P'$ as left A-modules; $P \cong P'$ as right A-modules; $P \cong P$ both as left and as right A-modules. This means that $\mathscr{J}(A)$ could equally well have been defined as the set of right A-isomorphism classes or as the set of left and right A-isomorphism classes (but not as the set of A^e-isomorphism classes).

In the theory of separable algebras the role of full matrix algebras over fields is played by the *split* algebras, i.e. algebras of the form $Hom_C(V, V)$ with V a finitely generated, faithful, projective C-module. For such algebras we give a fuller description of $\mathscr{J}(A)$.

DEFINITION. For any finitely generated, faithful, projective C-module V, let $\mathscr{J}(V)$ be the set of C-isomorphism classes of finitely generated projective C-modules W such that $\mathrm{Hom}_C(W, W)$ is a C-algebra isomorphic to $\mathrm{Hom}_C(V, V)$.

LEMMA 9. *$W \in \mathscr{J}(V)$ if and only if $W \cong V \otimes J$ as C-modules, for some C-module J with $\{J\} \in \mathscr{J}(C)$.*

Proof. Let $\{J\} \in \mathscr{J}(C)$. By [4, p. 210] there is a natural isomorphism as C-modules, and so also as C-algebras, $\mathrm{Hom}_C(V \otimes J, V \otimes J) \cong \mathrm{Hom}_C(V, V) \otimes \mathrm{Hom}_C(J, J)$. But $\mathrm{Hom}_C(J, J) = C$ since C is embedded in $\mathrm{Hom}_C(J, J)$ by its action on the module J, and since, for every M, $C_M \cong \mathrm{Hom}_{C_M}(J \otimes C_M, J \otimes C_M) \cong \mathrm{Hom}_C(J, J) \otimes C_M$ [4, Chap. VI, Ex. 11]. Thus $\{V \otimes J\} \in \mathscr{J}(V)$ for $\{J\} \in \mathscr{J}(C)$. Conversely, if $\{W\} \in \mathscr{J}(V)$ then W is a module over $A = \mathrm{Hom}_C(V, V)$. By [1, Prop. A. 3 and Prop. A. 6] there is a C-isomorphism $W \cong V \otimes J$ with $J = \mathrm{Hom}_A(V, W)$. Since V is projective and finitely generated as A-module, $J = \mathrm{Hom}_A(V, W)$ is projective and finitely generated as a C-module. Since both $\mathrm{Hom}_{C_M}(W \otimes C_M, W \otimes C_M) \cong \mathrm{Hom}_C(W, W) \otimes C_M$ and $\mathrm{Hom}_{C_M}(V \otimes C_M, V \otimes C_M) \cong \mathrm{Hom}_C(V, V) \otimes C_M$ are matrix rings of the same size over C_M, $V \otimes C_M$ and $W \otimes C_M$ are free C_M-modules on the same number of generators. Theefore $V \otimes C_M \cong W \otimes C_M \cong (V \otimes C_M) \otimes_{C_M} (J \otimes C_M)$, which forces $J \otimes C_M \cong C_M$, and so $J \in \mathscr{J}(C)$.

By Lemma 9 we can define a multiplication in $\mathscr{J}(V)$ by $(V \otimes J_1, V \otimes J_2) \to V \otimes J_1 \otimes J_2$. Then β': $\mathscr{J}(C) \to \mathscr{J}(V)$, given by $J \to V \otimes J$, is an epimorphism, and so $\mathscr{J}(V)$ is an abelian group with unit V.

LEMMA 10. $\operatorname{Ker}\beta = \operatorname{Ker}\beta'$. *Hence if* $A = \operatorname{Hom}_C(V, V)$, *the mapping* γ: $\mathscr{J}(A) \to \mathscr{J}(V)$ *given by* $A \otimes J \to V \otimes J$ *is an isomorphism making a commutative diagram*:

$$\begin{tikzcd} & \mathscr{J}(A) \arrow[d, "\gamma"] \\ \mathscr{J}(C) \arrow[ur, "\beta"] \arrow[dr, "\beta'"'] & \\ & \mathscr{J}(V) \end{tikzcd}$$

Proof. If $\{J\} \in \operatorname{Ker}\beta'$, the C-modules $V \otimes J$ and V are isomorphic. Then clearly the left A-modules[6] $A = \operatorname{Hom}_C(V, V)$ and $\operatorname{Hom}_C(V, V \otimes J)$ are isomorphic. However the latter module is isomorphic to

$$\operatorname{Hom}_C(V, V) \otimes \operatorname{Hom}_C(C, J) \cong A \otimes J,$$

and so $\{J\} \in \operatorname{Ker}\beta$.

Inversely, if $\{J\} \in \operatorname{Ker}\beta$, then by Theorem 7, $J = J_\sigma$ for some automorphism σ of A. We prove $V \otimes J_\sigma \cong V$ by localizing. We first compute $J_\sigma \otimes C_M$. From the definition of J_σ, ${}_\sigma A_1 \cong A \otimes J_\sigma$ and so $A \otimes J_\sigma \otimes C_M \cong {}_\sigma A_1 \otimes C_M = {}_{\sigma\otimes 1}(A \otimes C_M)_{1\otimes 1}$. By the uniqueness part of Proposition 4, $J_\sigma \otimes C_M \cong J_{\sigma\otimes 1}$. Furthermore, since C_M is local, $\mathscr{J}(C_M) = 1$ and thus by Theorem 7, $\sigma \otimes 1$ is an inner automorphism of $A \otimes C_M$. The last part of the proof of Lemma 5 then shows that $J_\sigma \otimes C_M = C_M u$ with u a unit in $A \otimes C_M$.

Next, since $J_\sigma \subset \operatorname{Hom}_C(V, V) = A$, there is a C-module homomorphism θ: $V \otimes J_\sigma \to V$ defined by $\theta(v \otimes j) = vj$. Then $\theta \otimes 1$ maps $V \otimes J_\sigma \otimes C_M$ to $V \otimes C_M$ and, in fact, if we write $J_\sigma \otimes C_M = C_M u$, $(\theta \otimes 1)(v \otimes cu) = (v \otimes c)u$ for v in V, c in C_M; here $(v \otimes c)u$ is defined because $u \in A \otimes C_M$ and $V \otimes C_M$ is an $(A \otimes C_M)$-module. Since u is a unit in $A \otimes C_M$, $\theta \otimes 1$ is an isomorphism. Hence if U and V are the kernel and cokernel of θ respectively $U \otimes C_M = V \otimes C_M = 0$ for all M. This proves that $U = V = 0$, and θ is an isomorphism. Hence $\{J\} \in \operatorname{Ker}\beta'$.

THEOREM 11. *If* $A = \operatorname{Hom}_C(V, V)$ *with* V *a faithful, finitely generated, projective C-module, the sequence*

$$1 \longrightarrow \mathscr{O}(A) \xrightarrow{\alpha} \mathscr{J}(C) \xrightarrow{\beta'} \mathscr{J}(V) \longrightarrow 1$$

is exact.

COROLLARY 12. *If* $\mathscr{J}(C) = 1$ *then not only is every automorphism of every central separable C-algebra inner* (i.e. $O(A) = 1$ *for all* A), *but also, for every split C-algebra* $\operatorname{Hom}_C(V, V)$, *the module* V *is uniquely determined* (i.e. $\mathscr{J}(V) = 1$ *for every* V, *and, in fact* $\mathscr{J}(A) = 1$ *for every* A). *Conversely, if for some central separable C-algebra* A [resp.

[6] We consider V a right A-module, so that $\operatorname{Hom}_C(V, X)$ becomes a left A-module.

split C-algebra $A = \mathrm{Hom}_C(V, V)$] *both* $\mathcal{O}(A)$ *and* $\mathcal{J}(A)$ *are trivial* [resp., $\mathcal{O}(A) = \mathcal{J}(V) = 1$] *then* $\mathcal{J}(C) = 1$ *and so* $\mathcal{O}(A) = \mathcal{J}(A) = 1$ *for every* A.

If we change the base ring C the exact sequences in Theorems 7 and 11 behave in the expected way: Specifically, if $C \to D$ is a ring homomorphism and if A is a central separable C-algebra then $A \otimes_C D$ is a central separable D-algebra [2, Corollary 1.6] and Theorem 7 yields the exact sequence

$$(2) \qquad 1 \longrightarrow \mathcal{O}(A \otimes D) \xrightarrow{\alpha_D} \mathcal{J}(D) \xrightarrow{\beta_D} \mathcal{J}(A \otimes D) \longrightarrow 1 .$$

LEMMA 13. *The homomorphism* $C \to D$ *gives rise to a homomorphism of complexes* (1) $\to$ (2).

Proof. The mapping $\mathcal{O}(A) \to \mathcal{O}(A \otimes D)$ is of course given by sending each automorphism class $[\sigma]$ in $\mathcal{O}(A)$ to $[\sigma \otimes 1]$ in $\mathcal{O}(A \otimes D)$. That the mapping $\{J\} \to \{J \otimes D\}$ yields a homomorphism $\mathcal{J}(C) \to \mathcal{J}(D)$ was already proved in § 2, and a similar argument shows that for $\{P\} \in \mathcal{J}(A)$ the mapping $P \to P \otimes D$ yields a homomorphism $\mathcal{J}(A) \to \mathcal{J}(A \otimes D)$. The desired commutativity properties of these maps with α, β, α_D and β_D are easily verified.

We remark that if $\mathcal{O}(A \otimes D) = 1$, then by Lemma 13, $\alpha\mathcal{O}(A) \subset \mathrm{Ker}(\mathcal{J}(C) \to \mathcal{J}(D))$. This combined with Lemma 2, Corollary 3, Theorems 7 and 11 and Lemma 13 yields

THEOREM 14. *Let* C *be an integral domain or a noetherian ring,* S *the complement of the set of zero-divisors (or more generally suppose* C *is any commutative ring,* S *a multiplicatively closed subset containing no zero divisors such that* $\mathcal{O}(A \otimes C_S) = 1$). *Then* $\mathcal{O}(A)$ *is isomorphic to the subgroup of the ideal class group,* $\mathcal{J}(C, S)$, *consisting of ideal classes* $[I]$ *such that* $IA \cong A$ *as left* A*-module.*

If besides $A \cong \mathrm{Hom}_C(V, V)$ *is the algebra of endomorphisms of a finitely generated, faithful, projective* C*-module* V, *then* $\mathcal{O}(A)$ *is also isomorphic to the subgroup of* $\mathcal{J}(C, S)$ *consisting of those ideal classes* $[I]$ *with* $IV \cong V$ *as* C*-modules.*

THEOREM 15. *Let* C *be a Dedekind ring,* $A = \mathrm{Hom}_C(V, V)$ *with* V *a finitely generated, projective module of rank* n. *Then* $\mathcal{O}(A)$ *is isomorphic to the subgroup of the ideal class group of* C *consisting of the ideal classes whose orders divide* n.

Proof. By classical results, [8], V is isomorphic to a direct sum $I_1 \oplus \cdots \oplus I_n$ of ideals with n and the class of $I_1 I_2 \cdots I_n$ uniquely determining the C-isomorphism class of V. Thus $IV \cong V$ if and only if

$I^n \prod I_i \cong \prod I_i$. Since $\prod I_i$ is an invertible ideal, $IV \cong V$ if and only if $I^n \cong C$, i.e. I^n is principal. Theorem 14 completes the proof.

REMARKS. (1) If C is any integral domain and V is a free C-module on n generators, the same proof shows that if $\{I\} \in Im\,\alpha$ then I^n is principal.

(2) In case V is free so that A is a matrix algebra over C, Theorem 15 was also proved by Kaplansky (unpublished).

(3) If C is the ring of integers of an algebraic number field, $\mathscr{I}(C, S)$ is well known to be a finite group. If $\mathscr{I}(C, S) \neq 1$, if n is an integer prime to the order of $\mathscr{I}(C, S)$, and if A is the algebra of $n \times n$ matrices over C, we obtain an example with $\mathscr{O}(A) = 1$ but $\mathscr{I}(C) \cong \mathscr{I}(A) \cong \mathscr{I}(V) \neq 1$. It is an open question whether $\mathscr{O}(A) = 1$ for every A implies $\mathscr{I}(C) = 1$.

REFERENCES

1. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc., **97** (1960), 1-24.

2. ———, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., **97** (1960), 367-409.

3. G. Azumaya, *On maximally central algebras*, Nagoya Math. J., **2** (1951), 119-150.

4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, 1956.

5. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publ., Vol. 37, Providence, 1956.

6. A. Rosenberg and D. Zelinsky, *On Amitsur's Complex*, Trans. Amer. Math. Soc., **97** (1960), 327-356.

7. J. P. Serre, *Modules projectifs et espaces fibrés à fibre vectorelle*, Séminaire Dubreil, Algèbre et théorie des nombres, Paris, 1958.

8. E. Steinitz, *Rechteckige Systeme und Moduln in algebraischen Zahlkörpern*, Math. Ann., **71** (1912), 328-354.

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