ON THE DIFFERENCE AND SUM OF BASIC SETS OF POLYNOMIALS

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1. If the two functions $f(z)$ and $g(z)$ are connected by the relation

$$g(z+1) - g(z) = f(z),$$

then $f(z)$ is called the difference of $g(z)$ and $g(z)$ is the sum of $f(z)$. These relations are denoted by

$$f(z) = \Delta g(z)\ ; \qquad g(z) = \mathscr{S} f(z)\ ,$$

and it is obvious that any function of period unity can be added to the sum of a given function. The authors considered recently [1] the difference set $\{u_n(z)\}$ and the sum set $\{v_n(z)\}$ of a given simple set of polynomials[1] $\{p_n(z)\}$. These sets are the simple sets defined by

$$u_n(z) = \Delta p_{n+1}(z)\ ; \qquad (n \geqq 0)\ , \tag{1.1}$$

$$v_0(z) = 1;\ v_n(z) = \mathscr{S} p_{n-1}(z)\ ; \qquad (n \geqq 1)\ , \tag{1.2}$$

and the indetermination in the sum set is removed by supposing that

$$v_n(0) = 0\ ; \qquad (n \geqq 1)\ . \tag{1.3}$$

The main result of the above mentioned work concerns the order δ and σ of the difference and sum sets respectively of a simple set of a given order ω. In fact, it has been shown that

$$\delta \leqq \max(1, \omega)\ , \tag{1.4}$$

and

$$\sigma \leqq \omega + 1\ . \tag{1.5}$$

Our aim in the present paper is to generalise these results for more general classes of basic sets of polynomials. It will be here shown that, with suitable modification of the definition of difference sets, the upper bound in (1.4) remains the same for the most general classes of basic sets of polynomials. As for the sum sets, it will be here proved that, in order to get a finite upper bound for the order of the sum set, a limitation on the class of basic sets is inevitable.

2. This section and the following one are devoted to the study of

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[1] The reader is supposed to be acquainted with the the theory of basic sets of polynomials as given by Whittaker [3].

the difference set $\{u_n(z)\}$ of a general basic set of polynomials $\{p_n(z)\}$. As an introduction we first consider the difference set $\{\vartheta_n(z)\}$ of the unit set (z^n). According to (1.1), this set is given by

$$\vartheta_n(z) = (z+1)^{n+1} - z^{n+1} ; \qquad (n \geqq 0) . \tag{2.1}$$

It has been shown [1; formula (2.14)] that this set admits the representation

$$z^n = \sum_{k=0}^{n} \lambda_{n,k} \vartheta_k(z) , \tag{2.2}$$

where[2]

$$|\lambda_{n,k}| < \frac{Kn!}{(k+1)!(2\pi)^{n-k}} ; \qquad (0 \leqq k \leqq n) . \tag{2.3}$$

Considering now the difference set $\{u_n(z)\}$ of a general basic set $\{p_n(z)\}$, the definition (1.1) has to be slightly modified in order to avoid a linear dependence among the polynomials of the resulting set. In fact, suppose that z^n admits the representation

$$z^n = \sum_k \pi_{n,k} p_k(z) ; \qquad (n \geqq 0) , \tag{2.4}$$

and in particular,

$$1 = \sum_k \pi_{0,k} p_k(z) . \tag{2.5}$$

A process of differencing, operating on this last relation, yields a linear relation between the involved differenced polynomials. For this reason a polynomial of the set $\{p_n(z)\}$ has to be eliminated. Suppose, in fact, that $\pi_{0,\mu}$ is the first non-zero coefficient in (2.5) then the difference set $\{u_n(z)\}$ is not defined by

$$\begin{cases} u_n(z) = \Delta p_n(z) & (0 \leqq n \leqq \mu - 1) \\ u_n(z) = \Delta p_{n+1}(z) & (n \geqq \mu) . \end{cases} \tag{2.6}$$

Thus the polynomial $p_\mu(z)$ is eliminated. In case of simple sets $\mu = 0$ and the definition (2.6) reduces to that in (1.1).

We first show[3] that the set of polynomials $\{u_n(z)\}$, as defined by (2.6) is basic. To this end we define the set $\{p_n^\dagger(z)\}$ by

$$\begin{aligned} p_n^\dagger(z) &= (1/z)\{p_n(z) - p_n(0)\} ; & (0 \leqq n \leqq \mu - 1) , \\ p_n^\dagger(z) &= (1/z)\{p_{n+1}(z) - p_{n+1}(0)\} ; & (n \geqq \mu) . \end{aligned} \tag{2.7}$$

It can be easily verified from the definitions (2.1), (2.6) and (2.7) that the set $\{u_n(z)\}$ is the product set

[2] In our notation K denotes positive finite numbers independent of n that not necessarily of the same value at different occurrences.

[3] The following discussion is due to Newns in his study on the derived sets [2; p. 465].

$$\{u_n(z)\} = \{p_n^\dagger(z)\}\{\vartheta_n(z)\} . \tag{2.8}$$

Hence, in order to prove that $\{u_n(z)\}$ is basic it is sufficient to show that the set $\{p_n^\dagger(z)\}$ is basic. In fact, rewriting (2.5) and using (2.7) we get

$$1 = \sum_k \pi_{0,k}(0) + z\Big\{\pi_{0,\mu}\mathscr{P}(z) + \sum_{k\geqq\mu} \pi_{0,k+1}p_k^\dagger(z)\Big\} ,$$

where $\mathscr{P}(z) = (1/z)\{p_\mu(z) - p_\mu(0)\}$. Since the first sum is equal to 1 it follows that

$$\mathscr{P}(z) = -\sum_{k\geqq\mu} \frac{\pi_{0,k+1}}{\pi_{0,\mu}} p_k^\dagger(z) . \tag{2.9}$$

Inserting (2.7) and (2.9) in (2.4), written for z^{n+1} we obtain the unique representation

$$z^{n+1} = \sum_k \pi_{n+1,k}(0) + z\Big\{\sum_{k=0}^{\mu-1} \pi_{n+1,k}p_k^\dagger(z) + \sum_{k\geqq\mu}\Big(\pi_{n+1,k+1} - \pi_{n+1,\mu}\frac{\pi_{0,k+1}}{\pi_{0,\mu}}\Big)p_k^\dagger(z)\Big\} .$$

As the first sum is zero it follows that the set $\{p_n^\dagger(z)\}$ is a basic set that admits the unique representation

$$z^n = \sum_k \pi_{n,k}^\dagger p_k^\dagger(z) , \tag{2.10}$$

where

$$\begin{cases} \pi_{n,k}^\dagger = \pi_{n+1,k} & (0 \leqq k \leqq \mu - 1) , \\ \pi_{n,k}^\dagger = \pi_{n+1,k+1} - \pi_{n+1,\mu}\dfrac{\pi_{0,k+1}}{\pi_{0,\mu}} & (k \geqq \mu,\ n \geqq 0) . \end{cases} \tag{2.11}$$

Hence the set $\{u_n(z)\}$ is the basic, as required.

3. We propose here to establish an upper bound for the order of the difference set of a given basic set of polynomials, and the following theorem shows that the bound given in (1.4) remains the same for general basic sets.

THEOREM 1. *Let* $\{p_n(z)$ *be a general basic set of polynomials of order* ω; *then the difference set* $\{u_n(z)\}$, *defined by* (2.6), *will be of order* δ, *where*

$$\delta \leqq \max(1, \omega) . \tag{3.1}$$

Proof. Let $(\gamma_{n,k})$ be the coefficients corresponding to those in (2.4) for the set $\{u_n(z)\}$. Hence, in view of (2.2), (2.8) and (2.10) it follows that

$$\gamma_{n,k} = \sum_{j=0}^{n} \lambda_{n,j}\pi_{j,k}^\dagger . \tag{3.2}$$

Adopting the usual notations for basic sets we write[4]

$$F_n(p; R) = \max_{i,j} \max_{|z|=R} \left| \sum_{k=1}^{j} \pi_{n,k} p_k(z) \right| ,$$

and $F_n(u; R)$ for the corresponding expression for the set $\{u_n(z)\}$. Since the set $\{p_n(z)\}$ is of order ω, then given any finite number $\omega_1 > \omega$ we shall have

$$F_n(p; R) < K n^{\omega_1 n} ; \qquad (n \geqq 1) . \tag{3.3}$$

Choose the integers s_n and t_n for the set $\{u_n(z)\}$ such that

$$F_n(u; R) = \max_{|z|=R} \left| \sum_{k=s_n}^{t_n} \gamma_{n,k} u_k(z) \right| ,$$

hence (3.2) implies that

$$F_n(u; R) = \max_{|z|=R} \left| \sum_{j=0}^{n} \lambda_{n,j} f_j(z) \right| , \tag{3.4}$$

where

$$f_j(z) = \sum_{k=s_n}^{t_n} \pi^{\dagger}_{j,k} u_k(z) .$$

Writing $g_j(z) = \sum_{k=s_n}^{t_n} \pi^{\dagger}_{j,k} p^{\dagger}_k(z)$, then it follows from (2.8) that, if

$$g_j(z) = \sum_k g_{j,k} z^k , \tag{3.5}$$

then

$$f_j(z) = \sum_k g_{j,k} \vartheta_k(z) . \tag{3.6}$$

Applying the relations (2.7) and (2.11) in $g_j(z)$ we easily obtain

$$\begin{aligned} g_j(z) = (1/z)\Big[&\sum_{k=s_n}^{\mu-1} \pi_{j+1,k} \{p_k(z) - p_k(0)\} \\ &+ \sum_{k=\mu+1}^{t_n+1} \left\{ \pi_{j+1,k} - \frac{\pi_{j+1,\mu}}{\pi_{0,\mu}} \pi_{0,k} \right\} \left\{ p_k(z) - p_k(0) \right\} \Big] . \end{aligned} \tag{3.7}$$

Putting

$$L_j(R) = \max_{|z|=R} | f_j(z) | ; \qquad \mathscr{M}_j(R) = \max_{|z|=R} | g_j(z) | ;$$
$$M(p_n; R) = \max_{|z|=R} | p_n(z) | ,$$

and observing that

[4] Because of the variety of sets considered here the dependence of the entity on the particular set is explicitly written, whenever it is possible; thus we shall, for example, write $M(p_n; R) = \max_{|z|=R} | p_n(z) |$.

$$\left|\sum_{k=m}^{n} \pi_{j+1,k} p_k(0)\right| \leqq \max_{|z|=R} \left|\sum_{k=m}^{n} \pi_{j+1,k} p_k(z)\right| \leqq F_{j+1}(p; R) ; \qquad (R > 0) ,$$

then, for any positive number ρ, (3.7) yields

$$\begin{aligned} \mathscr{M}_j(\rho) &\leqq (1/\rho)\left[4F_{j+1}(p; \rho) + \frac{|\pi_{j+1,\mu}|\, M(p_\mu; \rho)}{|\pi_{0,\mu}|\, M(p_\mu; \rho)} 2F_0(p; \rho)\right] \\ &\leqq (2/\rho) F_{j+1}(p; \rho)\left[2 + \frac{F_0(p; \rho)}{|\pi_{0,\mu}|\, M(p_\mu; \rho)}\right] \\ &< KF_{j+1}(p; \rho) . \end{aligned} \tag{3.8}$$

Suppose now that $\rho > R + 1$ and apply Cauchy's inequality for $g_j(z)$ as given in (3.5). Hence inserting (2.1) and (3.8) in (3.6) we obtain

$$\begin{aligned} L_j(R) &< KF_{j+1}(p; \rho) \sum_k \frac{M(\vartheta_k; R)}{\rho^k} \\ &< KF_{j+1}(p; \rho) \sum_k \frac{(R+1)^k}{\rho^k} < KF_{j+1}(p; \rho) . \end{aligned} \tag{3.9}$$

Substitution from (2.3), (3.3) and (3.9) in (3.4) now gives

$$\begin{aligned} F_n(u; R) &< K \sum_{j=0}^{n} |\lambda_{n,j}|\, F_{j+1}(p; \rho) \\ &< \frac{Kn!}{(2\pi)^{n+1}} \sum_{j=1}^{n+1} \frac{(2\pi)^j j^{\omega_1 j}}{j!} , \end{aligned}$$

and since $j! > (j/e)^j$; $(j \geqq 1)$, the above inequality yields

$$F_n(u; R) < \frac{Kn!}{(2\pi)^{n+1}} \sum_{j=1}^{n+1} (2e\pi)^j j^{j(\omega_1 - 1)} . \tag{3.10}$$

Suppose that $\omega \geqq 1$, then $\omega_1 > 1$ and (3.10) implies that

$$F_n(u; R) < K(n+1)! e^{n+1} (n+1)^{(n+1)(\omega_1 - 1)} ,$$

which ensures the order δ of the set $\{u_n(z)\}$ cannot exceed ω_1; and since ω_1 can be taken as near to ω as we please we deduce that $\delta \leqq \omega$.

Suppose that $\omega < 1$; then by suitable choice we shall have $\omega_1 \leqq 1$ and hence (3.10) gives

$$F_n(u; R) < K(n+1)! e^{n+1} ,$$

and thus $\delta \leqq 1$. We thus conclude that $\delta \leqq \max(1, \omega)$ and the proof of Theorem 1 is complete.

4. In the remaining sections the sum set $\{v_n(z)\}$ of a given set $\{p_n(z)\}$ is considered. As in the case of difference set we introduce by considering the sum set $\{\phi_n(z)\}$ of the unit set (z^n).

This set, according to the definitions (1.2) and (1.3), is given by

$$\begin{cases} \phi_0(z) = 1; \quad \phi_n(z+1) - \phi_n(z) = z^{n-1} \\ \qquad\qquad\qquad \phi_n(0) = 0\,; \qquad\qquad (n \geqq 1)\,. \end{cases} \tag{4.1}$$

It has been shown [1; formula (4.7)] that this set admits the representation

$$z^n = \sum_{k=1}^{n} \binom{n}{k-1} \phi_k(z)\,, \tag{4.2}$$

and that it accords to the inequalities [1; formulae (2.5), (2.6)]

$$\frac{(n-1)!}{(2\pi)^n} < M(\phi; R) < (n-1)!e^R\,; \qquad (n \geqq 1; R > 1)\,. \tag{4.3}$$

In case of sum sets the definitions (1.2) and (1.3) remain valid for general basic sets of polynomials. In fact, the sum set $\{v(z)\}$ of the set $\{p_n(z)\}$ is the basic set given by

$$v_0(z) = 1\,; \quad v_n(z) = \sum_k p_{n-1,k}\phi_{k+1}(z)\,; \qquad (n \geqq 1)\,, \tag{4.4}$$

where

$$p_n(z) = \sum_k p_{n,k} z^k\,, \tag{4.5}$$

and it admits the unique representation

$$z^n = \sum_k \varpi_{n,k} v_k(z)\,, \tag{4.6}$$

where

$$\begin{cases} \varpi_{0,0} = 1\,, \quad \varpi_{0,n} = 0; \ (n > 0)\,, \quad \varpi_{n,0} = 0, \ (n \geqq 1)\,, \\ \varpi_{n,k} = \sum_{j=1}^{n} \binom{n}{j-1} \pi_{j-1,k-1} \qquad (k \geqq 1, \ n \geqq 1)\,. \end{cases} \tag{4.7}$$

It should be noted that, if the class of the basic set $\{p_n(z)\}$ is not restricted, the order of the sum set $\{v_n(z)\}$ may be infinite, even if the order of the set $\{p_n(z)\}$ is zero. This fact is illustrated by the following example[5], which also suggests that, in order to ensure a finite upper bound for the order of the sum set, the basic set $\{p_n(z)\}$ should accord to the restriction that $D_n = 0(n)$, where D_n is the degree of the polynomial of highest degree in the representation

$$z_n = \sum_k \pi_{n,k} p_k(z)\,.$$

EXAMPLE. Let (ν_n) be an increasing sequence of even integers such that $\nu_n > 2n$ for all large n and $\lim_{n\to\infty} \nu_n/(n \log n) = 0$. Consider the

[5] This generalised example was suggested by the referee of this paper, as a substitute of two, originally given, particular examples.

basic set $\{p_n(z)\}$ given by

$$p_{2n}(z) = z^{2n}$$
$$p_{2n+1}(z) = z^{2n+1} + \{(2n+1)!\}^{\omega} z^{\nu_n} ,$$

where ω is any nonnegative number.

It is easily seen that this set is of order ω. Forming the sum set $\{v_n(z)\}$, (4.4) gives

$$\begin{aligned} v_0(z) &= 1 , \\ v_{2n+1}(z) &= \phi_{2n+1}(z) , \\ v_{2n+2}(z) &= \phi_{2n+2}(z) + \{(2n+1)!\}^{\omega} \phi_{\nu_n+1}(z) . \end{aligned} \tag{4.8}$$

In view of (4.2), it is easily verified that

$$z^{2n} = \sum_{j=1}^{2n} \binom{2n}{j-1} v_j(z) - \sum_{j=0}^{n-1} \binom{2n}{2j+1} \{(2j+1)!\}^{\omega} v_{\nu_j+1}(z) . \tag{4.9}$$

Writing, in view of (4.6),

$$\omega_n(v; R) = \sum_k | \varpi_{n,k} | M(v_k; R) , \tag{4.10}$$

and applying (4.3), equations (4.8) and (4.9) yield

$$\begin{aligned} \omega_{2n}(v; R) &> 2n\{(2n-1)!\}^{\omega} M_{\nu_{n-1}+1}; R\} \\ &> \{(2n-1)!\}^{\omega} (\nu_{n-1})!/(2\pi)^{\nu_{n-1}} . \end{aligned}$$

Hence, the order of the sum set $\{v_n(z)\}$ is

$$\begin{aligned} \sigma = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \omega_n(v; R)}{n \log n} &\geqq \limsup_{n\to\infty} \frac{\omega \log (2n-1)! + \log(\nu_{n-1})!}{2n \log 2n} \\ &\geqq \omega + \limsup_{n\to\infty} \nu_n/2n = \omega + \limsup_{n\to\infty} D_n/n , \end{aligned}$$

since

$$D_{2n} = 2n ; \qquad D_{2n+1} = \max (2n+1, \nu_n) .$$

The possibilities $\limsup_{n\to\infty} D_n/n = 1, \alpha, \infty$, where $1 < \alpha < \infty$, can be covered by choosing $\nu_n = 2n + 2$, $2[(n + \frac{1}{2})\alpha]$, $2(2n+1)[\sqrt{\log (2n+1)}]$ respectively, where $[x]$ has its usual meaning of "the integral part of x".

5. The above example shows also that the result formulated in the following theorem is best possible.

THEOREM 2. *Let ω be the order of the basic sets $\{p_n(z)\}$ which satisfies the condition that*

$$\limsup_{n\to\infty} D_n/n = \alpha < \infty , \tag{5.1}$$

then the sum set $\{v_n(z)\}$ will be of order $\sigma \leqq \omega + \alpha$.

Proof. Since the set $\{p_n(z)\}$ is of order ω, then for any finite number $\omega_1 > \omega$ we have

$$\omega_n(p; R) < Kn^{\omega_1 n}, \qquad (n \geqq 1), \tag{5.2}$$

where $\omega_n(p; R)$ is the Cannon sum for the set $\{p_n(z)\}$, given by

$$\omega_n(p; R) = \sum_k |\pi_{n,k}| M(p_k; R) .$$

Also, given any finite number $\alpha' > \alpha$, there exists, in view of (5.1), a positive integer n_0 such that

$$D_n < \alpha' n , \quad (n > n_0) ,$$

so that

$$(D_n)! < \Gamma(\alpha' n + 1) < K(\alpha' n)^{\alpha' n + \frac{1}{2}} , \quad (n > n_0) . \tag{5.3}$$

Now, let d_n be the degree of the polynomial $p_n(z)$; then combining (4.3) and (4.4) we obtain

$$\begin{aligned} M(v_n; R) &\leqq \sum_{k=0}^{d_{n-1}} | p_{n-1,k} \{ M(\phi_{k+1}; R) \\ &< K(d_{n-1} + 1)! M(p_{n-1}; R) ; \qquad (n \geqq 1,\ R > 1) . \end{aligned} \tag{5.4}$$

Applying (4.7) we get the familiar inequality

$$\omega_n(v; R) \leqq \sum_{k=0}^{n-1} \binom{n}{k} \sum_j |\pi_{k,j}| M(v_{j+1}; R) . \tag{5.5}$$

Inserting (5.2), (5.3) and (5.4) in (5.5) and mindful of the definition of the number D_n, it follows, for $n > n_0$, that

$$\begin{aligned} \omega_n(v; R) &< K \sum_{k=0}^{n-1} \binom{n}{k} (D_k + 1)! \omega_k(p; R) \\ &< K\Big[1 + \sum_{k=1}^{n_0} \binom{n}{k} (D_k + 1)! k^{\omega_1 k} + \sum_{k=n_0+1} \binom{n}{k} (\alpha' k + 1)^{\alpha' k + (3/2)} k^{k\omega_1}\Big] \\ &< K(\alpha' n + 1)^{\alpha' n + (3/2)} n^{\omega_1 n} \sum_{k=0}^{n} \binom{n}{k} = K 2^n (\alpha' n + 1)^{\alpha' n + (3/2)} n^{\omega_1 n} . \end{aligned}$$

This relation implies that the order σ of the sum set $\{v_n(z)\}$ does not exceed $\omega_1 + \alpha'$. Since ω_1 and α' can be arbitrarily chosen near to ω and α respectively, we conclude that $\sigma \leqq \omega + \alpha$; and Theorem 2 is therefore established.

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