THE CYCLIC CONNECTIVITY OF PLANE CONTINUA

F. BURTON JONES

Suppose that p and q are distinct points of the compact plane continuum M. If no point separates p from q in M and M is *locally connected*, then it is known [5] that M contains a simple closed curve which contains both p and q. But in the absence of local connectivity such a simple closed curve may fail to exist. Even if no point *cuts*[1] p from q in M, there does not necessarily exist in M a simple closed curve which contains both p and q. For example, no point of the continuum C indicated in Figure 1 cuts p from q in C, but C contains no simple closed curve whatsoever. However, if M is the continuum obtained by adding to C either of its complementary domains, there does exist in M a simple closed curve which contains both p and q. Here M fails to separate the plane and this is indicative of the general situation.

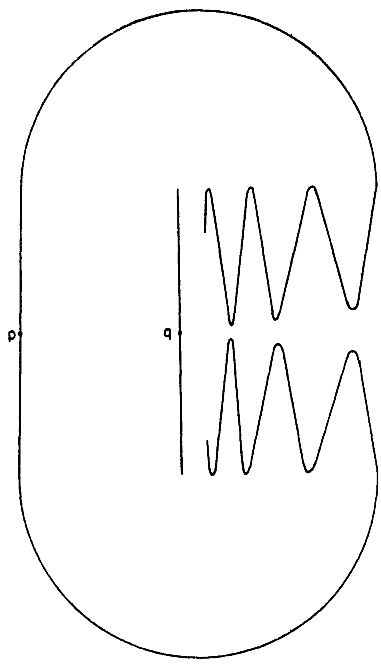

Fig. 1

LEMMA. *If p is a point of the compact subcontinuum M' of the plane S and L' is a nondegenerate compact continuum containing p*

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[1] A point x $(p \neq x \neq q)$ cuts p from q in M if every subcontinuum of M containing both p and q also contains x. Obviously a *separating* point is a cut point but for continua in general a cut point is not necessarily a separating point.

and lying in $(S - M') + p$ *such that* $L' - p$ *is connected, then there exists a connected open subset* D' *of* $S - M'$ *such that*

(1) $D' + p$ *contains* L',

(2) $D' + p$ *is a connected, locally connected, complete, metric space, and*

(3) $D' + p$ *is strongly regular* (*i.e.*, the author's Axiom 5_1* [1, p. 54] holds true in $D' + p$).

Proof. Let q denote a point of $L' - p$, let n denote a natural number such that $d(p, q) > 1/n$, and let $R_0, R_1, R_2, \cdots$ denote a sequence of circular regions centered on p of radii $1/n$, $1/n + 1$, $1/n + 2, \cdots$ respectively. Now for each integer i $(i > -1)$, add to M' every open interval I of the boundary C_i of R_i such that I contains no point of $L' + M'$ but has both of its end points in M', and call the resulting pointset N. Let D_1 denote the sum of the components of $(S - N)\cdot(S - \bar{R}_1)$ which contain points of L' and for each integer $i > 1$, let D_i denote the sum of the components of $(S - N)\cdot(R_{i-2} - \bar{R}_i)$ which contain points of L'. Furthermore let $D' = \sum D_i$. Certainly D' is open and since $L' - p$ is connected, D' is connected. Also it is easy to see that $D' + p$ contains L' and is a connected, complete, metric space. It remains only to show that $D' + p$ is strongly regular for it follows that such a space is locally connected [2, p. 623]. Obviously $D' + p$ is strongly regular at each point of D'. To see that $D' + p$ is strongly regular at p (relative to $D' + p$, of course) one has merely to observe that if k is a positive integer, the boundary of $p + \sum D_i\,(i > k)$ relative to $D' + p$ is a subset of the sum of those components of $(S - M')\cdot C_{k-1}$ which intersect L' and since L' contains no point of M' except p, this set of components is finite.

THEOREM. *Let* M *be a compact subcontinuum of the plane* S *which does not separate* S. *Then if* p *and* q *are distinct points of* M *and no point cuts* p *from* q *in* M, *there exists a simple closed curve* J *lying in* M *which contains both* p *and* q.

Proof. Three cases arise depending upon the location of p and q. If both p and q are inner points (non-boundary points) of M, then it follows from [3] that both p and q belong to the same component of the set of inner points of M. For this case the theorem is known to hold true (see for example [4], p. 124).

If both p and q are boundary points of M, then the argument outlined in [3] shows that M contains a compact continuum L which contains both p and q such that every point of $L - (p + q)$ is an inner point of M. Since L must contain a subcontinuum irreducible from p to q it is no loss of generality to assume that L itself has this property.

In this case $L - (p + q)$ is a connected subset of a component D of the set of inner points of M and the theorem follows with the help of the lemma in somewhat the same manner as the next case.

Finally, if q is an inner point of M and p is a boundary point of M, it follows from [3] that some component D of the set of inner points of M contains q and has p in its boundary. To show that $D + p$ contains a continuum L containing both p and q requires a modification of the argument given in [3].

Suppose that ε is a positive number such that $\varepsilon < d(p, q)$. Let $C_p(\varepsilon)$ denote a circle of radius ε centered on p and let C_q denote a straight line through q which is perpendicular to the line pq. There exists a simple domain $I(\varepsilon)$ which contains M such that if $J(\varepsilon)$ denotes the boundary of $I(\varepsilon)$, y is a boundary point of M, and z is a point of $I(\varepsilon) + J(\varepsilon)$, then $d[y, J(\varepsilon)] < \varepsilon$ and $d(z, M) < \varepsilon$. There exist arcs $T_p(\varepsilon)$ and $T_q(\varepsilon)$ in $C_p(\varepsilon)$ and C_q respectively such that each is minimal with respect to separating $I(\varepsilon) + J(\varepsilon)$, q belongs to $T_q(\varepsilon)$, and $T_p(\varepsilon)$ separates p from $T_q(\varepsilon)$ in $I(\varepsilon) + J(\varepsilon)$.

Since $T_p(\varepsilon)$ and $T_q(\varepsilon)$ have only their endpoints in $J(\varepsilon)$, and except for these points lie entirely in $I(\varepsilon)$, there exist in $J(\varepsilon)$ two nonintersecting unique arcs $A(\varepsilon)$ and $B(\varepsilon)$ such that $T_p(\varepsilon) + A(\varepsilon) + T_q(\varepsilon) + B(\varepsilon)$ is a simple closed curve $H(\varepsilon)$. Let $D(\varepsilon)$ denote the bounded complementary domain of $H(\varepsilon)$. If z is a point of $D(\varepsilon) + H(\varepsilon)$, then $d(z, M) < \varepsilon$. Any subcontinuum of M which contains $p + q$ contains a subcontinuum irreducible from $T_p(\varepsilon)$ to $T_q(\varepsilon)$ which lies in $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$.

Now let $L(\varepsilon)$ denote a continuum lying in $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$ which intersects both $T_p(\varepsilon)$ and $T_q(\varepsilon)$ such that if z belongs to $L(\varepsilon)$, then $d[z, A(\varepsilon)] = d[z, B(\varepsilon)]$. The continuum $L(\varepsilon)$ must exist; for if it did not, the set W of all points of $D(\varepsilon) + H(\varepsilon)$ equidistant from $A(\varepsilon)$ and $B(\varepsilon)$ would be the sum of two mutually separated sets one containing $W \cdot T_p(\varepsilon)$ and the other containing $W \cdot T_q(\varepsilon)$ and consequently some simple closed curve would separate $T_p(\varepsilon)$ from $T_q(\varepsilon)$ but at the same time would fail to contain a point of W which involves a contradiction. So there exists a simple infinite sequence α of values of ε such that $D(\varepsilon) + H(\varepsilon)$ converges to a subset of M, $T_q(\varepsilon) \to T_q$ and $L(\varepsilon) \to L$ as $\varepsilon \to 0$ in α. The set L has the following properties:

(a) L is a continuum containing both p and point of T_q,

(b) L is a subset of M, and

(c) every point of $L - (p + L \cdot T_q)$ is an inner point of M.

Properties (a) and (b) are evident. So it remains only to prove property (c).

Let x be a point of $L - (p + L \cdot T_q)$. Since x does not cut p from q in M, there exists a subcontinuum K of M which contains $p + q$ but not x. Let δ be a positive number such that $4\delta = d(x, K + T_q)$ and let

$U_\delta(x)$ and $U_{3\delta}(x)$ be the circular regions centered on x of radius δ and 3δ respectively. When ε (in α) is sufficiently small $[T_p(\varepsilon) + T_q(\varepsilon)] \cdot [U_{3\delta}(x)] = 0$ but $L(\varepsilon) \cdot U_\delta(x) \neq 0$. Let y be some point of $L(\varepsilon) \cdot U_\delta(x)$, let $r = \delta + d(x, y)$ and let $U_r(y)$ be a circular region of radius r and center y. Obviously $U_{3\delta}(x) \supset U_r(y) \supset U_\delta(x)$. So $[T_p(\varepsilon) + T_q(\varepsilon)] \cdot U_r(y) = 0$. If $A(\varepsilon) \cdot U_r(y) \neq 0$, let f be a point of $A(\varepsilon) \cdot U_r(y)$ such that $d(f, y) = d[y, A(\varepsilon)]$. But y belongs to $L(\varepsilon)$. Hence there exists in $U_r(y)$ a point g of $B(\varepsilon)$ such that $d(g, y) = d[g, B(\varepsilon)] = d(f, y)$. The sum of the straight line intervals yf and yg from y to f and from y to g respectively is an arc T_y lying in $U_r(y)$, having only its endpoints f and g in $H(\varepsilon)$, and containing the point y of $D(\varepsilon)$. Hence $T_y - (f + g) \subset D(\varepsilon)$ for clearly yf cannot intersect $B(\varepsilon)$ and yg cannot intersect $A(\varepsilon)$. But $T_y \cdot K = 0$ and K contains a continuum lying in $T_p(\varepsilon) + D(\varepsilon) + T_q(\varepsilon)$ irreducible from $T_p(\varepsilon)$ to $T_q(\varepsilon)$. Since the points f and g separate $T_p(\varepsilon)$ from $T_q(\varepsilon)$ in $H(\varepsilon)$ this involves a contradiction [4, Th. 17, p. 167]. Hence $U_r(y) \cdot H(\varepsilon) = 0$ and since y belongs to $D(\varepsilon)$, $U_r(y) \subset D(\varepsilon)$; so for sufficiently small values of ε (in α), $U_\delta(x) \subset D(\varepsilon)$. Consequently $U_\delta(x)$ is a subset of M and x is an inner point of M.

Now let C denote a circle which separates p from T_q. Obviously L intersects C. Hence L contains a subcontinuum L' irreducible from C to p. Let q' denote a point of $L' \cdot C$. Clearly $L' - p$ is a connected subset of D. Let M' denote the boundary of D. Since M' is a continuum and contains only the point p of L', by the lemma there exists a connected open subset D' of $S - M'$ which contains $L' - p$ and has the other properties of the set designated as D' in the lemma. It now follows from Theorem A of [1] that there exists a simple closed curve J' lying in $D' + p$ and containing $p + q'$. Since D' is a connected subset of $S - M'$ and contains a point of L, it follows that D' is a subset of D and that J' is a subset of M. Of course using J' it is now easy to construct a simple closed curve J which lies in $D + p$ and contains $p + q$.

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University of North Carolina