REAL COMMUTATIVE SEMIGROUPS ON THE PLANE

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A *real semigroup* is a topological semigroup containing a sub-semigroup R isomorphic to the multiplicative semigroup of real numbers, embedded so that 1 is an identity and 0 is a zero. This paper is devoted to a preliminary study of real commutative semigroups on the plane and especially to characterizing the product semigroup on $R \times R$. It leans heavily on the fundamental paper [5] of Mostert and Shields which in turn depends the paper [1] of Faucett who, among other things, characterized the multiplicative semigroup on the closed unit interval. Characterizations of the multiplicative semigroup of nonnegative real numbers and of all real numbers were given in [4] and [3] respectively. (In connection with the latter characterization, also see [2]).

Nothing like a complete description of all real commutative semigroups on the plane can be given at this time, even under the additional hypothesis that there are no (non-zero) nilpotent elements. A crude classification can be given however on the basis of the number and arrangement of the components of the maximal subgroup $H(1)$. If $H(1)$ is connected then the semigroup is necessarily the multiplicative semigroup of complex numbers. If $H(1)$ is not connected then the component G of the identity in $H(1)$ is always isomorphic to the two dimensional vector group. There can be precisely two components in $H(1)$; in this case, $H(1)$ may be dense or not. There are at least two instances of the former (see Examples 1 and 2 of § 6) and at least two instances of the latter (see Examples 3 and 4 of § 6). If there are more than two components, but there are no nilpotent elements, then the number of components of $H(1)$ is four and $H(1)/G$ is isomorphic to the four group. Example 5 of § 6 shows that even in this case $H(1)$ need not be dense and the suggestion is that there are many instances of this case. A characterization of the product semigroup on $R \times R$ appears in § 5.

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Preliminaries. The closure of a subset A of a topological space is denoted A^-. The set-theoretic difference of two sets A and B is denoted by $A \backslash B$.

A binary operation, or multiplication, is denoted by juxtaposition. By a semigroup S we mean a topological semigroup, that is, a Hausdorff space with a continuous associative multiplication. All semigroups

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in this paper are assumed to be commutative, though we shall occasionally list this hypothesis for emphasis. An isomorphism between two semigroups is a function which is both an algebraic isomorphism and a homeomorphism. If there is an identity, it will be denoted by 1. A zero will usually, though not always, be denoted by 0.

As in [5], $H(1)$ denotes the set of elements with inverses, and G denotes the component of the identity in $H(1)$. The boundary of G is denoted by L.

By the square of an element s is meant the product of s with itself, and is denoted by s^2. More generally, if n is a positive integer, s^n denotes the n-fold product of s with itself. If s has an inverse it is denoted by $1/s$. An element e is an idempotent if $e^2 = e$. In a semigroup with 0, a nilpotent element is an element $x \neq 0$ such that $x^n = 0$ for some positive integer n. (Some writers permit 0 to be a nilpotent element.)

For subsets A, B of S, $AB = \{ab: a \in A, b \in B\}$. If $x \in S$ then Ax is the abbreviation of $A\{x\}$. An ideal is a subset A such that $AS \subset A$. A sub-semigroup is a subset A such that $AA \subset A$.

Throughout this work, E will denote the Euclidean plane and R will denote a semigroup isomorphic to the multiplicative semigroup of real numbers. The set of all positive members of R is denoted by P and the set of negative members by N.

It is a standing assumption that E has the structure of a real (commutative) semigroup. That is, E satisfies the condition

(*) *E is endowed with the structure of a commutative topological semigroup which contains a sub-semigroup R isomorphic to the multiplicative semigroup of real numbers. Furthermore, the elements* 1 *and* 0 *of R act as an identity and a zero respectively for E.*

We intend to imply that a fixed isomorph of the real numbers has been chosen. (In this connection see Remark 5.4). Thus -1 is a well defined element so there is no harm in abbreviating $(-1)(x)$ to $-x$. For a subset $A \subset E$, $-A = \{-a: a \in A\}$. The element zero is sometimes referred to as the origin.

By coordinate-wise multiplication is meant the multiplication defined on $R \times R$ by the identity: $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$. That is, coordinate-wise multiplication is simply the name of multiplication on the product semigroup $R \times R$, but this language is convenient later. Under coordinate-wise multiplication, $R \times R$ satisfies condition (*) if R is identified with the set: $\{(x, x): x \in R\}$.

1. This section is devoted to a study of the embedding and separation properties of R and of certain aspects of the multiplication maps m_x, where m_x is the map of E into itself defined by the identity: $m_x(y) = xy$.

1.1 LEMMA. *Let $u \in R$. If $x \neq 0$ and $ux = x$ then $u = \pm 1$. Therefore m_x is a one-to-one function on P^- and on N^- which is one-to-one on R if and only if $x \neq -x$.*

Proof. If $u \neq \pm 1$ then either $u^n \to 0$ or $(1/u)^n \to 0$ so $u^n x \to 0$ or $(1/u)^n x \to 0$. However, $u^n x = x$ and $(1/u)^n x = x$ which is a contradiction. Therefore $u = \pm 1$. The remaining conclusions now follow easily.

1.2 THEOREM. *For every x, m_x is a closed map from R into E. Hence if $x \neq 0$ then m_x is a homeomorphism on P^- and on N^- which is a homeomorphism on R if and only if $-x \neq x$, or equivalently, if and only if Rx cuts E. In any case, Rx is closed subset of E.*

Proof. The assertions concerning the case $x = 0$ are trivial so assume $x \neq 0$. Suppose that t_n is a sequence in R such that $t_n x$ converges. If t_n does not converge then either t_n has two cluster points t, t', or a sub-sequence s_n of t_n satisfies: $1/s_n \to 0$. In the second case, $(1/s_n)(s_n x) \to 0$, so $x = 0$ which is a contradiction. In the first case, there are subsequences of t_n which converge to t and t' respectively so $tx = t'x$. Hence if either $t_n \geqq 0$ or $t_n \leqq 0$ or if $x \neq -x$ then $t = t'$ by Lemma 1.1. This is a contradiction so t_n converges. It follows, in any case, that m_x is a closed map on R and therefore Rx is a closed subset of E. Also, m_x is a homeomorphism on P^- and on N^- which is a homeomorphism on R if and only if $x \neq -x$.

If m_x is a homeomorphism on R then Rx cuts E since Rx is a closed subset of E. If m_x is not a homeomorphism on R then $x = -x$, $Rx = P^- x$ and hence Rx does not cut E. The proof the theorem is complete.

1.3 DEFINITION. If $x \neq 0$ then the *ray through* x, denoted $[x]$, is the set $P^- x$. The *open ray* through x, denoted $(x]$, is the set Px.

Evidently the union of any two distinct rays cuts E. In fact, if $[z]$ and $[w]$ are distinct rays and $T = [z] \cup [w]$ then $E \backslash T$ is the union of two disjoint (topological) planes which we are entitled to call sectors. For obviously if $[v]$ is a ray then $x \in (v]$ if and only if $(v] = (x]$, or equivalently, if and only if $(v] \cap (x] \neq \phi$. Thus if S is one of the components of $E \backslash T$ and $x \in S$ then $(x] \subset S$.

1.4 DEFINITION. An *open sector* is one of the components into which a pair of (not necessarily distinct) rays divides E, or the set $E \backslash \{0\}$. A *sector* denotes any one of the following sets:

(1) an open sector;

(2) an open ray;

(3) the union of an open sector with one or both of its bounding open rays.

According to this definition, a sector never contains the origin.

Due to the commutativity of multiplication, the maps m_x satisfy $m_x(ty) = tm_x(y)$ for all $t \in R$, $y \in E$. Hence if $m_x(y) \neq 0$ then $m_x((y])$ is an open ray. The squaring function χ (defined by the identity $\chi(x) = x^2$ for all $x \in E$) satisfies $\chi(tx) = t^2\chi(x)$ for all $t \in R$, $x \in E$. Hence if E has no nilpotent elements, χ maps open rays onto open rays. A useful property of such functions is that they map sectors into sectors. In order to see this, observe that the set of open rays forms a decomposition $\mathscr{D}$ of $E\backslash\{0\}$. It is easy to see that in the quotient topology, $\mathscr{D}$ is homeomorphic to a circle and that under the quotient map ν the sectors simply correspond to connected subsets of $\mathscr{D}$. If f is any map on a sector S of E which maps open says in S onto open rays of E, then f induces a map from $\nu(S)$ into $\mathscr{D}$. The image of f is simply the inverse image, under ν, of the image of some connected subset of $\mathscr{D}$, i.e., a sector. We record these facts for future reference in a form which is sufficiently general for our purposes.

1.5 THEOREM. *Suppose S is a sector and $f: S \to E$ is a map such that* (i) *$x \in S$ implies $f(x) \neq 0$;* (ii) *$x \in S$, $t \in R$ implies there exists a positive integer n such that $f(tr) = t^n f(x)$. Then $f(S)$ is a sector in E.*

We recall next the well know fact that the set of fixed points of an involution on a circle K must either be all of K, precisely two points, or empty. Our interest in involutions lies in the fact that if $x \in E$ and $x^2 = 1$, then m_x is an involution on E. Furthermore, if, as above, $\mathscr{D}$ denotes the decomposition on $E - \{0\}$ into open rays, m_x induces an involution on $\mathscr{D}$. It then follows easily that the set, $F(m_x)$ of fixed points (in E) of m_x is either all of E, the union of two distinct rays or zero alone.

Which of the several possibilities obtain in a given instance can sometimes be settled in the following way: Suppose a pair of distinct open rays $(z]$ and $(w]$ are interchanged by m_x, i.e., $m_x((z]) = (w]$. Let S_1, S_2 be the two sectors into which $[z] \cup [w]$ divides E. Observe that $m_x(S_1 \cup S_2) = S_1 \cup S_2$. If S_1 and S_2 are interchanged by m_x, i.e., $m_x(S_1) = S_2$, then only the origin is left fixed. If eithers S_1 or S_2 is mapped into itself by m_x, or in fact, if either $m_x(S_1) \cap S_1$ or $m_x(S_2) \cap S_2$ is non-empty, then $m_x(S_1) = S_1$, $m_x(S_2) = S_2$, and each of S_1 and S_2 contain an open ray of fixed points. In general, if $F(m_x) = [u] \cup [v]$, and if S, S' are the sectors into which $[u] \cup [v]$ divides E then S and S' are interchanged by m_x.

These results are summarized in the following important theorem.

1.6 THEOREM. *Suppose* $x \in E$ *and* $x^2 = 1$. *Then either*

(1) $F(m_x) = \{0\}$,

(2) $F(m_x) = [u] \cup [v]$ *for some pair of distinct rays* $[u]$ *and* $[v]$ *and* m_x *interchanges the two sectors into which* $[u] \cup [v]$ *divides* E, *or*

(3) $F(m_x) = E$. *In particular, suppose a pair of distinct rays* $[z]$ *and* $[w]$ *are interchanged by* m_x *and let* S_1, S_2 *denote the sectors into which* $[z] \cup [w]$ *divides* E. *Then*: $F(m_x) = \{0\}$ *if and only if* S_1 *and* S_2 *are interchanged, while* $F(m_x) = [u] \cup [v]$ *for some* $u \in S_1$, $v \in S_2$ *if and only if either* $m_x(S_1) \cap S_1 \neq \phi$ *or* $m_x(S_2) \cap S_2 \neq \phi$.

1.6.1 COROLLARY. *Suppose* $x \notin R$ *and that* Rx *cuts* E. *If* Rx *does not separate* 1 *from* -1 *then each sector of* $E \backslash Rx$ *contains an element* y *such that* $-y = y$.

Proof. Let $m(y) = -y$ for all $y \in E$. Let S_1 be the sector of $E \backslash Rx$ which contains 1. Then $1 \in m(S_1) \cap S_1$ and the desired conclusion follows from the theorem.

1.6.2 COROLLARY. *Let* U, V *denote the sectors of* $E \backslash R$. *Then either* (1) $-U = V$ *or* (2) $-U = U$. *Furthermore, condition* (1) *is equivalent to any one of the following conditions*:

(3) R *separates* x *and* $-x$ *for all* $x \notin R$;

(4) $-x \neq x$ *for any* $x \neq 0$;

(5) R *separates* x *and* $-x$ *for some* $x \notin R$;

(6) *There exists an* x *such that* Rx *cuts* E *and separates* 1 *and* -1;

(7) Rx *separates* 1 *and* -1 *for all* $x \notin R$;

(8) *for all* $x \neq 0$, Rx *cuts* E *and separates* y *and* $-y$ *if* $y \notin Rx$.

Condition (2) *is equivalent to* (9) *there exists an element* $u \in U$ *and* $v \in V$ *such that* $-u' = u'$ *if* $u' \in [u]$ *and* $-v' = v'$ *if* $v' \in [v]$.

Proof. Set $m(y) = -y$ for all $y \in E$. The various assertions follow directly from the theorem, the previous corollary or the fact that Ry separates x and $-x$ if and only if Rx separates $-y$ and y.

2. In this section we study the shape and nature of G, the component of the identity in the set $H(1)$ of elements having inverses with respect to 1. Recall that L denotes the boundary of G in E. According to [5], G and $H(1)$ are Lie groups and hence are open subsets of E. Furthermore, L is an ideal in $G \cup L$.

Evidently $0 \in L$. Suppose that L contains a non-zero element e. Since $P^- \subset G^-$, $P^-e \subset L$. However, $P^-e \neq L$. For if $P^-e = L$ then a closed subset of E, homeomorphic to the ray P^- forms the boundary of the open set G. On purely topological grounds, $G = E\backslash P^{-}e = E\backslash L$. Thus $-e \notin G$ so $-e \in P^{-}e$, whence $-e = e$. According to the result of the previous section, whenever multiplication by -1 has one ray of fixed points, it has another. That is, there exists $u \notin [e]$ such that $-u = u$. Therefore $u \in G$. However, if $u \in G$ and $-u = u$ then $-1 = 1$, which is absurd. Hence $L \neq [e]$, and we have proved

2.1 THEOREM. *The boundary L of G is not a ray.*

Continue to suppose L contains a non-zero element e. If $L = Re$ (in which case $e \neq -e$), then $G \cup L$ is a semigroup on a half-plane with G the open half plane. Therefore, by [5, § 4], G is isomorphic to the two dimensional vector group and 0 divides L into two subsets A and B such that $AB = 0$. The two sets A and B in this case must be Pe and Ne. Therefore $e^2 = 0$. That this case can occur is shown in Example 1 of the last section. However, it is impossible if E has no nilpotent elements.

2.2 THEOREM. *If E contains no nilpotent elements and L has more than one element then $L \neq Re$ for any $e \in L$.*

If L contains two distinct rays $[e_1]$, $[e_2]$ (as it must, as we have just seen, if E has no nilpotent elements) then $L = [e_1] \cup [e_2]$. For if there exists $e \in L\backslash([e_1] \cup [e_2])$ then $[e] \subset L$, $(e] \cap ([e_1] \cup [e_2]) = \phi$ and $[e]$ divides $G \cup L$ into two sectors. But then G would be contained in one of these sectors. Thus if $[e_i]$ formed one of the bounding rays of G, no point (save zero) of $[e_j]$ could be boundary point of G, which is a contradiction (here, $i, j = 1$ or 2 and $i \neq j$). Thus, if L contains two distinct rays then it is the union of them. Observe that this is even true in case $L = Re$ (for $Re = [e] \cup [-e]$). Therefore, in either case, $G \cup L$ is a semigroup on a half-plane in which G in the open half-plane. Therefore G is isomorphic to the two dimensional vector group and 0 divides L into two sets A and B so that $AB = 0$ and either

(i) $A^2 = B^2 = 0$,

(ii) A and B are groups, or

(iii) A is a group and $B^2 = 0$. Evidently $A = Pe_1$ if $e_1 \in A$ and $B = Pe_2$ if $e_2 \in B$. Thus we have the following

2.3 THEOREM. *If E is a real commutative semigroup and L has more than one point then L is the union of two distinct rays and G is isomorphic to the two dimensional vector group. If E contains no nilpotent elements then L contains two non-zero idempotents e_1, e_2 and $L = [e_1] \cup [e_2]$.*

It is apparent that $L = \{0\}$ if and only if $H(1) = G$, i.e. if and only if $H(1)$ has only one component. If $G = H(1)$ then G is not isomorphic to the two dimensional vector group. Therefore G contains a non-trivial compact subgroup [6; Theorem 41]. Hence, by [4], G is isomorphic to the product of the circle group and the multiplicative group of positive real numbers. It is then an easy matter to prove that E is isomorphic to the multiplicative semigroup of complex numbers. This result also follows from [2] as well as [4].

2.4 THEOREM. *If E is a real commutative semigroup then the following conditions are equivalent*:

(1) *E is isomorphic to the multiplicative semigroup of complex numbers;*

(3) $L = \{0\}$:

(3) *$H(1)$ is connected.*

3. The question was raised in [5] whether multiplication on G and L separately determines multiplication on $G \cup L$. We do not answer this question in general, but show that, under our hypotheses, if L contains two non-zero idempotents then there is only one semigroup on $G \cup L$. In particular, $G \cup L$ is isomorphic to the product semigroup on $P^- \times P^-$.

For $e \in L$, let $G_e = \{g \in G: ge = e\}$; G_e is the isotropy group of e in G. If $e \neq 0$ then G_e is isomorphic to the multiplicative group of positive real numbers. For $\dim G_e = \dim G - \dim (Ge) = \dim G - \dim (e] = 1$. Furthermore, G_e is a closed subset of G and G has no non-trivial compact subgroups. Since $Ge = Re$, Ge is not compact. These conditions imply that G_e is isomorphic to the multiplicative group of positive real numbers. (This proof is patterned after the proof of 3.6 of [5]).

3.1 LEMMA. *Let $e \in L$, $e \neq 0$. Then $e^2 = e$ if and only if $e \in G_e^-$.*

Proof. In general, if $x \in G_e^-$ then $e = ex$ so if $e \in G_e^-$ then $e^2 = e$.

Conversely, suppose $e^2 = e$. Then $P \neq G_e$ since $0 \in P^-$ and $e \neq 0$. If R_1 and R_2 are any one-parameter subgroups of G, $G = R_1 \cdot R_2$. Therefore $G = P \cdot G_2$. Since $e \in G^-$ there is a sequence t_n in P and a sequence $g_n \in G_e$ such that $t_n g_n \to e$. Therefore $t_n g_n e \to e$ so $t_n e \to e$. Thus $t_n \to 1$ so $(1/t_n)(t_n g_n) \to e$. Hence $g_n \to e$ and $e \in G_e^-$.

We are indebted to the referee for suggestions which lead to a considerable shortening of our proof of the following theorem.

3.2 THEOREM. *If L contains two non-zero idempotents then $G \cup L$ is isomorphic to $P^- \times P^-$.*

Proof. Let e_1, e_2 denote the non-zero idempotents of L and let G_i

denote the isotropy group of e_i, $i = 1, 2$. By the lemma, $G_1 \neq G_2$ for then $e_1e_2 = e_2$ and $e_2e_1 = e_1$. Thus $G = G_1G_2$ and hence $Ge_1 = G_2e_1$ and $Ge_2 = G_1e_2$. Moreover, $G_1^- = G_1 \cup \{e_1\}$ and $G_2^- = G_2 \cup \{e_2\}$ and the map from $G_1^- \times G_2^-$ to $G \cup L$ defined by $(x,y) \rightarrow xy$ is one-to-one and onto (clearly so on $G_1 \times G_2$, on $e_1 \times G_2$, on $G_1 \times e_2$ and on $\{e_1, e_2\}$ independently and hence everywhere). Further, if K is a compact subset of $G \cup L$ then Ke_1 and Ke_2 are compact subsets of P^-e_1 and P^-e_2 respectively. Therefore there are elements $g \in G_1$, $h \in G_2$ such that $ke_1 \leqq he_1$ and $ke_2 \leqq gh_1$ for all $k \in K$ (P^-e_1 and P^-e_2 are ordered in the natural way). If we let $[e_1, g]$ denote the subset of G_1^- between e_1 and g (and similarly for $[e_2, h]$), it follows that $K \subset [e_1, g]\cdot[e_2, h]$. Now our mapping sends $[e_1, g] \times [e_2, h]$ onto $[e_1, g]\cdot[e_2, h]$ and is a homeomorphism there. Thus K comes from a compact subset of $G_1^- \times G_2^-$. Hence, the mapping $(x, y) \rightarrow xy$ is a homeomorphism. Since P^-, G_1^- and G_2^- are isomorphic, the theorem follows.

As a corollary to this theorem and Theorem 2.3 we have

3.2.1 COROLLARY. *If E is a real commutative semigroup without nilpotent elements and L has more than one element then $G \cup L$ is isomorphic to the product semigroup on $P^- \times P^-$.*

4. If C is a component of $H(1)$ then $C = xG$ for $x \in C$, so that C is an open subset of E. Therefore $H(1)$ has at most a countable number of components. In this section we show that if E has no nilpotent elements then $H(1)$ has only a finite number of components, in fact, a maximum of four.

Suppose $H(1)$ has an infinite number of components $C_1, C_2, \cdots, C_n \cdots$. Choose $x_i \in C_i$ for each i. Let e_1, e_2 be elements lying in distinct open rays of L. Thus $C_i = x_iG$ and each C_i is a sector with bounding rays $(x_ie_1]$ and $(x_ie_2]$. Let K be a (topological) circle with the origin on the inside. Then each of the rays $(x_ie_1]$ intersects K in at least one point. Since a given ray $(x_ie_1]$ can form the common boundary of at most two of the components of $H(1)$, and since two open rays either identical or disjoint, it is possible to choose an infinite number of points s_n in K so that each s_n belongs to some open ray $(x_ie_1]$ and so that no two of the s_n belong to the some open ray. This set of points must have a cluster point $k \in K$. At most one of the s_n can belong to $(k]$. Assume $s_1 \notin (k]$. Then one of the two sectors of $E\backslash[k] \cup [s_1]$ contains infinitely many of the s_n. Call that one S_1; it does no harm to assume all of the s_n belong to S_1. The s_n can be re-numbered so that for each $n > 1$, s_{n+1} belongs to the sector S_n of$E\backslash[k] \cup [s_n]$ which does not contain s_1. Therefore $(s_{n+1}] \subset S_n$.

If U is a Euclidean neighborhood of k which contains s_n and s_{n+1}, for example, every open ray which is contained in S_n must intersect U. Therefore if $s_{n+1} \in (x_j e_1]$ then $(x_j e_2] \cap U \neq \phi$. Thus it is possible to choose a sequence of points $t_n \in Ee_2$ so that $t_n \to k$, while $s_n t_n = 0$ for every n. Hence $k^2 = 0$, so if E has no nilpotent elements then $H(1)$ has only a finite number of components.

Assume that E has no nilpotent elements. Set $G = C_0$, and let the remaining components be denoted $C_1, \cdots, C_n$. When L has more than one element, $n \neq 0$. According to Example 2 of § 6, $n = 1$ is possible. Assume that $n > 1$. Since $(-G)(-G) = G$, $H(1)/G$ has an element of order 2. Therefore $H(1)$ has an even number of components, so $n \geqq 3$. We show that $n = 3$ and that $H(1)/G$ is the four group.

First, order the rays that lie in the complement of G.

4.1 DEFINITION. If $x, y \notin G^-$ and if $(x]$ and $(y]$ are distinct, then $(x] < (y]$ provided x (and therefore $(x]$) is contained in the sector of $E \backslash [e_2] \cup [y]$ which does not contain G. For any $x \notin G^-$, $(e_2] < (x] < (e_1]$. If S_1 and S_2 are disjoint sectors contained in the complement of G then $S_1 < S_2$ if for every $x \in S_1$ and $y \in S_2$, $(x] < (y]$.

The collection of sectors $C_1, \cdots, C_n$ forms a linearly ordered collection of sectors according to this definition. Assume that they are numbered so that $C_1 < C_2 < \cdots < C_n$. Let D_i denote the sector between C_{i-1} and C_i for $i = 1, \cdots, n$ and let D_{n+1} denote the sector between C_n and $(e_1]$. The sectors D_i, $i = 1, \cdots, n+1$, are to include their bounding open rays.

Let χ denote the squaring function: $\chi(x) = x^2$. Since E has no nilpotent elements, χ maps sectors into sectors, according to Theorem 1.3. In addition, χ has the following special properties:

(1) $\chi(\bigcup_{i=0}^{n} c_i) = \bigcup_{i=0}^{n} C_i$;

(2) $\chi(\bigcup_{i=1}^{n+1} D_i) \subset \bigcup_{i=1}^{n+1} D_i$;

(3) $\chi(C_0) = C_0$ and $\chi(C_i) \cap C_i = \phi$ if $i > 0$. The latter part of (3) is due to the fact that $C_i = x_i G$ for $x_i \in C_i$. Therefore $\chi(C_i) = x_i^2 G$. But if $x_i^2 \in C_i$ then $x_i^2 = x_i g$ for some $g \in G$, so $x_i \in G$ which is a contradiction.

If x is sufficiently near e_2 then x^2 is near e_2. Therefore $\chi(D_1) \subset D_1$. Furthermore, $\chi(C_1)$ is a sector having boundary points in common with $\chi(D_1)$, and $\chi(C_1)$ is disjoint from $C_1 \cup D_1$. Thus $\chi(C_1)$ must have points in common with G, whence $\chi(C_1) = G$. It follows that C_1 contains an element whose square is 1. Hereafter, x_1 will denote this element; i.e., $x_1 \in C_1$ and $x_1^2 = 1$.

Choose $x_i \in C_i$ for $i = 2, \cdots, n$. Now

$$x_1(C_1 \cup C_2) = x_1(x_1 G \cup x_2 G) = G \cup x_1 x_2 G .$$

Set $S = C_1 \cup D_2 \cup C_2$. Then $x_1 S$ is a sector which contains $G \cup x_1 x_2 G$ and no other components of $H(1)$. Therefore either $x_1 x_2 G = x_1 G$ or $x_1 x_2 G = x_n G$. But $x_1 x_2 G = x_1 G$ implies $x_2 G = G$ which is a contradiction. Therefore $x_1 x_2 G = x_n G$. On the other hand, $x_2(x_1 G \cup G) = x_2 x_1 G \cup x_2 G$.

An argument similar to the preceding shows that x_2x_1G is either x_1G or x_3G. Since $x_2x_1 = x_1x_2$, $x_2x_1G \neq x_1G$. Therefore $x_2x_1G = x_3G$. But

$$x_2x_1G = x_1x_2G = x_nG \ .$$

Hence $n = 3$, so $H(1)$ has four components.

The argument used above to show that C_1 contains an element whose square is 1 applies with only trivial changes to C_3. Therefore we assume that $x_3^2 = 1$. Thus $H(1)/G$ has four elements, at least two of which have order two. Therefore $H(1)/G$ is the four group and $(C_i)(C_i) = G$ for $i = 0, 1, 2,$ or 3. We have proved the following result.

4.2 THEOREM. *If E is a commutative real semigroup without nilpotent elements then $H(1)$ has one, two or four components. When $H(1)$ has four components, $H(1)/G$ is the four group.*

5. Throughout this section E is assumed to be a real commutative semigroup without nilpotent elements. Thus if L has more than one element, as it will if $H(1)$ has more than one component, then L contains two non-zero idempotents and $G \cup L$ is isomorphic to $P^- \times P^-$.

Even if $H(1)$ has four components a question remains concering the location of $-G$. In a certain sense the question is irrelevant and $-G$ can be any one of C_1, C_2 or C_3; in this connection see Remark 5.4. However, the "correct" location for $-G$, viz. $-G = C_2$, is recognizable, modulo the choice of R.

5.1 THEOREM. *Assume that E has no nilpotent elements, and that L has more than one element. Then precisely one component of $H(1)$ lies between G and $-G$ if and only if R separates x and $-x$ for some x (equivalently, $x \neq 0$ implies $x \neq x$). In particular, if $x \neq 0$ implies $x \neq -x$ then $H(1)$ has four components.*

Proof. Let e_1 and e_2 denote the non-zero idempotents of L. If $x \neq -x$ for $x \neq 0$ then $(-e_1] < (-e_2]$ (in the sense of Definition 4.1). Let S denote the open second quadrant—that is the sector of elements x such that $(e_2] < (x] < (-e_1]$. Thus $E = G \cup (-G) \cup (S^-) \cup (-S^-)$. Let $C = \{x \in S: x^2 \in S^-\}$ and $D = \{x \in S: x^2 \in -S^-\}$. If $S \cap H(1) = \phi$ then $S = C \cup D$. Since C and D are closed subsets of S and S is connected, if both C and D are non-empty then $C \cap D \neq \phi$. But if $x \in C \cap D$ then $x^2 \in S^- \cap (-S^-)$ so $x^2 = 0$ which is a contradiction. Therefore either $S = C$ or $S = D$. If x is sufficiently near e_2 then $x^2 \notin -S^-$ while if x is sufficient near $-e_1$, then $x^2 \notin S^-$. Therefore $H(1) \cap S \neq \phi$ so that at leat one component of $H(1)$ is contained in S—that is to say, between G and $-G$. A similar argument shows that one component of $H(1)$ is contained in $-S$. Since $H(1)$ can have at most four components it must in this case have exactly four. Therefore precisely one component of $H(1)$ lies between G and $-G$.

Conversely, suppose precisely one component of $H(1)$ lies between G and $-G$; i.e., $-G = C_2$. Now $-C_1 \neq G$ and $-C_1 \neq -G$ so $-C_1 = C_3$. Therefore there exists $x \in E$ which is separated from $-x$ by R. The proof of the theorem is complete.

Under the hypotheses of the previous theorem the relative positions of $-e_1$, x_1 and $-e_2$ are known and according to this theorem $(x_1] < (-e_1]$. (Recall that x_1 is the member of C_1 such that $x_1^2 = 1$). However, which of $(x_1e_1]$ and $(x_1e_2]$ occurs first has not yet been determined. Since x_1G forms the component of $E \backslash [x_1e_1] \cup [x_1e_2]$ which does not contain G, $(x_1]$ is contained between $(x_1e_1]$ and $(x_1e_2]$. Let e be the member of the pair $\{e_1, e_2\}$ such that $(x_1e] < (x_1]$. Let S denote the sector of $E - [x_1e] \cup (e_1]$ which contains G. Let A be an arc from e_2 to x_1 which is contained entirely in $S \cup \{e_2\}$. Then e_2A is connected and cannot contain any elements of $H(1) \cup \{0\}$. Since $e_2 \in e_2A$, $e_2A \subset D_1$ so $(x_1e_2] < (x_1]$ and $e = e_2$.

The two rays $(e_1]$ and $(x_1e_1]$ are distinct and are interchanged under multiplication by x_1. Since $x_1(x_1G) = G$, each of the sectors of $E \backslash [e_1] \cup [x_1e_1]$ is mapped into itself by x_1 and therefore each of these sectors contains a ray of fixed points of x_1. Therefore there exists a (non-zero) point $z \in D_1$ such that if $y \in D_1$ then $x_1y = y$ if and only if $y \in [z]$. Either $(x_1e_2] = D_1 = (e_2]$ and $x_1e_2 = e_2$, or z is in the interior or D_1. Now in the proof preceding Theorem 4.2 (which only used hypotheses available here) it was shown that if $x \in D_1$ then $x^2 \in D_1$. Now $x_1z^2 = (x_1z)z = z^2$ so $z^2 \in (z]$. Therefore $z^2 = tz$ for some $t > 0$ so z/t is an idempotent in D_1 which is different from e_2 unless $x_1e_2 = e_2$, and which is different from zero in any case. In a similar fashion it can be shown that D_4 contains a non-zero idempotent which is different from e_1 unless $x_3e_1 = e_1$. (Where $x_3 \in C_3$ and $x_3^2 = 1$).

Therefore, if E has no nilpotent element, $x \neq 0$ implies $x \neq -x$ and if E has exactly four idempotents then $x_1e_2 = e_2$ and $x_3e_1 = e_1$. Furthermore, since $-x_1 = x_3$, we have $x_3e_2 = -e_2$ and $x_1e_1 = -x_1$. Thus

$$E = G^- \cup (-G^-) \cup C_1 \cup C_3$$

and it is now an easy matter to construct an isomorphism from E onto $R \times R$. (For this purpose, make use of Corollary 3.2.1).

There are several alternatives to the hypothesis that E contains exactly four idempotents. For if L contains more than one element and it is assumed that either $H(1)$ is dense in E or $L \cup (-L)$ is an ideal in E then $x_1e_2 = e_2$, $x_1e_1 = -e_1$, etc. and the conclusion that E is isomorphic to $R \times R$ follows as before. In virtue of the existence of non-zero idempotents in D_1 and D_4, another alternative is to assume that $x^2 \in G^-$

for all $x \in E$. Finally, since $(x_1e_2)^2 = e_2$, $(x_1e_1)^2 = e_1$, etc., it may be assumed that $x \notin L \cup -L$ implies $x^2 \notin L \cup (-L)$. We have proved the following result.

5.2 THEOREM. *If E is a real commutative semigroup without nilpotent elements and if $x \neq 0$ implies $x \neq -x$ then the following conditions are equivalent*:

(1) *E is isomorphic to $R \times R$;*

(2) *E has precisely four idempotents;*

(3) *L contains more than one element and $H(1)$ is dense in E;*

(4) *L contains more than one element and $x \in E$ implies $x^2 \in G^-$;*

(5) *L contains more than one element and $L \cup (-L)$ is an ideal in E;*

(6) *L contains more than one element and $x \notin L \cup (-L)$ implies $x^2 \notin L \cup (-L)$.*

The examples in § 6 show that none of the conditions (2)-(6) of the previous theorem implies E is isomorphic to $R \times R$ if the condition that $x \neq 0$ implies $x \neq -x$ is simply dropped. However, a certain weakening of this condition is possible.

Our proof of this fact makes use of Theorem 5.2.

5.3 THEOREM. *If E is a real commutative semigroup without nilpotent elements and if $x \in L\backslash\{0\}$ implies $x \neq -x$ then conditions* (1)-(4) *of the previous theorem are equivalent.*

Proof. Suppose E contains precisely four idempotents. Then L contains two non-zero idempotents e_1 and e_2. By hypothesis, $-e_1 \neq e_1$ and $-e_2 \neq e_2$. If $(-e_1] < (-e_2]$ then all of the hypotheses of the previous theorem are satisfied so E is isomorphic to $R \times R$.

Therefore suppose $(-e_2] < (-e_1]$. By theorem 5.1, either the sector between $(e_2]$ and $(-e_2]$ or the sector between $(-e_1]$ and $(e_1]$ is devoid of members of $H(1)$. There is little to distinguish between the two cases so we suppose the former holds; that is, $D_1 \cap H(1) = \phi$. Since the rays $(-e_1]$ and $(e_1]$ are interchanged under multiplication by -1 and $-(-G)=G$, there is a (non-zero)point $z \in D_1$ such that if $y \in D_1$, then $-y = y$ if and only if $y \in [z]$. As we have seen before, $z^2 \in D_1$, so D_1 contains a non-zero idempotent; this idempotent must be distinct from e_2 since $-e_2 = e_2$. But this means E has a fifth idempotent, which is a contradiction. Hence $(-e_1] < (-e_2]$ and (2) implies (1).

If L contains more than one element then E has at least four idempotents. If E contains five idempotents then there is an idempotent $e \notin G^- \cup (-G^-)$. Therefore P^-e and $Re_1 \cup Re_2$ divide E into five components whose union contains $H(1)$. But $H(1)$ can have at most four component so $H(1)$ is not dense in E. Therefore (3) implies (2).

Assume that (4) holds. Then G^- has four idempotents so since $x \in E$ implies $x^2 \in G^-$ there are no more. Therefore (4) implies (2). The remaining implications are obvious so the proof of the theorem is complete.

It is to be noticed that conditions (5) and (6) of Theorem 5.2 are not listed in Theorem 5.3. Example 4 of § 6 shows that (6) cannot be included while Example 5 shows that Condition (4) cannot.

5.4 REMARK. The previous two theorems yield criteria on E in order that it be isomorphic to $R \times R$ under a sort of canonical isomorphism—namely an isomorphism in which R maps onto the diagonal of $R \times R$. Conditions that there exist *some* isomorphism can be made somewhat weaker. Assume that $H(1)$ is known to have four components G, C_1, C_2 and C_3 where, as usual, $C_1 < C_2 < C_3$. Which one of these last three components is $-G$ depends on the particular choice of the isomorph of the real numbers in the beginning. For $-G$ is simply the component of $H(1)$ containing -1. Set $R_i = (Px_i) \cup P^-$ where $x_i \in C_i$ and $x_i^2 = 1$. Obviously R_i is a sub-semigroup of E which is isomorphic to R. Therefore by replacing a given choice of R by another (isomorphic) choice, it becomes apparent that every such semigroup on E in which $-G$ is a given C_i is isomorphic (under the identity isomorphism) to one in which $-G$ is C_j, $i, j = 1, 2$ or 3.

5.5 THEOREM. *If E is a real commutative semigroup without nilpotent elements in which for at least one $x \in L$, $-x \neq x$ then the following conditions are equivalent:*

(1) *E is isomorphic to $R \times R$;*
(2) *E has precisely four idempotents;*
(3) *$H(1)$ is dense in E;*
(4) *$x \in E$ implies $x^2 \in G^-$.*

Proof. The hypotheses imply that L has two non-zero idempotents e_1 and e_2 and we may suppose $-e_1 \neq e_1$. If $-e_2 \neq e_2$ then the hypotheses of Theorem 5.3 are satisfied and the conditions are already known to be equivalent.

Suppose $e_2 = -e_2$ and let S be the component of $E \backslash Re_1$ which does not contain G. There is $x \in S$ such that $-x = x$. Therefore $-x^2 = x^2$ so $x^2 \in (e_2] \cup (x]$.

If $x^2 \in (x]$ then S contains an idempotent so (2) cannot hold. If (4) holds then $x^2 \in G^-$ so $x^2 \in (e_2]$. Thus, if either (2) or (4) holds then$x^2 \in (e_2]$.

Therefore suppose $x^2 \in (e_2]$. Let C denote the sector of $E\backslash([x] \cup [e_1])$ which does not contain G. An argument analogous to that given in the proof of Theorem 5.1 shows that there is $y \in C$ such that $y^2 \in G \cup (-G)$. It follows that $H(1)$ has a third, and therefore a fourth, component.

Under the present circumstances, these components can be labeled G, $-G$, C_2 and C_3. Let x_2 be the member of C_2 such that $x_2^2 = 1$. Set $R_2 = P^- \cup (Px_2)$. Replacing R by R_2 yields a reals semigroup on E which satisfies the hypotheses of Theorem 5.3. Conditions (2) and (4) are unaffected by this change. Therefore (2) implies (1) and (4) implies (1).

If $H(1)$ is dense then $H(1) \cap S \neq \phi$ so again we see that $H(1)$ has four components. Hence replacement of R by R_2 as above yields a semigroup on E satisfying the hypotheses of Theorem 5.3. Therefore (3) implies (1).

The remaining implications are obvious.

6. **We conclude with several examples.** In addition to being examples of the various classes of semigroups mentioned in the introduction, they reveal a certain amount of indepencence among the conditions in Theorem 5.2.

It is convenient, in one way or another, to regard E as being coordinatized in the usual way. Thus we can speak readily of such terms as the y-axis, the open or closed first quadrant, etc.

All details concerning the proofs that the various multiplication are continuous and associative are omitted. However, many of the missing details are contained in the simple proof that the following canonical procedure for extending certain semigroups works: Suppose that r is an involution on E and that S is a closed sector in E so that $r(S) \cap S$ is either one of the bounding rays of S or the origin, and that r is the identity on $r(S) \cap S$. Assume that S is endowed with the structure of a commutative semigroup so that $r(S) \cap S$ is a subsemigroup. Let $T = S \cup r(S)$. Define multiplication on T as follows:

(1) if $x, y \in S$ then xy has its original meaning;

(2) if $x \in S$, $y \in r(S)$ then $xy = yx = r(xr(y))$;

(3) if $x, y \in r(S)$ then $xy = r(x)r(y)$. Then T is a commutative semigroup containing S as a sub-semigroup. We shall refer to this semigroup as the *extension of S to T by r.*

All of the following examples satisfy condition (*).

EXAMPLE 1. Let $G \cup L$ be the semigroup "II b." of [5; p. 387] but regard it as embedded in the closed first quadrant (multiplication no longer has any simple relation to the coordinates). In other words, we have a commutative semigroup with identity on the closed first quadrant so that G is the open first quadrant, L is the union of the nonnegative x and y-axes, $L^2 = 0$ and G^- has a sub-semigroup P^- which is isomorphic to the multiplicative semigroup of nonnegative real numbers, and $0 \in P^-$. Let $r(x, y) = (-y, -x)$ and let $S = G \cup L$. Then $T = r(S) \cup S$ is the union of the first and third quadrants, and $r(S) \cap S = \{0\}$.

Extend S to T by r. Next identify each point $(0, -y)$ with $(y, 0)$ and $(-x, 0)$ with $(0, x)$. The result is a semigroup on the plane which may be visualized as follows: G is the upper half-plane, $-G$ is the lower half-plane, R is the line $y = x$ and $L = Re$ where e is a point on the positive x-axis.

In Example 1, every condition listed in Theorem 5.2 (except condition (1), of course) is satisfied except the condition on nilpotent elements; $H(1)$ has two components.

EXAMPLE 2. Let $G \cup L$ be the closed first quadrant with coordinate-wise multiplication. Let $S = G \cup L$ and let r be the involution given in Example 1. Let $T = S \cup r(S)$ and extend S to T by r. Identify points as in Example 1. The result can be pictured as a semigroup on a plane in which G is the upper half-plane, $-G$ is the lower half-plane, R is the line $y = x$ and $L = Re_1 \cup Re_2$ where $e_1 = (1, 0)$ and $e_2 = (-1, 0)$. Both e_1 and e_2 are idempotents but $-e_1 = e_1$ and $-e_2 = e_2$.

In Example 2, all of the conditions listed in Theorem 5.2 (except Condition (1)) are met except the stipulation that $x \neq 0$ implies $-x \neq x$; $H(1)$ has two components.

EXAMPLE 3. Let $G \cup L$ be the closed first quadrant under coordinate-wise multiplication. Let $P = \{(x, x): x > 0\}$. Let A be the arc of the unit circle which lies in the closed fourth quadrant. Let $e_1 = (1, 0)$ and $z = (0, -1)$. Assume that A is endowed with the structure of a commutative semigroup so that e_1 is the identity and z is a zero for A. Regard the closed fourth quadrant Q as the product of A and P^- (with all points $0a$, $a \in A$, being identified with the origin). Let $S = G \cup L \cup Q$ and extend multiplication to L as follows: if $g \in G \cup L$, $x \in Q$ then $gx = xg = (ge_1)x$. Observe that $(ge_1)x$ is well defined since $ge_1 \in P^-e_1$, e_1 is the identity on Q and multiplication of any $x \in Q$ and $t \in P^-$ is the ordinary coordinate-wise product of x and t. It is a simple matter to check that S is a commutative semigroup containing $G \cup L$ as sub-semigroup. Let r be reflection about the y-axis. Then $E = S \cup r(S)$ is the plane and the extension of S to E by r yields a real semigroup with the following properties:

(1) E has no nilpotent elements;

(2) $H(1)$ has two components G and $-G$ and $G^- \cap (-G^-)$ is a ray;

(3) $H(1)$ is not dense in E.

EXAMPLE 4. Let $G \cup L$ be the closed first quadrant under coordinate-wise multiplication. Let P be as in Example 3 and let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let r be the involution $r(x, y) = (-y, -x)$. Set $S = G \cup L$ and $T = S \cup r(S)$. Extend S to T by r. Thus $-e_2 = (-1, 0)$ and $-e_1 = (0, -1)$. Let Q_1 and Q_2 denote the open second and fourth quadrants respectively. Let A_i be the arc of the unit circle contained in Q_i^-, $i = 1, 2$. Let $z_i \in A_i$ be the point on the line $y = -x$ in Q_i. Assume that A_i has the structure of a semigroup so that e_i is an identity, z_i is a zero and $(-e_i)^2 = e_i$, $i = 1, 2$. Regard Q_i^- is the product of A_i and P^-, indentifying all points $0a$, $a \in A_i$, with the origin as usual. Extend multiplication to E as follows: if $g \in G^-$ and $x \in Q_i$ then $gx = xg = (ge_i)x$ as in Example 3. Define multiplication by -1 to be reflection about the line $y = -x$ and if $g \in -G$, $x \in Q_i$, then

$$gx = xg = -(-gx) .$$

Finally, if $x \in Q_1$, $y \in Q_2$ define $xy = yx = 0$. Then E is a real semigroup with the following properties:

(1) E has no nilpotent elements;

(2) $H(1)$ has two conponents G and $-G$ and $G \cap (-G^-) = \{0\}$.

(3) $x \notin L \cup (-L)$ implies $x^2 \notin L \cup (-L)$. This example shows that in the proof of Theorem 5.1 the condition that R separate e_2 and $-e_2$ is essential.

EXAMPLE 5. Let H denote the semigroup on the closed right half-plane defined as follows:

(1) xy is the ordinary coordinate-wise product if either x or y belongs to the first quadrant; otherwise

(2) $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_2 \leqq 0$ and $y_2 \leqq 0$; define

$$xy = (x_1y_1, -(x_2y_2)) .$$

Let z be a point in the fourth quadrant on the line $y = -x$ and let $e_2 = (0, 1)$. Let C denote the closed sector bounded by $[z]$ and $[e_2]$ which contains the first quadrant. Shrink the set H to coincide with C and let S' be the semigroup induced on C by that on H. Let r' denote reflection about the line $y = -x$ and $T' = S' \cup r'(S')$; extend S' to T' by r'. Shrink T' to coincide with the closed right half-plane and let S denote the induced semigroup. Finally, let r be reflection about the y-axis. Let $T = S \cup r(S)$ and extend S to T by r. Choose R to lie in the first and third quadrants. Then $E = T$ becomes a real semigroup with the following properties:

(1) E has no nilpotent elements;

(2) L contains more than one element and $x \in L\backslash\{0\}$ implies $x \neq -x$;

(3) $L \cup (-L)$ is an ideal in E. However, E is not isomorphic to $R \times R$.

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