EXTENSIONS OF SHEAVES OF ASSOCIATIVE ALGEBRAS BY NONTRIVIAL KERNELS

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Introduction. Let X be a topological space, Λ a sheaf of associative algebras over X and A a sheaf of two-sided Λ-modules considered as a sheaf of algebras with trivial multiplication. It was shown in [1] that the group $F(\Lambda, A)$ of equivalence classes of algebra extensions of Λ with A as kernel occurs naturally in an exact sequence

$$\cdots \to H^1(X, A) \to F(\Lambda, A) \to \mathrm{Ext}^2(\Lambda, A) \to H^2(X, A) \to \cdots$$

where $H^*(X, A)$ denotes the Cech cohomology of X with coefficients in A. In this paper the same question will be discussed for the case in which A has a non-trivial multiplication. It will be shown that under appropriate hypothese $F(\Lambda, A)$ occurs in a similar exact sequence, except that in the other terms of the sequence, A must be replaced by the "bicenter" K_A of A. A precise statement of the main result of this paper is given in Theorem 2. The methods used here are an adaptation of those used by S. MacLane in [2].

1. The extension problem. Let R be a sheaf of rings on a topological space X. If C and D are sheaves of R-modules, then $Hom_R(C, D)$ will denote the sheaf of germs of R-homomorphisms of C into D and $Ext^n_R(C, D)$ will denote the nth derived functor of $Hom_R(C, D)$. If A is a sheaf of associative R-algebras, then, as usual, A^* will denote the opposite of R-algebras and $A^e = A \otimes_R A^*$ will denote the enveloping sheaf of A. A is a sheaf of A^e-modules, the operation of A^e on A being given by the formula $(\lambda \otimes \mu^*)(\gamma) = \lambda\gamma\mu$.

Now, let $M'_A = Hom_{A*}(A, A) \oplus Hom_A(A, A)$

where $\oplus$ denotes the direct sum. Then M'_A, being the direct sum of sheaves of rings, is itself a sheaf of rings and A can be considered as a sheaf of left and right M'_A-modules as follows: Let $\sigma = (\sigma_1, \sigma_2) \in M'_A$. Then the left action is given by $\sigma(a) = \sigma_1(a)$ and the right action by $(a)\sigma = \sigma_2(a)$. Let

$$M_A = \{\sigma \in M'_A \mid a(\sigma b) = (a\sigma) b \text{ for all } a, b \in A\}.$$

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Then M_A is a subsheaf of subrings of M'_A. M_A will be called the sheaf of germs of bimultiplications of A. Note that we cannot assert that A is a sheaf of M_A^e-modules since we do not know that $(\sigma a)\tau = \sigma(a\tau)$. If σ and τ satisfy this relation then they are called permutable bimultiplications. The natural ring homomorphisms $A \to Hom_{A*}(A, A)$ and $A \to Hom_A(A, A)$ given respectively by left and right multiplication induce a ring homomorphism $\mu: A \to M_A$ whose image is a sheaf of two-sided ideals. The kernel K_A of μ will be called the bicenter of A and the cokernel P_A of μ will be called the sheaf of germs of outer bimultiplications of A. P_A is a sheaf of rings and K_A is a sheaf of left and right P_A-modules. As above, K_A is not a sheaf of P_A^e-modules. Elements $\bar{\sigma}$ and $\bar{\tau}$ of P_A such that $(\bar{\sigma}a)\bar{\tau} = \bar{\sigma}(a\bar{\tau})$ for all $a \in K_A$ will be called permutable. Note that $\bar{\sigma}$ and $\bar{\tau}$ are permutable if and only if representative elements σ and τ in M_A are also permutable.

An extension of a sheaf Λ of R-algebras by a sheaf A of R-algebras is an exact sequence.

$$0 \to A \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \to 0 \tag{1}$$

of sheaves of R-algebras and R-algebra homomorphisms. As in [1], we shall say that such a sequence is locally trivial if there exists a covering $\mathscr{U} = \{U_\alpha\}$ of X such that the restriction of the sequence to each U_α splits as an exact sequence of sheaves of R-modules. Hence if (1) is locally trivial then there exist R-module homomorphisms $j_\alpha : \Lambda \mid U_\alpha \longrightarrow \Gamma \mid U_\alpha$ with $p\ j_\alpha =$ identity. Furthermore, since A is a sheaf of two-sided ideals in Γ, the map $\mu: A \longrightarrow M_A$ extends to a map $\mu_\Gamma: \Gamma \longrightarrow M_A$. Thus, we may define the composition

$$\theta_\alpha = (\text{coker}\ \mu) \circ \mu_\Gamma \circ j_\alpha : \Lambda \mid U_\alpha \to P_A \mid U_\alpha\ .$$

Since $(j_\beta - j_\alpha) : \Lambda \mid U_{\alpha\beta} \longrightarrow A \mid U_{\alpha\beta}$, we see that $\theta_\beta = \theta_\alpha$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Hence $\{\theta_\alpha\}$ determines an element $\theta \in \text{Hom}_R(\Lambda, P_A)$. We shall say that this θ is induced by the extension (1). Clearly θ is an algebra homomorphism whose image consists of permutable elements. Note that this implies that K_A is a sheaf of Λ^e-modules via the operation of P_A on K_A.

If $\theta \in \text{Hom}_R(\Lambda, P_A)$ is an algebra homomorphism whose image consists of permutable elements, then, with respect to the usual equivalence relation, we wish to classify the extensions which induce θ in the manner described above.

2. The complexes. From [1], we recall that a sheaf B of R-modules is said to be weakly R-projective if each stalk B_x is an R_x-projective module and it is said to be R-coherent if there exists a covering $\mathscr{U} = \{U_\alpha\}$ such that for each U_α there are integers p and q and R-homomorphisms so that the sequence

$$R^p \mid U_\alpha \longrightarrow R^q \mid U_\alpha \longrightarrow B \mid U_\alpha \longrightarrow 0$$

is exact. Also, as in [1], $C^*(X, B)$ will denote the direct limit over coverings $\mathscr{U}$ indexed by X of the Cech cohomology complexes $C^*(\mathscr{U}, B)$. If $S_0(\Lambda) = R$ and $S_n(\Lambda)$, $n > 0$ denotes the n-fold tensor product of Λ with itself, then we define

$$L^{i,j}(B) = C^i(X, Hom_R(S_j(\Lambda), B)) .$$

PROPOSITION 1. If X is paracompact Hausdorff and if Λ is weakly R-projective and R-coherent, then, for each $n \geqq 0$,

$$0 \longrightarrow L^{*,n}(K_A) \longrightarrow L^{*,n}(A) \xrightarrow{\mu^*} L^{*,n}(M_A) \xrightarrow{\pi^*} L^{*,n}(P_A) \longrightarrow 0$$

is an exact sequence of complexes, the mappings being those induced by the exact sequence of sheaves

$$0 \longrightarrow K_A \longrightarrow A \xrightarrow{\mu} M_A \xrightarrow{\pi} P_A \longrightarrow 0$$

Proof. In [1] it was shown that if Λ is weakly R-projective and R-coherent then so is $S_n(\Lambda)$ and hence the sheaves $Ext^i_R(S_n(\Lambda), B) = 0$ for $i > 0$, $n \geqq 0$ and for all B. Hence, for each $n \geqq 0$, there is an exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow Hom_R(S_n(\Lambda), K_A) \longrightarrow Hom_R(S_n(\Lambda), A) \longrightarrow Hom_R(S_n(\Lambda), M_A) \\ \longrightarrow Hom_R(S_n(\Lambda), P_A) \longrightarrow 0 . \end{aligned}$$

If X is paracompact Hausdorff then $C^*(X, -)$ is an exact functor and hence we get the indicated sequence of complexes.

We would like to consider each of the complexes $L^{i,j}(-)$ in the preceding proposition as a bicomplex in some manner which reflects a given structure of K_A as a sheaf of Λ^e-modules and which coincides with the usual structure of $Hom_R(S_n(\Lambda), -)$ as a complex. This is too much to ask, but such a structure on $L^{i,j}(A)$ can be approximated as follows: Let $\theta \in \mathrm{Hom}(\Lambda, P_A)$ be an algebra homomorphism whose image consists of permutable elements. If θ is regarded as an element of $L^{0,1}(P_A)$, then by exactness there is an element $\sigma \in L^{0,1}(M_A)$ such that $\pi_*(\sigma) = \theta$. Let σ be represented by cocycle $\{\sigma_\alpha\}$ on some sufficiently fine covering $\mathscr{U}$. Given this date, we can define a "coboundary" operator δ_σ on $L^{m,n}(A)$ by the following formula. Let $k \in L^{m,n}(A)$ be represented by a cochain $\{k_{\alpha_0,\dots,\alpha_m}\}$ on $\mathscr{U}$. Then

$$\begin{aligned} \delta_\sigma k_{\alpha_0,\dots,\alpha_m}(\lambda_1,\dots,\lambda_{n+1}) &= \sigma_{\alpha_0}(\lambda_1)k_{\alpha_0,\dots,\alpha_m}(\lambda_2,\dots,\lambda_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i k_{\alpha_0,\dots,\alpha_m}(\lambda_1,\dots,\lambda_i\lambda_{i+1},\dots,\lambda_{n+1}) \\ &+ (-1)^{n+1} k_{\alpha_0,\dots,\alpha_m}(\lambda_1,\dots,\lambda_n)\sigma_{\alpha_0}(\lambda_{n+1}) . \end{aligned}$$

We shall see that the restriction of δ_σ to $L^{i,j}(K_A)$ is in fact a good coboundary operator.

In order to investigate the properties of δ_σ and the relations between δ_σ and the Cech coboundary operator $\hat{\delta}$, we must introduce some more notation.

(2.1) To avoid constantly writing variables we make the following convention: If r is a function of p variables and s is a function of q variables, both with values in an algebra, then $r\cdot s$ is the function of $p+q$ variables defined by

$$r\cdot s(\lambda_{1,\ldots,}\lambda_{p+q}) = r(\lambda_{1,\ldots,}\lambda_p)\cdot s(\lambda_{p+1,\ldots,}\lambda_{p+q}) .$$

(2.2) m will denote ambiguously the multiplication in all of the algebras which appear here.

(2.3) Since θ is an algebra homomorphism,
$\pi_*(\sigma_\alpha\cdot\sigma_\alpha - \sigma_\alpha\circ m) = 0$ Hence there exists an $f \in L^{0,2}(A)$ which is represented by a cochain $\{f_\alpha\}$ on $\mathscr{U}$ such that

$$\mu_* f_\alpha = \sigma_\alpha\cdot\sigma_\alpha - \sigma_\alpha\circ m .$$

(2.4) Since $\pi_*(\hat{\delta}\sigma) = \hat{\delta}\,\pi_*(\sigma) = 0$, there exists an $h \in L^{1,1}(A)$ which is represented by a cochain $\{h_{\alpha\beta}\}$ on $\mathscr{U}$ such that

$$\mu_* h_{\alpha\beta} = (\hat{\delta}\sigma)_{\alpha\beta} .$$

(2.5) If $\sigma' \in L^{0,1}(M_A)$ also satisfies $\pi_*(\sigma') = \theta$, then $\pi_*(\sigma' - \sigma) = 0$ and hence there exists a $\bar{\sigma} \in L^{0,1}(A)$ which is represented by a cochain $\{\bar{\sigma}_\alpha\}$ on $\mathscr{U}$ such that

$$\mu_*\,\bar{\sigma}_\alpha = \bar{\sigma}'_\alpha - \bar{\sigma}_\alpha .$$

Using these notations the following result in easily checked:

PROPOSITION 2. If $k \in L^{m,n}(A)$ is represented by $\{k_{\alpha_0,\ldots,\alpha_m}\}$ on $\mathscr{U}$, then

(2.6) $\delta_\sigma\delta_\sigma k_{\alpha_0,\ldots,\alpha_m} = f_{\alpha_0}\cdot k_{\alpha_0,\ldots,\alpha_m} - k_{\alpha_0,\ldots,\alpha_m}\cdot f_{\alpha_0}$

(2.7) $\delta_\sigma(\hat{\delta}\,k)_{\alpha_0,\ldots,\alpha_{m+1}} = (\hat{\delta}\,\delta_\sigma\,k)_{\alpha_0,\ldots,\alpha_{m+1}} - h_{\alpha_0,\,\alpha_1}\,k_{\alpha_1,\ldots,\alpha_{m+1}}$
$\qquad - (-1)^{n+1}k_{\alpha_1,\ldots,\alpha_{m+1}}h_{\alpha_0,\alpha_1}$

(2.8) $\delta_{\sigma'}\,k_{\alpha_0,\ldots,\alpha_m} = \delta_\sigma k_{\alpha_0,\ldots,\alpha_m} + \bar{\sigma}_{\alpha_0}\,k_{\alpha_0,\ldots,\alpha_m} + (-1)^{n+1}k_{\alpha_0,\ldots,\alpha_m}\,\bar{\sigma}_{\alpha_0}$.

COROLLARY. *$L^{i,j}(K_A)$ is a bicomplex with respect to the pair of differential operators $\hat{\delta}$, δ_σ. The total differential operator is given by*

$$\delta = (-1)^{j+1}\hat{\delta} + \delta_\sigma .$$

This differential operator depends only on θ.

Finally, we shall need to know something about the behavior of $\hat{\delta}$ on products of low dimensional cochains, where Cech cochains are multiplied by multiplying the values (suitably restricted when necessary) on corresponding elements of the nerve of a covering according to the convention of 2.1. It is easy to verify the following statements by explicit calculation.

PROPOSITION 3. If $r \in L^{0,p}(A)$ and $s \in L^{0,q}(A)$ are represented on $\mathscr{U}$ by $\{r_\alpha\}$ and $\{s_\alpha\}$ respectively, then $\hat{\delta}(r \cdot s) \in L^{1,p+q}(A)$ and

$$\text{(2.9)} \quad \hat{\delta}(r\cdot s)_{\alpha\beta} = (\hat{\delta}r)_{\alpha\beta}\cdot s_\alpha + r_\alpha \cdot (\hat{\delta}s)_{\alpha\beta} + (\hat{\delta}r)_{\alpha\beta}\cdot(\hat{\delta}s)_{\alpha\beta} .$$

If $t \in L^{1,p}(A)$ and $u \in L^{1,q}(A)$ are represented on $\mathscr{U}$ by $\{t_{\alpha\beta}\}$ and $\{u_{\alpha\beta}\}$ respectively then $\hat{\delta}(t\cdot u) \in L^{2,p+q}(A)$ and

$$\text{(2.10)} \quad (\hat{\delta}(t\cdot u))_{\alpha\beta\gamma} = (\hat{\delta}t)_{\alpha\beta\gamma}\cdot u_{\alpha\gamma} + t_{\alpha\gamma}\cdot(\hat{\delta}u)_{\alpha\beta\gamma} + (\hat{\delta}t)_{\alpha\beta\gamma}\cdot(\hat{\delta}u)_{\alpha\beta\gamma} - t_{\alpha\beta}\cdot u_{\beta\gamma} - t_{\beta\gamma}\cdot u_{\alpha\beta} .$$

Finally, if $r \in L^{m,p}(A)$ and $s \in L^{m,q}(A)$ then δ_σ satisfies the good coboundary formula.

$$\text{(2.11)} \quad \delta_\sigma(r\cdot s) = (\delta_\sigma r)\cdot s + (-1)^p r\cdot \delta_\sigma s .$$

3. The obstruction. We shall regard the complex $L^{i,j}(K_A)$ as being filtered by the second degree and we define $F^p(L) = \sum_{j \geqq p} L^{i,j}(K_A)$. In analogy with the proceedings of [1], the classical results for extensions of algebras suggest that each algebra homomorphism $\theta \in \operatorname{Hom}(\Lambda, P_A)$ whose range consists of permutable elements determines an "obstruction" in $H^3(F^1(L))$; this obstruction being zero if and only if there exists an extension which induces θ in the manner described in § 1. A representative cocycle for such a cohomology class would be an element of $L^{2,1}(K_A) \oplus L^{1,2}(K_A) \oplus L^{0,3}(K_A)$.

Let $\sigma \in L^{0,1}(A)$ satisfy $\pi_*\sigma = \theta$ and let

$f \in L^{0,2}(A)$ and $h \in L^{1,1}(A)$ be defined as in 2.3 and 2.4. Then the components of a representative cocycle of the "obstruction" to θ are defined as follows:

(i) Since $\mu_*(\hat{\delta}h) = \hat{\delta}\mu_* h = 0$, there exists an element $a \in L^{2,1}(K_A)$ which is represented by a cochain $\{a_{\alpha\beta\gamma}\}$ on $\mathscr{U}$ such that

$$a_{\alpha\beta\gamma} = (\hat{\delta}h)_{\alpha\beta\gamma}$$

(ii) A standard elementary calculation shows that $\mu_*(\delta_\sigma f) = 0$.

Hence there exists an element $c \in L^{0,3}(K_A)$ which is represented by a cochain $\{c_\alpha\}$ on $\mathscr{U}$ such that $c_\alpha = \delta_\sigma f_\alpha$.

(iii) An equally elementary calculation shows that $\mu_*[\hat{\delta} f - \delta_\sigma h - h \cdot h] = 0$. Hence there exists an element $b \in L^{1,2}(K_A)$ which is represented by a cochain $\{b_{\alpha\beta}\}$ on $\mathscr{U}$ such that

$$b_{\alpha\beta} = -(\hat{\delta} f)_{\alpha\beta} + \delta_\sigma h_{\alpha\beta} + h_{\alpha\beta} \cdot h_{\alpha\beta}$$

THEOREM 1. *Let $s = a \oplus b \oplus c$. Then s is a cocycle of $F^1(L)$ whose cohomology class depends only on θ.*

DEFINITION. The cohomology class of s will be denoted by $Ob(\theta)$ and will be called the obstruction to θ.

THEOREM 2. *Let X be paracompact Hausdorff and let Λ be weakly R-projective and R-coherent. Then $Ob(\theta) = 0$ if and only if there is an extension of Λ by A which induces θ. If $OB(\theta) = 0$, then the set $F_\theta(\Lambda, A)$ of equivalence classes of extensions which induce θ is in one-to-one correspondence with the set of elements of the group $H^2(F^1K)$, and hence the following two sequences are exact.*

(1) $0 \longrightarrow H^1[\mathrm{Hom}_R(S_*(\Lambda), K_A)] \longrightarrow \mathrm{Ext}^1_{\Lambda^e}(\Lambda, K_A) \longrightarrow H^1(X, K_A)$
$\longrightarrow F_\theta(\Lambda, A) \longrightarrow \mathrm{Ext}^2_{\Lambda^e}(\Lambda, K_A) \longrightarrow H^2(X, K_A) \longrightarrow$

(2) $0 \longrightarrow H^2[\mathrm{Hom}_R(S_*(\Lambda), K_A)] \longrightarrow F_\theta(\Lambda, A) \longrightarrow$
$H^1(X, \mathrm{Hom}_R(\Lambda, K_A)) \longrightarrow$

Proof of Theorem 1. It is clear that $\hat{\delta} a = \hat{\delta}\hat{\delta} h = 0$, and, by 2.6, that $\delta_\sigma c = \delta_\sigma \delta_\sigma f = 0$. Thus, to prove that s is a cocycle we must show that $\hat{\delta} b = \delta_\sigma a$ and that $\hat{\delta} c = -\delta_\sigma b$. To derive the first expression, we have by definition that

$$(\hat{\delta} b)_{\alpha\beta\gamma} = -(\hat{\delta}\hat{\delta} f)_{\alpha\beta\gamma} + (\hat{\delta}\delta_\sigma h)_{\alpha\beta\gamma} + \hat{\delta}(h \cdot h)_{\alpha\beta\gamma}$$

The first term is zero and the second and third terms can be expanded by 2.7 and 2.10 respectively. After obvious cancellations, this yields

$$(\hat{\delta} b)_{\alpha\beta\gamma} = \delta_\sigma(\hat{\delta} h)_{\alpha\beta\gamma} + (\hat{\delta} h)_{\alpha\beta\gamma} \cdot h_{\alpha\gamma} + h_{\alpha\gamma} \cdot (\hat{\delta} h)_{\alpha\beta\gamma} + (\hat{\delta} h)_{\alpha\beta\gamma} \cdot (\hat{\delta} h)_{\alpha\beta\gamma}$$

Since $\hat{\delta} h = a \in L^{2,1}(K_A)$, on a sufficiently fine covering multiplication by $(\hat{\delta} h)_{\alpha\beta\gamma}$ is zero and hence $\hat{\delta} b = \delta_\sigma a$. Similarly, since $c = \delta_\sigma f$, $\hat{\delta} c$ can be expanded by the equation, 2.7, for commuting $\hat{\delta}$ and δ_σ. The resulting expression can be simplified by using equations 2.6 and 2.11 and the definition of b in (ii). This yields easily that

$$(\hat{\delta} c)_{\alpha\beta} = \delta_\sigma[(\hat{\delta} f)_{\alpha\beta} - \delta_\sigma h_{\alpha\beta} - h_{\alpha\beta} \cdot h_{\alpha\beta}] = -\delta_\sigma b_{\alpha\beta} .$$

Thus s is a cocycle.

The definition of s depends on the choices of b, h, and f. We shall show that changing any of these changes s by a coboundary and that any cocycle cohomologous to s can be obtained by such a choice.

Suppose that h' satisfies $\mu_* h' = \hat{\delta}\sigma$ and f' satisfies $\mu_* f' = \sigma\cdot\sigma - \sigma\circ m$. Then $h' - h = \bar{h} \in L^{1,1}(K_A)$ and $f' - f = \bar{f} \in L^{0,2}(K_A)$. If s' denotes the cocycle corresponding to σ, h' and f', then it is easy to see that

$$s' - s = (\hat{\delta} + \delta_\sigma)\bar{h} + (-\hat{\delta} + \delta_\sigma)\bar{f} = \delta(\bar{h} + \bar{f}) .$$

Conversely, if $\bar{h}\oplus\bar{f}$ is any 2-cochain of $F^1(L)$, then $h + \bar{h}$ and $f + \bar{f}$ are admissable liftings of $\hat{\delta}_\sigma$ and $\sigma\cdot\sigma - \sigma\circ m$ respectively and this change alters s by $\delta(\bar{h}\oplus\bar{f})$. Hence, in this manner we obtain all cocycles cohomologous to s.

It remains to show that if $\pi_*\sigma' = \theta$, then h' and f' can be chosen so that the corresponding cocycle $s' = a' + b' + c' = s$. Since $\pi_*(\sigma'\cdot -\sigma) = 0$, there is a $\bar{\sigma} \in L^{0,1}(A)$ such that $\mu_*\bar{\sigma} = \sigma' - \sigma$. Let $h' = h + \hat{\delta}\bar{\sigma}$ and $f' = f + \delta_\sigma\bar{\sigma} + \bar{\sigma}\cdot\bar{\sigma}$. Then it is immediate that h' and f' are liftings of $\hat{\delta}\sigma'$ and $\sigma'\cdot\sigma' - \sigma'\cdot m$ respectively and that $a' = \hat{\delta}h' = a$. The difference $\delta_{\sigma'}f' - \delta_\sigma f$ can be expressed by 2.8. Using 2.6 and 2.11, it is easily seen that this difference is zero and hence $c' = c$. The only difficult point is to show that $b' = b$. By definition

$$b' = -\hat{\delta}f' + \delta_{\sigma'}h' + h'\cdot h'$$

Using the definitions of f' and h' and rearranging terms, we arrive at the equality

$$\begin{aligned} b_{\alpha\beta}' - b_{\alpha\beta} = {} & [\delta_\sigma\hat{\delta}\bar{\sigma}_{\alpha\beta} - \hat{\delta}\delta_\sigma\bar{\sigma}_{\alpha\beta}] + [\bar{\sigma}_\alpha\cdot h_{\alpha\beta} + h_{\alpha\beta}\cdot\bar{\sigma}_\alpha + \hat{\delta}\bar{\sigma}_{\sigma\beta}\cdot h_{\alpha\beta} + h_{\alpha\beta}\cdot\hat{\delta}\bar{\sigma}_{\alpha\beta}] \\ & + [\bar{\sigma}_\alpha\cdot\hat{\delta}\bar{\sigma}_{\alpha\beta} + \hat{\delta}\bar{\sigma}_{\alpha\beta}\cdot\bar{\sigma}_\alpha + \hat{\delta}\bar{\sigma}_{\alpha\beta}\cdot\hat{\delta}\bar{\sigma}_{\alpha\beta} - \hat{\delta}(\bar{\sigma}\cdot\bar{\sigma})_{\alpha\beta}] . \end{aligned}$$

The third bracket is zero by the formula 2.9 for the Cech coboundary of a product and the first bracket equals $-h_{\alpha\beta}\cdot\bar{\sigma}_\beta - \bar{\sigma}_\beta\cdot h_{\alpha\beta}$ by the rule 2.7 for interchanging $\hat{\delta}_\sigma$ and δ. Hence the sum of the first two brackets is zero and therefore $b' = b$.

Proof of Theorem 2. Suppose $0 \longrightarrow A \xrightarrow{i} \Gamma \xrightarrow{p} \Lambda \longrightarrow 0$ is an extension. By Proposition 3.1 of [1], the hypotheses imply that any such extension is locally trivial considered as an extension of sheaves of R-modules. Hence there exists a covering $\mathscr{U} = \{U_\alpha\}$ which carries R-module homomorphisms $j_\alpha\cdot\Lambda \mid U_\alpha \longrightarrow \Gamma \mid U_\alpha$ with $p\cdot j_\alpha =$ identity. If $\sigma_\alpha : \Lambda \mid U_\alpha \rightarrow M_A \mid U_\alpha$ is defined by $[\sigma_\alpha(\lambda)](a) = j_\alpha(\lambda)\cdot a$ and $(a)[\sigma_\alpha(\lambda)] = a\cdot j_\alpha(\lambda)$ then $\{\sigma_\alpha\}$ determines an element $\sigma \in L^{0,1}(M_A)$ which is a lifting of the homomorphism θ induced as in § 1 by the given extension. If we define $h_{\alpha\beta} = j_\beta - j_\alpha$ and $f_\alpha = j_\alpha j_\alpha - j_\alpha\circ m$, then the corresponding elements $h \in L^{1,1}(A)$ and $f \in L^{0,\hat{2}}(A)$ satisfy $\mu_* h = \delta\sigma$ and $\mu_* f = \sigma\cdot\sigma -$

$\sigma \circ m$. Elementary calculations show that for this choice of h and f we get that $s = a \oplus b \oplus c = 0$ and hence $Ob(\theta) = 0$.

Conversely, if $Ob(\theta) = 0$, then on some sufficiently fine covering $\mathscr{U}$, we may choose $\{f_\alpha\} \in \hat{C}^0(\mathscr{U}, Hom_R(S_2(\Lambda), A))$ and $\{h_{\alpha\beta}\} \in \hat{C}^1(U, Hom_R(\Lambda, A))$ so that $\delta_\sigma f_\alpha = 0$, $(\hat{\delta} h)_{\alpha\beta\gamma} = 0$ and $(\hat{\delta} f)_{\alpha\beta} = \delta_\sigma h_{\alpha\beta} + h_{\alpha\beta} \cdot h_{\alpha\beta}$. As in [1], we define Γ to be the sheaf which is the quotient of $\bigcup_\alpha (A \oplus \Lambda) \mid U_\alpha$ by the relation

$$(a + h_{\alpha\beta}(\lambda), \lambda)_\alpha \sim (a, \lambda)_\beta \text{ for } (a, \lambda) \in A \oplus \Lambda \mid U_{\alpha\beta} .$$

Multiplication in Γ is given by the formula

$$(a, \lambda)_\alpha \cdot (a', \lambda')_\alpha = (aa' + \sigma_\alpha(\lambda)a' + a\sigma_\alpha(\lambda) + f_\alpha(\lambda, \lambda'), \lambda\lambda')_\alpha .$$

It is easy to show that this multiplication is associative since $\delta_\sigma f = 0$ and that it agrees with the equivalence relation since $\hat{\delta} f = \delta_\sigma h + h \cdot h$.

It follows then, exactly as in MacLane [2] that the set of equivalence classes of extensions which realize a given θ with $Ob(\theta) = 0$ is in one-to-one correspondence with the set of elements of the group $H^2(F^1(L))$. The exact sequences are derived exactly as in [1] from the exact sequences of complexes

$$0 \longrightarrow F^1L \longrightarrow F^0L \longrightarrow E_0^{*,0} \longrightarrow 0$$

and

$$0 \longrightarrow F^2L \longrightarrow F^1L \longrightarrow E_0^{*,1} \longrightarrow 0 .$$

4. Examples. (1) If $K_A = 0$ then all obstructions are zero and all terms involving K_A in the exact sequence containing $F_\theta(\Lambda, A)$ are zero. Hence there is a unique extension of Λ by A which induces a given $\theta \in \mathrm{Hom}_R(\Lambda, P_A)$. As in MacLane [2], this extension can be described as the "graph" of θ; i.e., the pull-back of the pair of maps θ: $\Lambda \longrightarrow P_A$, π: $M_A \longrightarrow P_A$.

(2) If $K_A = A$, then the map μ: $A \longrightarrow M_A$ is the zero map and hence $M_A = P_A$. Consequently, if $\theta \in \mathrm{Hom}_R(\Lambda, P_A)$ is given, then σ may be chosen equal to θ and so $\delta\sigma$ and $\sigma \cdot \sigma - \sigma \circ m$ are both zero. Therefore, any cocycle $f \oplus h \in L^{0,2}(A) \oplus L^{1,1}(A)$ is a lifting of these two terms. It follows that $Ob(\theta) = 0$ and that $F_\theta(\Lambda, A) = H^2(F^1L)$. Thus the results of [1] are a special case of the results of this paper.

(3) We wish to discuss more thoroughly a remark in § 3.3 of [1]. Let X be paracompact Hausdorff and let Λ be a weakly R-projective and R-coherent sheaf of R-algebras. Suppose that A is a sheaf of R-algebras and that

$$0 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow 0$$

is an exact sequence of R-modules. Let $\mathscr{U} = \{U_\alpha\}$ be a sufficiently fine covering of X and let $\{j_\alpha\} \in \hat{C}^0(\mathscr{U}, Hom_R(\Lambda, \Gamma))$ determine the locally trivial structure of Γ and let $h_{\alpha\beta} = (\hat{\delta} j)_{\alpha\beta}$. An algebra homomorphism $\theta \in \mathrm{Hom}_R(\Lambda, P_A)$ whose image consists of permutable elements will be called compatible with the locally trivial structure of Γ if there exists a lifting $\sigma \in L^{0,1}(M_A)$ of θ which is represented by a cochain $\{\sigma_\alpha\}$ on $\mathscr{U}$ such that $\mu_* h = \hat{\delta}\sigma$. Furthermore, an element $f \in L^{0,2}(A)$ will be called a multiplication compatible with θ and h if $\mu_* f = \sigma \cdot \sigma - \sigma \circ m$, $\hat{\delta} f = \delta_\sigma h + h \cdot h$ and $\delta_\sigma f = 0$. The set of equivalence classes with respect to the usual equivalence relation of such multiplications will be denoted by $F_{\theta,h}(\Lambda, A)$. We wish to calculate $F_{\theta,h}(\Lambda, A)$.

Proceeding as in § 2, let $f \in L^{0,2}(A)$ be a cochain such that $\mu_* f = \sigma \cdot \sigma - \sigma \circ m$. Corresponding to $f \oplus h$ there is an obstruction cocycle $s(h) = c \oplus b \oplus 0$. The only relevant changes of $s(h)$ are given by varying f by an element $\bar{f} \in L^{0,2}(K_A)$. Such a change alters s by a coboundary in F^2L. Hence we obtain the result:

THEOREM. *Corresponding to θ and h, there is an obstruction cohomology class $Ob(\theta, h) \in H^3(F^2L)$ which is zero if and only if there exists a multiplication compatible with θ and h. If $Ob(\theta, h) = 0$ then $F_{\theta,h}(\Lambda, A)$ is in one-to-one correspondence with the elements of the group $H^2[Hom_R(S_*(\Lambda), K_A)]$.*

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