THE SOCHOCKI-PLEMELJ FORMULA FOR THE FUNCTIONS OF TWO COMPLEX VARIABLES

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Introduction. In the case of one complex variable the following theorems are well known [3]:

1. Let C be a rectifiable oriented Jordan arc or curve and $f(\zeta)$ an integrable function defined on C, analytic at a point $z_0 \in C$ (in case C is an arc we suppose z_0 is different from both endpoints of C). Then the function

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

possesses the left and right limit $F_l(z_0)$ and $F_r(z_0)$, respectively, when the point ζ approaches to the point z_0 remaining permanently on one side of C and the relation

$$F_l(z_0) - F_r(z_0) = f(z_0)$$

holds.

2. Under the same conditions concerning the curve C suppose $f(\zeta)$ satisfies at every point $\zeta_1 \in C$ the Hölder condition

$$|f(\zeta) - f(\zeta_1)| \leqq M|\zeta - \zeta_1|^\alpha\,, \qquad M > 0\,, \qquad 0 < \alpha \leqq 1\,.$$

Then $F(z)$ possesses at almost every point $z_0 \in C$ the left and right limit when the point ζ approaches to z_0 along a non-tangent path to C and

$$F_l(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + \frac{1}{2}f(z_0)\,,$$
$$F_r(z_0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta - \frac{1}{2}f(z_0)\,.$$

The improper integral on the right hand side is taken in the Cauchy sense.

The aim of the present note is to extend these theorems to the theory of functions of two complex variables.[1] We start with Bergman's integral formula [1], [2] which generalizes the Cauchy formula for the case of functions of several variables. It would be very interesting to obtain similar results starting with other integral formulas which are similar to Bergman's formula e.g. A. Weil's formula [6] or later forms

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[1] Analogous results about the limits of exterior differential forms have been obtained by C. H. Look and T. D. Chung, see [4].

of it, see [5].

The case of a bicylinder. Let D be a bicylinder bounded by the hypersurfaces

$$\begin{matrix} z_1 - e^{i\lambda_1} = 0 \,, & |z_2| \leqq 1 \\ z_2 - e^{i\lambda_2} = 0 \,, & |z_1| \leqq 1 \end{matrix} \qquad \lambda_j \in [0, 2\pi]$$

and let $f(\zeta_1, \zeta_2)$ be an integrable function defined on the distinguished boundary surface d of D

$$(z_1 = e^{i\lambda_1}) \times (z_2 = e^{i\lambda_2}) \,.$$

1. Suppose that $f(\zeta_1, \zeta_2)$ is analytic at a point $z_1^0, z_2^0 \in D$. We consider the function

$$(1) \qquad F(z_1, z_2) = -\frac{1}{4\pi^2}\iint_d \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1 d\zeta_2 \,.$$

Since $f(\zeta_1, \zeta_2)$ is analytic at the point $z_1^0, z_2^0 \in d$, there exists a small bicylinder B which contains z_1^0, z_2^0 inside and such that $f(\zeta_1, \zeta_2)$ is analytic in $\bar{B}$. B is bounded by the hypersurfaces

$$\begin{matrix} z_1 - z_1^0 - r_1 e^{i\lambda_3} = 0 \,, & |z_2| \leqq r_2 \\ z_2 - z_2^0 - r_2 e^{i\lambda_4} = 0 \,, & |z_1| \leqq r_1 \,. \end{matrix} \qquad r_j > 0 \,, \quad \lambda_{j+2} \in [0, 2\pi] \,, \quad j = 1, 2 \,.$$

Suppose that the point z_1, z_2 belongs to DB, the intersection of D and B. Then using the integral formula for the function $f(\zeta_1, \zeta_2)$ and the domain DB we obtain

$$(2) \qquad \begin{aligned} f(z_1, z_2) = &-\frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_2\bar{B}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_2 d\zeta_1 \\ &-\frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_4\bar{B}} - \frac{1}{4\pi^2}\int_{d_2\bar{B}}\int_{d_3\bar{B}} - \frac{1}{4\pi^2}\int_{d_3\bar{B}}\int_{d_4\bar{B}} \,, \end{aligned}$$

where d_j, $j = 1, 2, 3, 4$ denotes the positive oriented circle $z_j - e^{i\lambda_j} = 0$ and $z_j - z_j^0 - r_j e^{i\lambda_{j+2}} = 0$, $j = 1, 2$, respectively. (The integrands missing in the formula (2) are equal to that of the first integral.)

From (1) and (2) results

$$(3) \quad \begin{aligned} F(z_1, z_2) = &-\frac{1}{4\pi^2}\int_{d_1 - d_1\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1}\int_{d_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2}\, d\zeta_2 - \frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_2 - d_2\bar{B}} \\ &+ f(z_1, z_2) + \frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_4\bar{B}} + \frac{1}{4\pi^2}\int_{d_2\bar{B}}\int_{d_3\bar{B}} + \frac{1}{4\pi^2}\int_{d_3\bar{B}}\int_{d_4\bar{B}} \,. \end{aligned}$$

Let z_1, z_2 approach to the point z_1^0, z_2^0 remaining inside the bicylinder D, then

$$f(z_1, z_2) \to f(z_1^0, z_2^0) \,,$$

$$(4) \qquad \frac{1}{4\pi^2}\int_{d_3\bar{B}}\int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \to \frac{1}{4\pi^2}\int_{d_3\bar{B}}\int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)d\zeta_1 d\zeta_2}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)} \,.$$

Using the Cauchy formula for the domain which lies on the z_2-plane and is bounded by the curves $d_2\bar{B}$ and $d_4\bar{B}$, we obtain

$$(4') \qquad \int_{d_2\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = 2\pi i f(\zeta_1, z_2) - \int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \,.$$

When the point z_1, z_2 tends to z_1^0, z_2^0 it results from (4′)

$$\lim_{z_1, z_2 \to z_1^0, z_2^0} \int_{d_2\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 = 2\pi i f(\zeta_1, z_2^0) - \int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 \,.$$

The convergence is uniform with respect to $\zeta_1 \in d_3\bar{B}$, therefore,

$$(5) \qquad \lim_{z_1, z_2 \to z_1^0, z_2^0} \frac{1}{4\pi^2}\int_{d_2\bar{B}}\int_{d_3\bar{B}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2$$
$$= \frac{1}{4\pi^2}\int_{d_3\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0}\left\{2\pi i f(\zeta_1, z_2^0) - \int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2\right\} .$$

In a similar way we obtain the formula

$$(6) \qquad \lim_{z_1, z_2 \to z_1^0, z_2^0} \frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2$$
$$= \frac{1}{4\pi^2}\int_{d_4\bar{B}} \frac{d\zeta_2}{\zeta_2 - z_2^0}\left\{2\pi i f(z_1^0, \zeta_2) - \int_{d_3\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_1 - z_1^0} d\zeta_1\right\} .$$

For the first and second term on the right hand side of (3) we obtain the limits ($z_1, z_2 \in DB$):

$$(7') \qquad \lim_{z_1, z_2 \to z_1^0, z_2^0}\left\{- \frac{1}{4\pi^2}\int_{d_1 - d_1\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1}\int_{d_2} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2\right\}$$
$$= \left\{- \frac{1}{4\pi^2}\int_{d_1 - d_1\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0}\left\{2\pi i f(\zeta_1, z_2^0) - \int_{d_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2\right\}\right.$$
$$\left. + \int_{d_2 - d_2\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2\right\}$$

and

$$(7'') \qquad \lim_{z_1, z_2 \to z_1^0, z_2^0}\left\{- \frac{1}{4\pi^2}\int_{d_1\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1}\int_{d_2 - d_2\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2\right\}$$
$$= - \frac{1}{4\pi^2}\int_{d_2 - d_2\bar{B}} \frac{d\zeta_2}{\zeta_2 - z_2^0}\left\{2\pi i f(z_1^0, \zeta_2) - \int_{d_3\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_1 - z_1^0} d\zeta_1\right\} .$$

From (3), (4), (5), (6), (7′) and (7″) results

$$F_i(z_1^0, z_2^0) = \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ z_1, z_2 \in D}} F(z_1, z_2) = f(z_1^0, z_2^0) + \frac{1}{2\pi i}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0}\, d\zeta_1 + \frac{1}{2\pi i}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 - \frac{1}{4\pi^2}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0} \cdot \int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \,. \tag{8}$$

When the point z_1, z_2 does not belong to D and tends to z_1^0, z_2^0, we obtain three values for the exterior limit $F_{ik}(z_1^0, z_2^0)$, $k = 1, 2, 3$, (in this case we need to put 0 instead of $f(z_1, z_2)$ in (2) and similar changes ought to be made in (7′) and (7″))

$$\begin{aligned} F_{i1}(z_1^0, z_2^0) &= \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ |z_1|>1,\ |z_2|>1}} F(z_1, z_2) \\ &= -\frac{1}{4\pi^2}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \,; \\ F_{i2}(z_1^0, z_2^0) &= \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ |z_1|<1,\ |z_2|>1}} F(z_1, z_2) \\ &= -\frac{1}{4\pi^2}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \\ &\quad + \frac{1}{2\pi i}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \,; \\ F_{i3}(z_1^0, z_2^0) &= \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ |z_1|>1,\ |z_2|<1}} F(z_1, z_2) \\ &= -\frac{1}{4\pi^2}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{d\zeta_1}{\zeta_1 - z_1^0}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \\ &\quad + \frac{1}{2\pi i}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0}\, d\zeta_1 \,. \end{aligned} \tag{9}$$

Therefore

$$\begin{aligned} F_i(z_1^0, z_2^0) - F_{i1}(z_1^0, z_2^0) &= f(z_1^0, z_2^0) + \frac{1}{2\pi i}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0}\, d\zeta_1 \\ &\quad + \frac{1}{2\pi i}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \,; \\ F_i(z_1^0, z_2^0) - F_{i2}(z_1^0, z_2^0) &= f(z_1^0, z_2^0) + \frac{1}{2\pi i}\int_{a_1 - a_1\bar{B} - a_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0}\, d\zeta_1 \,; \\ F_i(z_1^0, z_2^0) - F_{i3}(z_1^0, z_2^0) &= f(z_1^0, z_2^0) + \frac{1}{2\pi i}\int_{a_2 - a_2\bar{B} - a_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0}\, d\zeta_2 \,. \end{aligned}$$

REMARK. The formulas (10) can be transformed as follows.

According to the well-known formula for the function $f(\zeta_1, z_2^0)$ of one complex variable ζ_1, which is analytic at the point $\zeta_1 = z_1^0$, we have (see [3])

$$G_i^{(z_2^0)}(z_1^0) = \lim_{\substack{z_1 \to z_1^0 \\ |z_1|<1}} \frac{1}{2\pi i}\int_{d_1} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1} d\zeta_1 = \frac{1}{2\pi i}\int_{d_1 - d_1\bar{B} - d_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0} d\zeta_1 + f(z_1^0, z_2^0) .$$

Suppose the radius r_1 of the circle d_3 tends to 0, then

$$\lim_{r_1 \to 0} \frac{1}{2\pi i}\int_{d_1 - d_1\bar{B} - d_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0} d\zeta_1 = G_i^{(z_2^0)}(z_1^0) - f(z_1^0, z_2^0) .$$

Similarly, we have

$$\lim_{r_2 \to 0} \frac{1}{2\pi i}\int_{d_2 - d_2\bar{B} - d_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_1 = G_i^{(z_1^0)}(z_2^0) - f(z_1^0, z_2^0) .$$

On the other hand, we have

$$\lim_{r_1 \to 0} \frac{1}{2\pi i}\int_{d_1 - d_1\bar{B} - d_3\bar{B}} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1^0} d\zeta_2 = G_l^{(z_2^0)}(z_1^0) = \lim_{\substack{z_1 \to z_1^0 \\ |z_1|>1}} \frac{1}{2\pi i}\int_{d_1} \frac{f(\zeta_1, z_2^0)}{\zeta_1 - z_1} d\zeta_1$$

$$\lim_{r_2 \to 0} \frac{1}{2\pi i}\int_{d_2 - d_2\bar{B} - d_4\bar{B}} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2^0} d\zeta_2 = G_l^{(z_1^0)}(z_2^0) = \lim_{\substack{z_2 \to z_2^0 \\ |z_2|>1}} \frac{1}{2\pi i}\int_{d_2} \frac{f(z_1^0, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 .$$

Therefore,

$$F_i(z_1^0, z_2^0) - F_{l1}(z_1^0, z_2^0) = G_i^{(z_2^0)}(z_1^0) + G_i^{(z_1^0)}(z_2^0) ,$$
$$(= f(z_1^0, z_2^0) + G_l^{(z_2^0)}(z_1^0) + G_l^{(z_1^0)}(z_2^0))$$

(10*) $$F_i(z_1^0, z_2^0) - F_{l2}(z_1^0, z_2^0) = G_i^{(z_2^0)}(z_1^0) , \qquad (= f(z_1^0, z_2^0) + G_l^{(z_2^0)}(z_1^0))$$
$$F_i(z_1^0, z_2^0) - F_{l3}(z_1^0, z_2^0) = G_i^{(z_1^0)}(z_2^0) , \qquad (= f(z_1^0, z_2^0) + G_l^{(z_1^0)}(z_2^0)) .$$

2. Suppose now that the function $f(\zeta_1, \zeta_2)$ is not analytic at z_1^0, z_2^0 but satisfies the condition

(11) $$|f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)| \leqq M \cdot |\zeta_1 - z_1^0|^{\alpha_1} \cdot |\zeta_2 - z_2^0|^{\alpha_2} ,$$
$$M > 0, \ \alpha_j > 0, \ j = 1, 2 .$$

We have

(12) $$F(z_1, z_2) = -\frac{1}{4\pi^2}\iint_d \frac{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2$$
$$-\frac{1}{4\pi^2}\iint_d \frac{f(z_1^0, z_2^0)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 .$$

Since $f(z_1^0, z_2^0)$ is analytic, we can apply the formulas (8) and (9) to the second term of (12).

According to the assumption (11) the improper integral

$$J(z_1^0, z_2^0) = -\frac{1}{4\pi^2}\iint_d \frac{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)}\, d\zeta_1 d\zeta_2$$

exists. Let $\rho_j(z_j, d_j)$ be the Euclidean distance of the point z_j from the circle d_j, $j = 1, 2$. We shall show that the limit

$$\lim_{z_1, z_2 \to z_1^0, z_2^0} J(z_1, z_2) = \lim_{z_1, z_2 \to z_1^0, z_2^0} \left\{-\frac{1}{4\pi^2}\iint_d \frac{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1 d\zeta_2\right\} = J(z_1^0, z_2^0)$$

exists when the point z_1, z_2 tends to z_1^0, z_2^0 in such a way that the rations $|z_j - z_j^0| : \rho_j(z_j, d_j)$, $j = 1, 2$, are bounded, i.e.,

$$\frac{|z_j - z_j^0|}{\rho_j(z_j, d_j)} < A\,, \qquad A > 0,\ j = 1, 2\,. \tag{13}$$

In fact, we have

$$\begin{aligned} J(z_1, z_2) - J(z_1^0, z_2^0) &= -\frac{1}{4\pi^2}\iint_d \frac{[f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)]}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)} \\ &\quad \cdot \frac{[(\zeta_1 - z_1^0)(\zeta_2 - z_2^0) - (\zeta_1 - z_1)(\zeta_2 - z_2)]}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_1 d\zeta_2 \\ &= -\frac{1}{4\pi^2}\left\{\int_{d_1\bar{B}}\int_{d_2\bar{B}} + \int_{d_1 - d_1\bar{B}}\int_{d_2\bar{B}} + \int_{d_1 - d_1\bar{B}}\int_{d_2 - d_2\bar{B}} + \int_{d_2 - d_2\bar{B}}\int_{d_1\bar{B}}\right\} \end{aligned}$$

(the integrands missing in the formula (14) are equal to that of the first term). The third term on the right hand side tends to 0 when $z_1, z_2 \to z_1^0, z_2^0$. The first term can be written in the form

$$\begin{aligned} -\frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_2\bar{B}} &\frac{[f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)]}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)} \\ &\cdot \left\{\frac{z_1 - z_1^0}{\zeta_1 - z_1} + \frac{z_2 - z_2^0}{\zeta_2 - z_2}\left(1 + \frac{z_1 - z_1^0}{\zeta_1 - z_1}\right)\right\} d\zeta_2 d\zeta_1\,. \end{aligned} \tag{15}$$

Suppose the radii r_j, $j = 1, 2$, of the bicylinder B are so small that for $\zeta_j \in d_j\bar{B}$, $j = 1, 2$, we have $|\zeta_j - z_j^0| \leqq \delta$, where $\delta > 0$ is an arbitrary fixed number. Let z_1, z_2 satisfy the condition (13), then using (11) we obtain

$$\begin{aligned} &\left| -\frac{1}{4\pi^2}\int_{d_1\bar{B}}\int_{d_2\bar{B}} \frac{[f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)]}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)} \cdot \left\{\frac{z_1 - z_1^0}{\zeta_1 - z_1} + \frac{z_2 - z_2^0}{\zeta_2 - z_2}\left(1 + \frac{z_1 - z_1^0}{\zeta_1 - z_1}\right)\right\} d\zeta_2 d\zeta_1 \right| \\ &\leqq \frac{1}{4\pi^2} M\{A + A(1 + A)\}\int_{d_1\bar{B}}\int_{d_2\bar{B}} \frac{|d\zeta_1||d\zeta_2|}{|\zeta_1 - z_1^0|^{1-\alpha_1}|\zeta_2 - z_2^0|^{1-\alpha_2}} < \text{const.}\ \delta^{\alpha_1 + \alpha_2}\,. \end{aligned}$$

Therefore, for sufficiently small fixed $\delta > 0$ and z_1, z_2 sufficiently near to z_1^0, z_2^0 the first and third term on the right hand side of (14) are arbitrary small. Similarly, the remaining two terms of (14) tend to 0 when $\delta \to 0$.

For the difference between the interior and exterior limits of $F(z_1 z_2)$ we obtain the same formulas (10), (10*) assuming that z_1, z_2 tends to z_1^0, z_2^0 in such a way that the conditions (13) are satisfied.

The interior limit $F_i(z_1^0, z_2^0)$ is equal to $J(z_1^0, z_2^0)$ plus the terms of the right hand side of (8). Similarly, we obtain three values of the exterior limits $F_{lj}(z_1^0, z_2^0)$, $j = 1, 2, 3$, adding $J(z_1^0, z_2^0)$ to the terms of the right hand side of (9).

A general domain with the distinguished boundary surface. Suppose the given domain D is bounded by three[2] analytic hypersurfaces (for definitions see [1], [2])

$$\Phi_j(z_1, z_2, \lambda_j) = 0 , \qquad\qquad j = 1, 2, 3 ,$$

and let z_1^0, z_2^0 be a fixed point which lies on the part of the intersection d_{12} of the hypersurfaces $\Phi_1(z_1, z_2, \lambda_1) = 0$, $\Phi_2(z_1, z_2, \lambda_2) = 0$ which belongs to the boundary of D. We assume that z_1^0, z_2^0 does not belong to the hypersurface $\Phi_3(z_1, z_2, \lambda_3) = 0$.

1. Let $f(z_1, z_2)$ be a continuous function defined on the distinguished boundary surface d of D, analytic at the point z_1^0, z_2^0. We consider the function $F(z_1, z_2)$ defined by Bergman's integral formula[3] [2]

$$(16) \qquad F(z_1, z_2) = -\frac{1}{4\pi^2}\iint_{d_{12}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \cdot \frac{\{\Phi_1(z_1, \zeta_2 \lambda_1)\Phi_2(z_1, z_2, \lambda_2) - \Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, \zeta_2, \lambda_2)\}}{\Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, z_2, \lambda_2)} d\zeta_1, d\zeta_2 - \frac{1}{4\pi^2}\iint_{d_{13}} - \frac{1}{4\pi^2}\iint_{d_{23}} ,$$

z_1, z_2 lies outside $\Phi_j(z_1, z_2, \lambda_j) = 0$, $j = 1, 2, 3$, and $d_{j,k}$ denotes the part of intersection of the hypersurfaces $\Phi_j(z_1, z_2, \lambda_j) = 0$, $\Phi_k(z_1, z_2, \lambda_k) = 0$ which belongs to the boundary of D.

Suppose the analytic hypersurface $\Phi_4(z_1, z_2, \lambda_4) = 0$ intersects the hypersurfaces $\Phi_1(z_1, z_2, \lambda_1) = 0$, $\Phi_2(z_1, z_2, \lambda_2) = 0$ and define a new domain $B \subset D$ which is bounded by segments of $\Phi_1(z_1, z_2, \lambda_1) = 0$, $\Phi_2(z_1, z_2, \lambda_2) = 0$ and $\Phi_4(z_1, z_2, \lambda_4) = 0$. Further, suppose that the point z_1^0, z_2^0 does neither belong to the intersection of $\Phi_1 = 0$, $\Phi_4 = 0$ nor to that of $\Phi_2 = 0$, $\Phi_4 = 0$, and lies on the boundary of B. Let B be sufficiently small so that $f(\zeta_1, \zeta_2)$ is analytic in $\bar{B}$.

Let z_1, z_2 be an arbitrary point in B. Using Bergman's integral

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[2] For simplicity we assume that the number of the boundary surfaces is 3, but the considerations are valid for the general case.

[3] The integrands of the second and third integrals equal to those of the first with Φ_1 and Φ_2 replaced by Φ_1, Φ_3 and Φ_2, Φ_3, respectively.

formula representing the function $f(z_1, z_2)$ in the domain B, we obtain (comp. footnote 2)

$$(17)\quad f(z_1, z_2) = -\frac{1}{4\pi^2}\iint_{d_{12}\bar{B}} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \cdot \frac{\{\Phi_1(z_1, \zeta_2, \lambda_1)\Phi_2(z_1, z_2, \lambda_2) - \Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, \zeta_2, \lambda_2)\}}{\Phi_1(z_1, z_2, \lambda_1)\Phi_2(z_1, z_2, \lambda_2)} d\zeta_1 d\zeta_2 - \frac{1}{4\pi^2}\iint_{d_{14}\bar{B}} - \frac{1}{4\pi^2}\iint_{d_{24}\bar{B}} .$$

Since $\iint_{d_{12}} = \iint_{d_{12}\bar{B}} + \iint_{d_{12}-d_{12}\bar{B}}$ it follows from (16) and (17)

$$(18)\quad F(z_1, z_2) = -\frac{1}{4\pi^2}\iint_{d_{12}-d_{12}\bar{B}} + f(z_1, z_2) + \frac{1}{4\pi^2}\iint_{d_{14}\bar{B}} + \frac{1}{4\pi^2}\iint_{d_{24}\bar{B}} - \frac{1}{4\pi^2}\iint_{d_{13}} - \frac{1}{4\pi^2}\iint_{d_{23}} .$$

If the point z_1, z_2 lies outside the domain B and the hypersurfaces $\Phi_j = 0$, $j = 1, \cdots, 4$, we ought to substitute 0 for $f(z_1, z_2)$ in (18).

Consider the integrals on the right hand side of (18). As long as the point z_1, z_2 does not lie on any of the hypersurfaces $\Phi_j(z_1, z_2, \lambda_j) = 0$, $j = 1, 2, 3, 4$, we have $\Phi_j(z_1, z_2, \lambda_j) \neq 0$. According to the assumption under which the Bergman integral formula was proved (see [2]) the functions

$$(19)\quad \psi_{jk}(z_1, z_2, \zeta_1, \zeta_2, \lambda_j, \lambda_k) = \frac{\Phi_j(z_1, \zeta_2, \lambda_j)\Phi_k(z_1, z_2, \lambda_k) - \Phi_j(z_1, z_2, \lambda_j)\Phi_k(z_1, \zeta_2, \lambda_k)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} , \quad j, k = 1, \cdots, 4$$

are continuous provided that $\zeta_1, \zeta_2 \in d$ and z_1, z_2 does not lie on the distinguished boundary surface d of D. (It can happen that $\zeta_1 = z_1$ or $\zeta_2 = z_2$, but the case $\zeta_1, \zeta_2 = z_1, z_2$ is excluded.)

We denote by λ_1^0 and λ_2^0 the values of the parameters λ_1 and λ_2 which correspond to the point z_1^0, z_2^0, i.e., $\Phi_1(z_1^0, z_2^0, \lambda_1^0) = 0$, $\Phi_2(z_1^0, z_2^0, \lambda_2^0) = 0$.

Let $z_1 = z_1^0$, $z_2 = z_2^0$, then the integrals in (18) are improper since the factors $\Phi_1^{-1}(z_1^0, z_2^0, \lambda_1)$ and $\Phi_2^{-1}(z_1^0, z_2^0, \lambda_2)$ are indefinite for $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$, respectively. The functions $\psi_{jk}(z_1^0, z_2^0, \zeta_1, \zeta_2, \lambda_j, \lambda_k)$ are continuous for $(\zeta_1, \zeta_2) \in d_{12} - d_{12}\bar{B} + d_{14}\bar{B} + d_{24}\bar{B} + d_{13} + d_{23}$ (according to Bergman's assumption) because the point ζ_1, ζ_2 does not coincide with z_1^0, z_2^0.

In general, the integrals on the right hand side of (18) are divergent for $(z_1, z_2) = (z_1^0, z_2^0)$.

Suppose the functions $\Phi_j(z_1^0, z_2^0, \lambda_j)$, $j = 1, 2$, satisfy the conditions

$$(19^*)\quad |\Phi_j(z_1^0, z_2^0, \lambda_j)| \geqq A|\lambda_j - \lambda_j^0|^\alpha , \qquad A > 0,\ 0 < \alpha < 1 ,$$

then $F(z_1^0, z_2^0)$ exists. We denote by $\rho(z_1, z_2; z_1^0, z_2^0)$ the Euclidean distance between the points z_1, z_2 and z_1^0, z_2^0 and by $\rho_j(z_1, z_2; \Phi_j)$ the distance of the point z_1, z_2 to the hypersurface $\Phi_j = 0$. If the functions $\Phi_j(z_1, z_2, \lambda_j)$, $j = 1, 2$, satisfy the conditions

$$(20) \qquad |\Phi_j(z_1, z_2, \lambda_j)| \geqq A|\lambda_j - \lambda_j^0|^\alpha, \qquad 0 < \alpha < \tfrac{1}{2},$$

for z_1, z_2 belonging to Δ, where Δ is defined by the inequalities

$$(20^*) \qquad \Delta : 0 < \frac{\rho(z_1, z_2; z_1^0, z_2^0)}{\rho_j(z_1, z_2; \Phi_j)} < M, \qquad M > 0,\ j = 1, 2,$$

then

$$(21) \qquad \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ z_1, z_2 \in D\Delta}} F(z_1, z_2) = F_i(z_1^0, z_2^0).$$

The proof of (21) is similar to that given in § 1.

Similarly, if the point z_1, z_2 lies outside the domain D and tends to z_1^0, z_2^0 there exists the exterior limit

$$(21^*) \qquad \lim_{\substack{z_1, z_2 \to z_1^0, z_2^0 \\ z_1, z_2 \notin D;\ z_1, z_2 \in \Delta}} F(z_1, z_2) = F_l(z_1^0, z_2^0)$$

provided that (20) and (20*) hold. The difference of both limits is equal to $f(z_1^0, z_2^0)$:

$$(22) \qquad F_i(z_1^0, z_2^0) - F_l(z_1^0, z_2^0) = f(z_1^0, z_2^0).$$

REMARK. Under the conditions (20), (20*) there exists one interior and only one exterior limit of the function $F(z_1, z_2)$ for $z_1, z_2 \to z_1^0, z_2^0$.

2. Suppose now the function $f(\zeta_1, \zeta_2)$ is not analytic at the point z_1^0, z_2^0 but satisfies the condition

$$(23) \qquad |f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)| \leqq A|\zeta_1 - z_1^0||\zeta_2 - z_2^0|, \qquad A > 0.$$

The function $F(z_1, z_2)$ can be represented as follows

$$(24) \qquad F(z_1, z_2) = \sum_{1 \leqq j < k \leqq 3} -\frac{1}{4\pi^2}\iint_{d_{jk}} [\{f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)\} + f(z_1^0, z_2^0)] \cdot \frac{\psi_{jk}(z_1, z_2, \zeta_1, \zeta_2, \lambda_j, \lambda_k)}{\Phi_j(z_1, z_2, \lambda_1)\Phi_k(z_1, z_2, \lambda_2)} d\zeta_1 d\zeta_2.$$

Since $f(z_1^0, z_2^0) = \text{const.}$ is an analytic function, we can apply to the latter terms in (24) the results obtained in § 1. Under the conditions (20), (20*) there exists the exterior and interior limit of those terms.

Consider the first term in (24). If the function

$$\psi_{12}(z_1^0, z_2^0, \zeta_1, \zeta_2, \lambda_1, \lambda_2) = \frac{\Phi_1(z_1^0, \zeta_2, \lambda_1)\Phi_2(z_1^0, z_2^0, \lambda_2) - \Phi_1(z_1^0, z_2^0, \lambda_1)\Phi_2(z_1^0, \zeta_2, \lambda_2)}{(\zeta_1 - z_1^0)(\zeta_2 - z_2^0)}$$

is continuous for $\zeta_1, \zeta_2 \in d_{12}$, the integral

$$(25) \qquad -\frac{1}{4\pi^2}\iint_{d_{12}} \frac{[f(\zeta_1, \zeta_2) - f(z_1^0, z_2^0)]\cdot\psi_{12}(z_1^0, z_2^0, \zeta_1, \zeta_2, \lambda_1, \lambda_2)}{\Phi_1(z_1^0, z_2^0, \lambda_1)\Phi_2(z_1^0, z_2^0, \lambda_2)} d\zeta_1 d\zeta_2$$

exists provided that $\Phi_1(z_1^0, z_2^0, \lambda_1)$ and $\Phi_2(z_1^0, z_2^0, \lambda_2)$ satisfy the condition (19*). If in addition $\psi_{12}(z_1, z_2, \zeta_1, \zeta_2, \lambda_1, \lambda_2)$ is continuous for $\zeta_1, \zeta_2 \in d_{12}$ and $z_1, z_2 \to z_1^0, z_2^0$ and if $\Phi_1(z_1, z_2, \lambda_1)$, $\Phi_2(z_1, z_2, \lambda_2)$ satisfy (20), (20*), there exists the limit of (25) for $z_1, z_2 \to z_1^0, z_2^0$. In the case where $\psi_{12}(z_1, z_2, \zeta_1, \zeta_2, \lambda_1, \lambda_2)$ is not continuous for $\zeta_1, \zeta_2 \in d_{12}$ and $z_1, z_2 \to z_1^0, z_2^0$ we use the condition (23). Then the limit of (25) exists provided that $z_1, z_2 \to z_1^0, z_2^0$ under the conditions (20), (20*).

Observe that for the difference between the interior and exterior limit of $F(z_1, z_2)$ we obtain the formula (22).

3. If one of the hypersurfaces $\Phi_j(z_1, z_2, \lambda_j) = 0$, $j = 1, 2$, depends on one of the variables z_1, z_2, e.g., if $\Phi_1(z_1, z_2, \lambda_1)$ is independent from z_2

$$(26) \qquad \Phi_1(z_1, z_2, \lambda_1) = z_1 - \varphi(\lambda_1) ,$$

then the integrand in the first term on the right hand side of (18) can be represented in the form

$$(27) \qquad \omega_{12} = \frac{f(\zeta_1, \zeta_2)[\Phi_2(z_1, z_2, \lambda_2) - \Phi_2(z_1, \zeta_2, \lambda_2)]}{(\zeta_1 - z_1)(\zeta_2 - z_2)\Phi_2(z_1, z_2, \lambda_2)} .$$

According to Bergman's assumption (26) is continuous for $\zeta_1, \zeta_2 \in d$, $\zeta_2 = z_2$, $\zeta_1 \neq z_1$. For $z_1 = z_1^0$, $z_2 = z_2^0$ the integral $\iint_{d_{12}-d_{12}\bar{B}}$ in (18) and the remaining integrals are improper. If $\Phi_2(z_1^0, z_2^0, \lambda_2)$ satisfies the condition (19*) it is sufficient to take into account the singulartiy due to the factor $(\zeta_1 - z_1^0)^{-1}$.

According to (26) the first coordinate of every point ζ_1, ζ_2 of d belongs to the curve $C_1 : z_1 = \varphi(\lambda_1)$. Suppose, the double integral over $d_{12} - d_{12}\bar{B}$ can be represented as follows

$$(28) \qquad \iint_{d_{12}-d_{12}\bar{B}} \omega_{12} d\zeta_1 d\zeta_2 = \int_{C_1} \frac{d\zeta_1}{\zeta_1 - z_1} \int_{[d_{12}-d_{12}\bar{B}]_{z_2}} \frac{f(\zeta_1, \zeta_2)[\Phi_2(z_1, z_2, \lambda_2) - \Phi_2(z_1, \zeta_2, \lambda_2)]}{(\zeta_2 - z_2)\Phi_2(z_1, z_2, \lambda_2)} d\zeta_2 ,$$

where $[d_{12} - d_{12}\bar{B}]_{z_2}$ is the projection of the set $d_{12} - d_{12}\bar{B}$ on the z_2 plane. Under the conditions (19*) assuming that

$$\frac{f(\zeta_1, \zeta_2)[\Phi_2(z_1, z_2, \lambda_2) - \Phi_2(z_1, \zeta_2, \lambda_2)]}{\Phi_2(z_1, z_2, \lambda_2)}$$

is analytic at the point z_1^0, z_2^0 the integral (28) possesses one interior and two exterior limits when $z_1, z_2 \to z_1^0 z_2^0$. Similarly, the integrals $\iint_{d_{13}}$ and $\iint_{d_{14}}$ in (18) possesses one interior and two exterior limits.

In the case where $\Phi_1(z_1, z_2, \lambda_1) = z_1 - \varphi(\lambda_1)$, $\Phi_2(z_1, z_2, \lambda_2) = z_2 - \psi(\lambda_2)$, we obtain the same result as for a bicylinder.

REMARK. The Sochocki-Plemelj formula (22) was proved for a special class of domains-domains with the distinguished boundary surface. The basic tool was the Bergman's integral formula (16). It arises the problem to generalize the Bergman formula for more general domains with maximal manifold (Bergman-Silov boundary) and to extend the Sochocki-Plemelj formula for such domains.

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