BEST FIT TO A RANDOM VARIABLE BY A RANDOM VARIABLE MEASURABLE WITH RESPECT TO A σ-LATTICE

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1. Introduction and summary. Let $(\Omega, \mathscr{S}, \mu)$ be a probability space and f a random variable, an $\mathscr{S}$-measurable function from Ω into the space R of real numbers. Let $\mathscr{S}_0$ be a sub-σ-algebra of $\mathscr{S}$. Let f be integrable; that is, let its expectation $E(f)$ exist. Then the Radon-Nikodym Theorem yields an $\mathscr{S}_0$-measurable function g, the conditional expectation of f given $\mathscr{S}_0$: $g = E(f \mid \mathscr{S}_0)$. The conditional expectation g is, in a strong sense to be made precise below, the best fit to f by an $\mathscr{S}_0$-measurable function. The purpose of the present note is to show that there corresponds to f a function with the same minimizing properties when an arbitrary sub-σ-lattice $\mathscr{L}$ takes the place of $\mathscr{S}_0$.

The conditional expectation $g = E(f \mid \mathscr{S}_0)$ has the property that

$$\int (f - g)h d\mu = 0$$

for $\mathscr{S}_0$-measurable h such that the integral exists. It is then immediate that

$$\int (f - h)^2 d\mu = \int (f - g)^2 d\mu + \int (g - h)^2 d\mu \, .$$

More generally, the squared difference may be replaced by the W. H. Young form $\Delta_\Phi(\circ, \circ)$ determined by an arbitrary convex function Φ (see §2):

$$\int \Delta_\Phi(f, h) d\mu = \int \Delta_\Phi(f, g) d\mu + \int \Delta_\Phi(g, h) d\mu$$

for $\mathscr{S}_0$-measurable h, provided appropriate integrals exist. (The function $\Delta_\Phi(\circ, \circ)$ is nonnegative and vanishes when the arguments are equal.) Thus, for every Φ, $g = E(f \mid \mathscr{S}_0)$ is the solution of the minimizing problem: given f, to minimize $\int \Delta_\Phi(f, h) d\mu$ in the class of $\mathscr{S}_0$-measurable functions. The conditional expectation therefore enjoys a powerful claim to be the "best" fit to f by an $\mathscr{S}_0$-measurable function. (Blackwell [3] has remarked that for square-integrable functions, the conditional expectation may be regarded as a projection in Hilbert space.)

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Let now $\mathscr{L}$ be a sub-σ-lattice of $\mathscr{S}$: $\mathscr{L}$ is a class of sets in $\mathscr{S}$ containing the void set ϕ and the whole space Ω, and closed under countable intersections and countable unions. Let h be called $\mathscr{L}$-measurable if for every real t $\{\omega \in \Omega: h(\omega) < t\} \in \mathscr{L}$. It will be shown that given an integrable function f, there exists an $\mathscr{L}$-measurable g such that

$$(1.1) \qquad \int (f-h)^2 d\mu \geqq \int (f-g)^2 d\mu + \int (g-h)^2 d\mu \, ,$$

and, indeed, such that

$$(1.2) \qquad \int \Delta_\Phi(f, h) d\mu \geqq \int \Delta_\Phi(f, g) d\mu + \int \Delta_\Phi(g, h) d\mu$$

for every Φ, provided appropriate integrals exist. Thus g is the "best" fit to f in the class of $\mathscr{L}$-measurable functions. (When f is square-integrable, g may be interpreted in L^2 as the point in the cone of $\mathscr{L}$-measurable functions nearest to the given point f.) To determine g requires the specification not only of f but also of the probability measure μ. Thus it seems appropriate to regard f (and g) as random variables. On the other hand, the "best fit" to a sum need not be sum of the "best fits", so a designation of g as a "conditional expectation given $\mathscr{L}$" does not seem completely appropriate.

Methods used in this paper require that μ be totally finite. It would be of interest to relax this restriction.

The problem of maximum likelihood estimation of parameters subject to order restrictions led to a study of the problem of minimizing $\int \Delta_\Phi(f, h) d\mu$ in a special case ([5], § 4). In that special case, Ω is n-dimensional euclidean space, and $\mathscr{L}$ is the class of sets in $\mathscr{S}$ such that $L \in \mathscr{L}$, $(v_1, v_2, \cdots, v_n) \in L$, $u_1 \leqq v_1$, $u_2 \leqq v_2$, $\cdots$, $u_n \leqq v_n \Rightarrow (u_1, u_2, \cdots, u_n) \in L$. Members of $\mathscr{L}$ were called "lower layers". Methods known from the Radon-Nikodym theory were used, but the connection was not clearly understood. It is the purpose of the present paper not only to replace n-dimensional euclidean space by an arbitrary space Ω, and the class of "lower layers" by an arbitrary σ-lattice, but also to formulate the results so as to include conditional expectation given a sub-σ-field as the special instance occurring when $\mathscr{L}$ is a σ-field.

Special cases occurring in maximum likelihood estimation of ordered parameters are treated in [1], [4], [6], [7] and [8]. In the situation treated in [5], inequality (1.1) was found independently by G. M. Ewing[1] and by W. T. Reid[1]; special cases appear in [4] and [9].

Section 2 of the present paper is devoted to definitions. The problem for square-integrable f is treated as a problem in Hilbert space in § 3.

[1] Private communication.

Results on the minimum problem for arbitrary classes of functions are obtained in § 4, and used in § 5 to yield the principal results, Theorem 5.1 and Theorem 5.2, for integrable f and measurable f. It is shown in § 6 that, given a partial ordering on, Ω, a σ-lattice $\mathscr{L}$ can be introduced such that the $\mathscr{L}$-measurable functions are precisely the order-preserving functions. Application to certain problems of maximum likelihood estimation of a multi-dimensional parameter is mentioned in § 7. It is also remarked that (1.2) may be used in a modification of the proof of the Rao-Blackwell Theorem on sufficient statistics[2].

2. **Definitions.** Let Φ be a convex function of a real variable. Set $G_\Phi \equiv_D \{u: \Phi(u) < \infty\}$. (Symbols $\equiv_D$ and $\Longleftrightarrow_D$ will be used in defining the symbol or relation which appears on the right.) Define (cf. [10])

(2.1) $$\Psi(z) \equiv_D \sup_u [uz - \Phi(u)] .$$

Then (W. H. Young's inequality)

(2.2) $$0 \leqq \Phi(u) + \Psi(z) - uz \leqq \infty , \qquad u, z \text{ real.}$$

The function Ψ is convex, and Φ and Ψ are conjugate in the sense of W. H. Young.

For $u \in G_\Phi$, let $\varphi(u)$ denote the left derivative of Φ at u; φ is continuous from the left.

Consider the graph of $\Phi(u)$ in the cartesian (u, w) plane: $w = \Phi(u)$. For fixed z, the form $zu - \Phi(u)$ represents the vertical directed distance from the graph of Φ to the line $w = zu$. If $z = \varphi(u_0)$ for a number $u_0 \in G_\Phi$ then the directed distance $u\varphi(u_0) - \Phi(u)$ is maximized for $u = u_0$, since the line $w = u\varphi(u_0)$ is parallel to a line of support at u_0. Therefore

(2.3) $$\Phi(u) + \Psi[\varphi(u)] - u\varphi(u) \equiv 0 , \qquad u \in G_\Phi .$$

For $u, v \in G_\Phi$, define

(2.4) $$\begin{cases} \Delta_\Phi(u, v) \equiv_D \Phi(u) + \Psi[\varphi(v)] - u\varphi(v) \\ \qquad\qquad = \Phi(u) - \Phi(v) - (u - v)\varphi(v) . \end{cases}$$

(The subscript Φ will often be omitted.) This form has an obvious geometric interpretation relative to the graph of Φ. It follows from (2.2) and (2.3) that

(2.5) $$\Delta(u, v) \geqq 0 , \qquad \Delta(u, u) = 0 , \qquad u, v \in G_\Phi .$$

Also

[2] That there is a connection between (1.2) and the Rao-Blackwell Theorem was suggested to the writer by Cand. Mag. Brøns of the Statistics Institute, University of Copenhagen.

$$(2.6)\qquad \begin{cases} \Delta(u, v) = \int_{\{t:\ v \leqq t < u\}} (u - t) d\varphi(t) & \text{if } v \leqq u\ , \\ \Delta(u, v) = \int_{\{t:\ u \leqq t < v\}} (t - u) d\varphi(t) & \text{if } v \geqq u\ . \end{cases}$$

For $u, v, w \in G_\Phi$, (2.4) yields

$$(2.7)\qquad \Delta(u, w) = \Delta(u, v) + \Delta(v, w) + (u - v)[\varphi(v) - \varphi(w)]\ .$$

Let $(\Omega, \mathscr{S}, \mu)$ be a probability measure space. Let ϕ denote the void set. For $A \subset \Omega$, let A^c denote its complement $\Omega - A$. For $\mathscr{S}$-measurable, real functions f, h with ranges in G_Φ, and for $A \in \mathscr{S}$, define

$$(2.8)\qquad J_\Phi(f, h; A) \equiv_D \int_A \Delta_\Phi(f, h) d\mu\ .$$

(The subscribt Φ will often be omitted.) Define also

$$(2.9)\qquad J(f, h) \equiv_D J(f, h; \Omega)\ .$$

From (2.5),

$$(2.10)\qquad 0 \leqq J(f, h; A) \leqq J(f, h) \leqq \infty\ .$$

3. **Fitting a square-integrable function.** Let $\mathscr{L}$ be a sub-σ-lattice of $\mathscr{S}$; that is, let $\phi \in \mathscr{L}$, $\Omega \in \mathscr{L}$, $\mathscr{L} \subset \mathscr{S}$, and let $\mathscr{L}$ be closed under countable unions and intersections. Let $\mathscr{C}(\mathscr{L})$ denote the class of real-valued functions h on Ω such that $\{\omega: h(\omega) < t\} \in \mathscr{L}$ for real t. "Fitting" a given function f refers to the problem of minimizing $J_\Phi(f, h)$ for $h \in \mathscr{C}(\mathscr{L})$. It will be shown that, broadly speaking, given f there is a function $g \in \mathscr{C}(\mathscr{L})$, independent of Φ, which minimizes $J_\Phi(f, \circ)$ in $\mathscr{C}(\mathscr{L})$ for every Φ. For this function g, indeed,

$$J_\Phi(f, h) \geqq J_\Phi(f, g) + J_\Phi(g, h)$$

for $h \in \mathscr{C}(\mathscr{L})$. In the present approach to the problem, the square-integrable function f is regarded as an element of the Hilbert space of square-integrable functions. (In [11] von Neumann approached the Radon-Nikodym Theorem via Hilbert space.)

Let $\mathscr{H}$ be a real Hilbert space, and $\mathscr{C}$ a closed convex cone in $\mathscr{H}$: $\mathscr{C}$ is closed; $x \in \mathscr{C}$, $a \geqq 0 \Rightarrow ax \in \mathscr{C}$; and $x \in \mathscr{C}$, $y \in \mathscr{C} \Rightarrow x + y \in \mathscr{C}$. The following theorem and argument are familiar ([12], p. 120) when $\mathscr{C}$ is a linear subspace, and perhaps in the present more general situation as well.

The inner product in $\mathscr{H}$ will be denoted by $(\circ, \circ)$ and the norm by $\|\circ\|$.

THEOREM 3.1. *If* $f \in \mathscr{H}$ *then there exists a* $g \in \mathscr{C}$ *such that*

$(f-g, h) \leqq 0$ for all $h \in \mathscr{C}$. If there exists $f_0 \neq 0$ in $\mathscr{H}$ such that $(f, f_0)f_0/\|f_0\|^2 \in \mathscr{C}$, then $(f-g, g) = 0$.

If $\mathscr{C}$ is a linear subspace of $\mathscr{H}$ it follows that $(f-g, h) = 0$ for $h \in \mathscr{C}$. It seems of interest to note, as Blackwell has remarked [3], that in this special case Theorem 3.1 yields at once the conditional expectation of a square-integrable random variable. Let $\mathscr{S}_0$ be a sub-σ-algebra of $\mathscr{S}$, $\mathscr{H}$ the class L^2 of square-integrable functions, and $\mathscr{C}$ the subclass of square-integrable, $\mathscr{S}_0$-measurable functions. The function g furnished by the theorem is then $E(f \mid \mathscr{S}_0)$, for $\int fhd\mu = \int ghd\mu$ for $h \in \mathscr{C}$, and in particular when h is the indicator (characteristic) function of a set in $\mathscr{S}_0$.

Proof of Theorem 3.1. Let N denote the set of all elements of $\mathscr{H}$ of the form $f-h$ for $h \in \mathscr{C}$. Since $\mathscr{C}$ is closed, so is N. Since $\mathscr{C}$ is convex, so is N, for $\lambda(f-h_1) + \mu(f-h_2) = f - (\lambda h_1 + \mu h_2) \in N$ if $0 \leqq \lambda \leqq 1$, $\lambda + \mu = 1$, $h_1, h_2 \in \mathscr{C}$. It follows ([12], Theorem 3, p. 120) that N has an element k of smallest norm. Set $g \equiv_D f - k$; then $g \in \mathscr{C}$. Let $h \in \mathscr{C}$; then if $a \geqq 0$, $g + ah = (a+1)[g/(a+1) + ah/(a+1)] \in \mathscr{C}$. Therefore

$$\|k\|^2 \leqq \|f-(g+ah)\|^2 = \|k - ah\|^2$$
$$= \|k\|^2 - 2a(k, h) + a^2\|h\|^2 .$$

Suppose there exists $h \in \mathscr{C}$ such that $(k, h) > 0$. Set $a = (k, h)/\|h\|^2$, and find $\|k\|^2 \leqq \|k\|^2 - (k, h)^2/\|h\|^2$, a contradiction. Therefore $(k,h) \leqq 0$ for $h \in \mathscr{C}$, the first conclusion of the theorem.

The second conclusion, $(f-g, g) = 0$, is obvious if $g = 0$. In approaching this conclusion for $g \neq 0$, it is first shown that $g \neq 0$ and $(f, g) \geqq 0$ imply $(f-g, g) = 0$. Set $b \equiv_D (f-g, g)/\|g\|^2 = [(f, g) - \|g\|^2]/\|g\|^2 \geqq -1$. Then $g + bg = (1+b)g \in \mathscr{C}$. Hence $\|k\|^2 \leqq \|f-(g+bg)\|^2 = \|k - bg\|^2 = \|k\|^2 - (k, g)^2/\|g\|^2$, so that $(f-g, g) = (k, g) = 0$. It remains to verify that the hypotheses of the theorem imply $(f, g) \geqq 0$. Set $a = (f, f_0)/\|f_0\|^2$. Since by hypothesis $af_0 \in \mathscr{C}$,

$$\|k\|^2 = \|f-g\|^2 \leqq \|f - af_0\|^2 ,$$

or

$$\|f\|^2 - 2(f, g) + \|g\|^2 \leqq \|f\|^2 - 2a(f, f_0) + a^2\|f_0\|^2 ,$$

so that

$$2(f, g) \geqq \|g\|^2 + (f, f_0)^2/\|f_0\|^2 \geqq 0 .$$

This completes the proof of Theorem 3.1

Let L^2 denote the class of square-integrable functions, and set

$\mathscr{C}_1(\mathscr{L}) = L^2 \cap \mathscr{C}(\mathscr{L})$; $\mathscr{C}_1(\mathscr{L})$ is the class of those $\mathscr{L}$-measurable functions which are square-integrable.

LEMMA 3.1. *If $f \in L^2$, there exists $g \in \mathscr{C}_1(\mathscr{L})$ such that*

$$\int (f-h)^2 d\mu \geqq \int (f-g)^2 d\mu + \int (g-h)^2 d\mu \tag{3.1}$$

for all $h \in \mathscr{C}_1(\mathscr{L})$; g is unique a.e. (μ).

Inequality (3.1) is of the form (1.2) for $\Phi(u) \equiv u^2/2$.

Proof of Lemma 3.1. Lemma 3.1 results from the application of Theorem 3.1 to the Hilbert space L^2, in which the inner product is defined by $(f_1, f_2) \equiv_D \int f_1 f_2 d\mu$ for $f_1, f_2 \in L^2$. In this application the closed convex cone $\mathscr{C}$ of Theorem 3.1 is identified with $\mathscr{C}_1(\mathscr{L})$. It is readily verified that $\mathscr{C}_1(\mathscr{L})$ is a convex cone. Also $\mathscr{C}_1(\mathscr{L})$ is closed in L^2, for if $\| h_n - h \|^2 \rightarrow 0$ as $n \rightarrow \infty$, then $\{h_n\}$ converges to h in measure, and a subsequence converges to h a.e. (μ); but the limit of a sequence of $\mathscr{L}$-measurable functions is also $\mathscr{L}$-measurable. Let g be the element of $\mathscr{C}_1(\mathscr{L})$ guaranteed by Theorem 3.1. Then

$$\int (f-g) h d\mu \leqq 0 \tag{3.2}$$

for $h \in \mathscr{C}_1(\mathscr{L})$. Further, every constant function is in $\mathscr{C}_1(\mathscr{L})$. Therefore the second hypothesis of Theorem 3.1 is satisfied for $f_0 \equiv_D 1$. It follows that

$$\int (f-g) g \, d\mu = 0 \, , \tag{3.3}$$

so that

$$\int (f-g)(g-h) d\mu \geqq 0 \, . \tag{3.4}$$

Inequality (3.1) is now immediate. The uniqueness a.e. (μ) of g is evident from (3.1).

For a real-valued function φ of a real variable, and a function h from Ω into the real line R, let φh denote the composite function: for $\omega \in \Omega$, $\varphi h(\omega) \equiv_D \varphi[h(\omega)]$. Inequality (3.4) is the special instance of

$$\int (f-g)(\varphi g - \varphi h) d\mu \geqq 0 \, , \tag{3.5}$$

in which $\varphi(u) \equiv u$. From (2.7) it follows that (3.5) is equivalent to

(1.2), given the existence of appropriate integrals. Conditions will now be investigated under which, given f, the same function g satisfies (3.5) for functions φ other than the identity well. Lemma 3.2, below, is phrased more generally than is required for the present application.

Let W be a vector lattice ([2], Chapter XV), so that

$$a, b \in W \Rightarrow a \vee b + a \wedge b = a + b \tag{3.6}$$

(here $a \vee b$ and $a \wedge b$ denote respectively the l.u.b. and g.l.b. of the two elements a and b of W). (For (3.6) it is sufficient that W be a commutative lattice-ordered group; ([2], p. 219).) Let $\mathscr{D}$ be a class of order-preserving maps of W into itself, which is a lattice under the induced partial ordering: $\varphi_1 \leqq \varphi_2 \Longleftrightarrow_D \varphi_1(w) \leqq \varphi_2(w)$ for all $w \in W$ ("$\leqq$" denotes the ordering relation on the partially ordered set W). Let $\mathscr{E}$ be a subclass of $\mathscr{D}$. An intersection of lattices is a lattice, and the intersection of all lattices containing $\mathscr{E}$ is the smallest lattice, $\mathscr{E}^*$, containing $\mathscr{E}$. It may be constructed as follows. For an arbitrary subclass $\mathscr{F}$ of $\mathscr{D}$, define $T\mathscr{F}$ as the class of all elements of $\mathscr{D}$ of the form $\varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$ for $\varphi_1, \varphi_2 \in \mathscr{F}$. Then

$$\mathscr{E}^* = \lim_n T^n \mathscr{E} = \bigcup_n T^n \mathscr{E}.$$

LEMMA 3.2. *Let L be a nonnegative* (*or non-positive*) *linear functional on $\mathscr{D}$. Then $L = 0$ on $\mathscr{E}$ implies $L = 0$ on $\mathscr{E}^*$.*
(This may be regarded as a special instance of the proposition that in a normed lattice the elements of zero norm form a lattice.)

Proof. It suffices to show that $\mathscr{F} \subset \mathscr{D}$ and $L = 0$ on $\mathscr{F}$ imply $L = 0$ on $T\mathscr{F}$. But this is immediate from (3.6) and the assumed linearity and constancy of sign of L.

Lemma 3.2 is applied in proving Theorem 3.2.

THEOREM 3.2. *Let $f \in L^2$ and let g be given by Lemma* 3.1. *Let Φ be convex, let $\varphi g \in L^2$, and let the range of f be in G_Φ. Then the range of g is in G_Φ* (i.e., *there is a determination of g in the equivalence class determined by Lemma* 3.1 *whose range is in G_Φ*),

$$\int (f - g)(\varphi g - \varphi h) d\mu \geqq 0 , \tag{3.7}$$

and

$$J_\Phi(f, h) \geqq J_\Phi(f, g) + J_\Phi(g, h) \tag{3.8}$$

for all $h \in \mathscr{C}(\mathscr{L})$ such that the range of h is in G_Φ and such that $\varphi h \in L^2$.

Proof. Setting h in (3.2) first equal to 1 then equal to -1 yields the result that

$$\int (f-g)d\mu = 0 \,. \tag{3.9}$$

From (3.3) and (3.9) it follows that

$$\int (f-g)(ag+b)d\mu = 0 \,.$$

for real a and b. In applying Lemma 3.2, take for W the real line (a vector lattice) R. For fixed f and hence fixed g, take for $\mathscr{D}$ the class of non-decreasing functions ψ defined on R such that $\psi g \in L^2$. One verifies that $\mathscr{D}$ is a lattice. For $\psi \in \mathscr{D}$, set $L(\psi) \equiv_D \int (f-g)\psi g d\mu$. L is clearly a linear functional on $\mathscr{D}$; from (3.2) it follows that L is non-positive. Let $\mathscr{E}$ denote the subclass of $\mathscr{D}$ consisting of functions ψ of the form $\psi(y) \equiv ay + b$, $a \geqq 0$. For arbitrary real c and d with $c < d$, define ψ_1 by $\psi_1(y) = 0$ for $y \leqq c$, $\psi_1(y) = (y-c)/(d-c)$ for $c < y \leqq d$, $\psi_1(y) = 1$ for $y > d$. Then $\psi_1 \in T^2\mathscr{E}$. By Lemma 3.2, $L(\psi_1) = 0$. Let t be an arbitrary real number. For $n=1, 2, \cdots$, set $c_n = t$, $d_n = t + 1/n$, and define ψ_n as ψ_1 was defined above, with c and d replaced by c_n and d_n respectively. Let ψ_0 denote the step-function: $\psi_0(y) = 0$ for $y \leqq t$, $\psi_0(y) = 1$ for $y > t$. Then $L(\psi_0) = \lim_{n\to\infty} L(\psi_{1_n}) = 0$. That is,

$$\int_{\{\omega:g(\omega)>t\}} [f(\omega) - g(\omega)]d\mu(\omega) = 0 \,.$$

It follows that for every Borel set B of real numbers,

$$\int_{\{\omega:g(\omega)\in B\}} [f(\omega) - g(\omega)]d\mu(\omega) = 0 \,. \tag{3.10}$$

(Equation (3.10) may be interpreted thus: $g = E(f \mid g)$.)

It can be seen as follows that the conclusion that the range of g is in G_φ is a consequence of (3.10). Suppose, for example, that $f(\omega) < a$ for $\omega \in \Omega$. Then

$$a\mu\{g \geqq a\} \leqq \int_{\{g\geqq a\}} g d\mu = \int_{\{g\geqq a\}} f d\mu < a\mu\{g \geqq a\} \,,$$

unless $\mu\{g \geqq a\} = 0$.

It now follows from (3.10) that $\int (f-g)\varphi g d\mu = 0$. Also, if the range of h is in G_φ and if $\varphi(h) \in L^2$, it follows from (3.2) (with h there replaced by φh) that $\int (f-g)\varphi h d\mu \leqq 0$. Equation (3.7) is then immediate. The proof of Theorem 3.2 is completed by the observation that (3.8) is a consequence of (3.7) and (2.7).

4. Minimizing $J(f, \circ)$. Some theorems on minimzing $J(f, \circ)$ in arbitrary classes of $\mathscr{S}$-measurable functions are given in this section. In § 5 the result of Theorem 3.2 is extended to arbitrary integrable f, using the results of the present section.

LEMMA 4.1. *Let Φ be convex. Let f, h_1, h_2 be $\mathscr{L}$-measurable functions with ranges in G_Φ. Set $E \equiv_D \{\omega\colon h_1(\omega) < h_2(\omega)\}$, and for real t set $E(t) \equiv_D \{\omega\colon h_1(\omega) \leqq t < h_2(\omega)\}$. Then*

$$(4.1) \qquad -\infty \leqq J_\Phi(f, h_2; E) - J_\Phi(f, h_1; E) = \int d\varphi(t) \int_{E(t)} [t - f(\omega)] d\mu(\omega) \leqq \infty\ ,$$

provided either $J_\Phi(f, h_1; E) < \infty$ or $J_\Phi(f, h_2; E) < \infty$.

Proof. From (2.8) and (2.6),

$$J(f, h; A) = \int_{A \cap \{\omega: h(\omega) < f(\omega)\}} d\mu(\omega) \int_{\{t: h(\omega) \leqq t < f(\omega)\}} [f(\omega) - t] d\varphi(t) + \int_{A \cap \{\omega: f(\omega) < h(\omega)\}} d\mu(\omega) \int_{\{t: f(\omega) \leqq t < h(\omega)\}} [t - f(\omega)] d\varphi(t)\ .$$

Since Δ is nonnegative (inequality (2.5)), Fubini's Theorem ([12], Corollary, p. 95) applies, to yield

$$(4.2) \qquad J(f, h; A) = \int d\varphi(t) \int_{A \cap \{\omega: h(\omega) \leqq t < f(\omega)\}} [f(\omega) - t] d\mu(\omega) + \int d\varphi(t) \int_{A \cap \{\omega: f(\omega) \leqq t < h(\omega)\}} [t - f(\omega)] d\mu(\omega)\ .$$

Set $A = E$ and h first equal to h_2, then equal to h_1. Lemma 4.1 then follows, using the observation that

$$E \cap \{h_1 \leqq t < f\} = E \cap \{h_2 \leqq t < f\} \cup E \cap \{f > t\} \cap \{h_1 \leqq t < h_2\}$$

and

$$E \cap \{f \leqq t < h_2\} = E \cap \{f \leqq t < h_1\} \cup E \cap \{f \leqq t\} \cap \{h_1 \leqq t < h_2\}\ .$$

THEOREM 4.1. *Let $\mathscr{C}$ be a class of $\mathscr{S}$-measurable functions, and f a given, fixed $\mathscr{S}$-measurable function. A sufficient condition that g minimize $J_\Phi(f, \circ)$ in $\mathscr{C}$ for all Φ such that the range of f is in G_Φ is that g be bounded by $\inf_\omega f(\omega)$ and $\sup_\omega f(\omega)$, and that*

$$(4.3) \qquad \int_{\{\omega: g(\omega) \leqq t < h(\omega)\}} [f(\omega) - t] d\mu(\omega) \leqq 0 \quad \text{and} \quad \int_{\{\omega: h(\omega) \leqq t < g(\omega)\}} [t - f(\omega)] d\mu(\omega) \leqq 0$$

hold for all real t and every $h \in \mathscr{C}$. If $\mathscr{C}$ is a lattice under the partial ordering $h_1 \leqq h_2 \Longleftrightarrow_D h_1(\omega) \leqq h_2(\omega)$ for $\omega \in \Omega$, then (4.3) *is also necessary.*

Proof of sufficiency. For $h \in \mathscr{C}$, set

$$B_1 \equiv_D \{\omega\colon g(\omega) < h(\omega)\} ,$$
$$B_2 \equiv_D \{\omega\colon g(\omega) > h(\omega)\} ,$$
$$B_3 \equiv_D \{\omega\colon g(\omega) = h(\omega)\} .$$

Then

$$J(f, g) = \sum_{i=1}^{3} J(f, g; B_i)$$

and

$$J(f, h) = \sum_{i=1}^{3} J(f, h; B_i) .$$

Clearly $J(f, g; B_3) = J(f, h; B_3)$. In Lemma 4.1 set $h_1 = g$, $h_2 = h$, so that E becomes B_1 and $E(t)$ becomes $\{\omega\colon g(\omega) \leqq t < h(\omega)\}$. From (4.1) and (4.3) follows

$$0 \leqq \int d\varphi(t) \int_{\{\omega\colon g(\omega) \leqq t < h(\omega)\}} [t - f(\omega)] d\mu(\omega) = J(f, h; B_1) - J(f, g; B_1) \leqq \infty .$$

Interchanging the roles of g and h in the application of Lemma 4.1 yields

$$0 \geqq \int d\varphi(t) \int_{\{\omega\colon h(\omega) \leqq t < g(\omega)\}} [t - f(\omega)] d\mu(\omega) = J(f, g; B_2) - J(f, h; B_2) \geqq -\infty .$$

Subtraction gives $0 \leqq J(f, h) - J(f, g) \leqq \infty$, completing the proof of the sufficiency of condition (4.3).

Proof of necessity. Let t_0 be a real number, and define $\Phi_0(t) \equiv_D |t - t_0|/2$, so that $\varphi_0(t)$ has a unit jump at t_0, with $\varphi_0(t_0) = -1/2$. Applying Lemma 4.1 first with $h_2 = h$, $h_1 = g$, $E = \{g < h\}$ and then with $h_2 = g$, $h_1 = h$, $E = \{h < g\}$, one has

$$\text{(4.4)} \quad \begin{aligned} -\infty &\leqq J_{\Phi_0}(f, h) - J_{\Phi_0}(f, g) \\ &= \int_{\{\omega\colon g(\omega) \leqq t_0 < h(\omega)\}} [t_0 - f(\omega)] d\mu(\omega) + \int_{\{\omega\colon h(\omega) \leqq t_0 < g(\omega)\}} [f(\omega) - t_0] d\mu(\omega) . \end{aligned}$$

If g minimizes $J_{\Phi_0}(f, \circ)$ in $\mathscr{C}$, then the left member is nonnegative for every $h \in \mathscr{C}$. Given $h \in \mathscr{C}$, define $h_1 \equiv_D g \wedge h$, and replace h in (4.4) by h_1. One finds

$$0 \leqq J_{\Phi_0}(f, h_1) - J_{\Phi_0}(f, g) = \int_{\{\omega\colon h(\omega) \leqq t_0 < g(\omega)\}} [f(\omega) - t_0] d\mu(\omega) ,$$

verifying the second of inequalities (4.3). Similarly, setting $h_1 = g \vee h$ yields the first, completing the proof of Theorem 4.1.

Let f be a given $\mathscr{S}$-measurable function, and $\mathscr{C}$ a class of $\mathscr{S}$-

measurable functions. Consider the following two properties of a function $g \in \mathscr{C}$ which is bounded by $\inf_\omega f(\omega)$ and $\sup_\omega f(\omega)$, and for which $\int |f - g| \, d\mu < \infty$.

For real t and $h \in \mathscr{C}$,

$$\begin{aligned} &\int_{\{\omega:\, g(\omega) \leqq t < h(\omega)\}} [g(\omega) - f(\omega)] d\mu(\omega) \geqq 0 \,, \\ &\int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [f(\omega) - g(\omega)] d\mu(\omega) \geqq 0 \,. \end{aligned} \tag{4.5}$$

For all Φ such that the range of f is in G_Φ and all $h \in \mathscr{C}$ with range in G_Φ,

$$J_\Phi(f, h) \geqq J_\Phi(f, g) + J_\Phi(g, h) \,. \tag{4.6}$$

THEOREM 4.2. *Let f be a given $\mathscr{S}$-measurable function. Suppose that* $\inf_\omega f(\omega) \leqq g(\omega) \leqq \sup_\omega f(\omega)$ *for* $\omega \in \Omega$ *and that* $\int |f - g| \, d\mu < \infty$. *Then* (4.5) $\Longleftrightarrow$ (4.6).

Proof that (4.5) $\Rightarrow$ (4.6). Let $h \in \mathscr{C}$, let Φ be convex, and let f, h have ranges in G_Φ. Set $B_1 \equiv_D \{\omega: g(\omega) < h(\omega)\}$, $B_2 \equiv_D \{\omega: h(\omega) < g(\omega)\}$. Set

$$a \equiv_D \int d\varphi(t) \int_{\{\omega:\, g(\omega) \leqq t < h(\omega)\}} [t - g(\omega)] d\mu(\omega) \geqq 0$$

and

$$b \equiv_D \int d\varphi(t) \int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [g(\omega) - t] d\mu(\omega) \geqq 0 \,.$$

In (4.2), replace f by g and A by Ω, to find

$$J(g, h) = a + b \,.$$

Applying (4.5) and Lemma 4.1, one has

$$a \leqq \int d\varphi(t) \int_{\{\omega:\, g(\omega) \leqq t < h(\omega)\}} [t - f(\omega)] d\mu(\omega) = J(f, h; B_1) - J(f, g; B_1)$$

and

$$b \leqq \int d\varphi(t) \int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [f(\omega) - t] d\mu(\omega) = J(f, h; B_2) - J(f, g; B_2) \,,$$

provided either $J(f, h) < \infty$ or $J(f, g) < \infty$. If both are infinite, (4.6) is granted. If at least one is finite, then

$$J(g, h) = a + b \leqq J(f, h) - J(f, g) \,.$$

Since $J(g, h) \geqq 0$, $J(f, g)$ must then be finite, and (4.6) follows.

Proof that (4.6) $\Rightarrow$ (4.5). From (4.6) and (2.7) it follows that

$$\int (f-g)(\varphi g - \varphi h) d\mu \geqq 0$$

when $h \in \mathscr{C}$, and when the ranges of f and h are contained in G_Φ, provided the integral exists. Let t be a real number, and set $\Phi(u) \equiv_D -(u-t)$ for $u \leqq t$, $\Phi(u) \equiv_D 0$ for; $u > t$. Then

$$\int (f-g)(\varphi g - \varphi h) d\mu = -\int_{\{g \leqq t < h\}} (f-g) d\mu + \int_{\{h \leqq t < g\}} (f-g) d\mu ,$$

the integrals existing by hypothesis. Given $h \in \mathscr{C}$, set $h_1 \equiv_D g \wedge h$. Then

$$0 \leqq \int (f-g)(\varphi g - \varphi h_1) d\mu = \int_{\{h \leqq t < g\}} (f-g) d\mu .$$

The proof of the first member of (4.5) is similar.

5. Fitting an integrable function in $\mathscr{C}(\mathscr{L})$. Let f be integrable. For positive M, N, define

(5.1) $$f_{M,N} \equiv_D [-M \vee f] \wedge N ,$$

and

(5.2) $$f_M \equiv_D \lim_{N \to \infty} f_{M,N} ,$$

so that

(5.3) $$f = \lim_{M \to \infty} f_M .$$

For fixed M, N, the function $f_{M,N}$ is square-integrable. Lemma 3.1 makes correspond to $f_{M,N}$ a square-integrable, $\mathscr{L}$-measurable function $g_{M,N}$. It will first be shown that

(5.4) $$g_M \equiv_D \lim_{N \to \infty} g_{M,N}$$

and

(5.5) $$g \equiv_D \lim_{M \to \infty} g_M$$

exist. The principal result of the paper will then be proved:

THEOREM 5.1. *If f is integrable and if the range of f is in G_Φ, then*

$$J_\Phi(f, h) \geqq J_\Phi(f, g) + J_\Phi(g, h)$$

for every $h \in \mathscr{C}(\mathscr{L})$ whose range is in G_Φ.

The proof follows several preliminary lemmas.

LEMMA 5.1. *Let $f \in L^2$ and let g be given by Lemma 3.1. Let t be real, and let $h \in \mathscr{C}(\mathscr{L})$. Then*

$$\begin{aligned}(5.6)\qquad &\int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [f(\omega) - t]d\mu(\omega) \\ &\quad > \int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [f(\omega) - g(\omega)]d\mu(\omega) \geqq 0 ,\end{aligned}$$

$$\begin{aligned}(5.7)\qquad &\int_{\{\omega:\, g(\omega) \leqq t < h(\omega)\}} [f(\omega) - t]d\mu(\omega) \\ &\quad \leqq \int_{\{\omega:\, g(\omega) \leqq t < h(\omega)\}} [f(\omega) - g(\omega)]d\mu(\omega) \leqq 0 ,\end{aligned}$$

provided, in (5.6), *that the indicated set has positive measure.*

Proof. Set $\Phi(u) \equiv_D -(u - t)$ for $u \leqq t$, $\Phi(u) \equiv_D 0$ for $u > t$. Set $h_1 \equiv_D g \wedge h$. Then $\varphi h_1 \in \mathscr{C}(\mathscr{L})$. application of (3.2) with h replaced by φh_1 yields

$$\int_{\{\omega:\, g(\omega) \wedge h(\omega) \leqq t\}} [f(\omega) - g(\omega)]d\mu(\omega) \geqq 0 .$$

Also, by (3.10),

$$\int_{\{\omega:\, g(\omega) \leqq t\}} [f(\omega) - g(\omega)]d\mu(\omega) = 0 .$$

Since $\{g \wedge h \leqq t\} = \{g \leqq t\} \cup \{h \leqq t < g\}$, it follows that

$$\int_{\{\omega:\, h(\omega) \leqq t < g(\omega)\}} [f(\omega) - g(\omega)]d\mu(\omega) \geqq 0 .$$

The first of inequalities (5.6) is clear. The proof of (5.7) is similar.

COROLLARY 5.1. *Let $f_i \in L^2$ and let g_i be determined by f_i through Lemma 3.1, $i = 1, 2$. If $f_1(\omega) \leqq f_2(\omega)$ for $\omega \in \Omega$, then there are determinations of g_1, g_2 such that $g_1(\omega) \leqq g_2(\omega)$ for $\omega \in \Omega$.*

Proof. Suppose that for some real t, $\mu\{\omega: g_2(\omega) \leqq t < g_1(\omega)\} > 0$. From (5.6) and (5.7) it follows that

$$\begin{aligned}&\int_{\{\omega:\, g_2(\omega) \leqq t < g_1(\omega)\}} [f_2(\omega) - t]d\mu(\omega) \leqq 0 \\ &\quad < \int_{\{\omega:\, g_2(\omega) \leqq t < g_1(\omega)\}} [f_1(\omega) - t]d\mu(\omega) \\ &\quad \leqq \int_{\{\omega:\, g_2(\omega) \leqq t < g_1(\omega)\}} [f_2(\omega) - t]d\mu(\omega) ,\end{aligned}$$

a contradiction. Thus for every real t, $\mu\{g_2 \leqq t < g_1\} = 0$, so that $g_1 \leqq g_2$ a.e. (μ). One may then suppose g_1, g_2 so chosen that the inequality is satisfied everywhere.

Frow Corollary 5.1 it follows that for fixed M the sequence $g_{M,N}$ is monotone, as is also the sequence g_M. The existence of the limits g_M and g is then guaranteed.

THEOREM 5.2. *If g is $\mathscr{S}$-measurable and if the range of f is in G_Φ, then*

$$J_\Phi(f, h) \geqq J_\Phi(f, g) + J_\Phi(g, h)$$

for all bounded $h \in \mathscr{C}(\mathscr{L})$ with range in G_Φ.

Proof. From the geometric interpretation (cf. (2.4)) of Δ and the boundedness of h it is clear that for fixed M there exists N_0 such that $\Delta[f_{M,N}(\omega), h(\omega)]$ is non-decreasing in N for $N > N_0$, $\omega \in \Omega$. Also there exists M_0 such that $\Delta[f_M(\omega), h(\omega)]$ is non-decreasing in M for $M > M_0$, $\omega \in \Omega$. Therefore

$$\begin{cases} J(f_M, h) = \lim_{N\to\infty} J(f_{M,N}, h) , \\ J(f, h) = \lim_{M\to\infty} J(f_M, h) . \end{cases} \tag{5.8}$$

By Theorem 3.2,

$$J(f_{M,N}, h) \geqq J(f_{M,N}, g_{M,N}) + J(g_{M,N}, h) ;$$

hence

$$\liminf_{N\to\infty} J(f_{M,N}, h) \leqq \liminf_{N\to\infty} J(f_{M,N}, g_{M,N}) + \liminf_{N\to\infty} J(g_{M,N}, h) .$$

By Fatou's lemma,

$$\liminf_{N\to\infty} J(f_{M,N}, g_{M,N}) \geqq J(f_M, g_M)$$

and

$$\liminf_{N\to\infty} J(g_{M,N}, h) \geqq J(g_M, h) .$$

Therefore

$$\liminf_{N\to\infty} J(f_{M,N}, h) \geqq J(f_M, g_M) + J(g_M, h) .$$

From (5.8) it now follows that

$$J(f_M, h) \geqq J(f_M, g_M) + J(g_M, h) .$$

A repetition of the argument yields

$$J(f, h) \geqq J(f, g) + J(g, h) ,$$

completing the proof of Theorem 5.2.

LEMMA 5.3. *If f is integrable, so is g.*

Proof. Let $E_{MN} \equiv_D \{\omega: g_{M,N}(\omega) \geqq 0\}$. The application of (3.10) to $f_{M,N}, g_{m,n}$ gives $\int_{E_{MN}} g_{M,N} d\mu = \int_{E_{MN}} f_{M,N} d\mu$. Therefore

$$\int_{E_{MN}} |g_{M,N}| d\mu = \int_{E_{MN}} g_{M,N} d\mu$$
$$= \int_{E_{MN}} f_{M,N} d\mu \leqq \int_{E_{MN}} |f_{M,N}| d\mu \leqq \int_{E_{MN}} |f| d\mu .$$

Similarly

$$\int_{E^c_{NM}} |g_{M,N}| d\mu = \int_{E^c_{MN}} - g_{M,N} d\mu$$
$$= \int_{E^{-c}_{MN}} - f_{M,N} d\mu \leqq \int_{E^c_{MN}} |f_{M,N}| d\mu \leqq \int_{E^c_{MN}} |f| d\mu .$$

Addition gives

$$\int |g_{M,N}| d\mu \leqq \int |f| d\mu ,$$

and the integrability of $|g| = \lim_M \lim_N |g_{M,N}|$ follows.

Proof of Theorem 5.1. By hypothesis and Lemma 5.3, both f and g are integrable. Passage to the limit yields (4.5). By Theorem 3.2, $g_{M,N}$ is bounded by $\inf_\omega f_{M,N}(\omega)$ and $\sup_\omega f_{M,N}(\omega)$; therefore also $\inf_\omega f(\omega) \leqq g(\omega) \leqq \sup_\omega f(\omega)$, $\omega \in \Omega$. The conclusion of Theorem 5.1 now follows from Theorem 4.2.

6. σ-lattices determined by partial orderings on Ω. The problem of minimizing $J(f, \circ)$ in $\mathscr{D}(\mathscr{L})$ was discussed in § 4 of [5] for the special case in which Ω is a euclidean space E_n, and in which a partial ordering on E_n is given by

$$\omega = (\omega_1, \cdots, \omega_n) \leqq \xi = (\xi_1, \cdots, \xi_n) \Longleftrightarrow_D \omega_1 \leqq \xi_1, \omega_2 \leqq \xi_2, \cdots, \omega_n \leqq \xi_n .$$

In [5], classes $\mathscr{L}$ and $\mathscr{U}$ of $\mathscr{S}$-measurable sets were introduced as follows: $L \in \mathscr{L} \Longleftrightarrow_D \xi \in L$, $\omega \leqq \xi \Rightarrow \omega \in L$; $U \in \mathscr{U} \Longleftrightarrow_D U^c \in \mathscr{L}$. The approach in [5] to the minimum problem was through an analoue of the Hahn-Jordan decomposition theorem. The present investigation began with the realization that the methods apply equally well when $\mathscr{L}$ is an arbitrary σ-lattice of sets in $\mathscr{S}$. Indeed, such an approach forms an alternative to that developed in the preceding sections. The present section is devoted to the remark that, given a partial ordering on Ω, the class of $\mathscr{S}$-measurable, order-preserving maps from Ω into R coincides with the class $\mathscr{C}(\mathscr{L})$ for a suitably defined σ-lattice $\mathscr{L}$.

Given a σ-lattice $\mathscr{L} \subset \mathscr{S}$, $\mathscr{C}(\mathscr{L})$ denotes the class of functions h such that for every real $t\{\omega: h(\omega) < t\} \in \mathscr{L}$. For a partial ordering $\mathscr{P}(\leqq)$ of Ω, define $\mathscr{P}^*$ as the class of $\mathscr{S}$-measurable, order-preserving maps of Ω into R. Define also $\mathscr{L}(\mathscr{P})$ as the class of $\mathscr{S}$-measurable sets A such that $\xi \in A$, $\omega \leqq \xi \Rightarrow \omega \in A$. The class $\mathscr{L}(\mathscr{P})$ is a σ-lattice.

The following theorem may be proved by straightforward application of the definitions.

THEOREM 6.1. $\mathscr{C}[\mathscr{L}(\mathscr{P})] = \mathscr{P}^*$.

In should perhaps also be remarked that given a class $\mathscr{C}$ of $\mathscr{S}$-measurable functions, one can determine as follows a σ-lattice $\mathscr{L}$ of $\mathscr{S}$-measurable sets such that $\mathscr{C}$ is embedded in the class $\mathscr{C}(\mathscr{L})$ of $\mathscr{L}$-measurable sets. Define a partial ordering $\mathscr{P}(\mathscr{C})$: $\omega \leqq \xi \Longleftrightarrow_D h(\omega) \leqq h(\xi)$ for all $h \in \mathscr{C}$. Then set $\mathscr{L} = \mathscr{L}[\mathscr{P}(\mathscr{C})]$.

7. Concluding remarks. Let X_0 be a random vector, and $\tau = (\tau_1, \cdots, \tau_n)$ a point of euclidean n-space E_n. Define

$$\Psi(\tau) \equiv_D \log E(e^{X_0 \cdot \tau}) .$$

The function Ψ is convex, defined on a convex subset G_Ψ of E_n. For τ in G_Ψ, $\exp\{x \cdot \tau - \Psi(\tau)\}$ $(x \in E_n)$ is the density function with respect to the distribution of X_0 of a member of the exponential family (Darmois-Koopman class, Koopman-Pitman class, or Laplacian family) of distributions generated by X_0.

For $i = 1, 2, \cdots, k$, let $\tau^i \in G_\Psi$. Let independent random samples of sizes $N_1, \cdots, N_k$ be taken from the distributions corresponding to $\tau^1, \cdots, \tau^k$ respectively. Let $\bar{x}^i$ denote the (vector) sample mean of the sample from the ith population. Then the logarithm of the joint density function is

$$\sum_{i=1}^{k} N_i(\bar{x}^i \cdot \tau^i) - \Psi(\tau^i) . \tag{7.1}$$

For $n = 1$, let Φ denote the convex function conjugate to Ψ in the sense of W. H. Young (§ 2); and define θ^i by $\tau^i = \varphi(\theta^i)$, $i = 1, 2, \cdots, k$. A problem of maximum likelihood estimation of the parameters $\theta^1, \cdots, \theta^k$ is a problem of maximizing (7.1), or equivalently of minimizing, for given $\bar{x}^1, \cdots, \bar{x}^k$,

$$\sum_{i=1}^{k} N_i[\Phi(\bar{x}^i) + \Psi(\tau^i) - \bar{x}^i \tau^i] . \tag{7.2}$$

Let Ω be a space of k distinct points $\omega^1, \cdots, \omega^k$, and μ a measure assigning measure N_i/N to ω^i, $i = 1, 2, \cdots, k$, where $N = \sum_{i=1}^k N_i$. Define $f(\omega^i) = \bar{x}^i$, $h(\omega^i) = \theta^i$, $i = 1, 2, \cdots, k$. The sum (7.2) can then be written $NJ_\Phi(f, h)$. The problem of minimizing (7.2) subject to a partial ordering on $\theta^1, \theta^2, \cdots, \theta^k$ is thus a special instance of the problem treated in this paper. (This special problem has been treated in [5], [6], [7], and [1], and a special case in [4].)

Certain problems involving n-dimensional parameters with $n > 1$ reduce to the one-dimensional case.

1°. Suppose the components $X_{10}, \cdots, X_{n0}$ of X_0 are independent. Then $\Psi(\tau)$ is of the form $\sum_{j=1}^{n} \Psi_j(\tau_j)$. The form to be minimized can be written

$$\sum_{i=1}^{k} N_i \left[\sum_{i=1}^{n} \Phi_j(\bar{x}_j^i) + \Psi_j(\tau_j^i) - \bar{x}_j^i \tau_j^i \right] ,$$

or $\sum_{j=1}^{n} J_{\Phi_j}(f_j, h_j)$. In effect, the components of the n-dimensional parameter can be estimated separately.

The methods of the present paper appear to extend naturally to situations involving convex functions of several real variables only for functions Φ of the form $\sum_{j=1}^{n} \Phi_j$; and for such functions the one-dimensional treatment suffices. Much of the material in § 3 is meaningful also when Φ is an arbitrary convex function of several real variables; but for such functions generalizations of Theorms 5.1 and 5.2 have escaped the author.

2°. Suppose that order restrictions are applied only to the first components $\tau_1^1, \cdots, \tau_1^k$ of $\tau^1, \cdots, \tau^k$, and that the other components are required to be independent of i:

$$\tau_2^1 = \cdots = \tau_2^k, \tau_3^1 = \cdots = \tau_3^k, \cdots, \tau_n^1 = \cdots = \tau_n^k . \tag{7.3}$$

The minimizing values of $\tau_1^1, \cdots, \tau_1^k$ must minimize also the function of them obtained when the parameters τ_j^i $j = 2, 3, \cdots, n, i = 1, 2, \cdots, k$, are replaced by their minimizing values. But this function is of the form (7.2) (one-dimensional problem) for a certain function Φ depending on the minimizing values of the τ_j^i $(j = 2, 3, \cdots, n, i = 1, 2, \cdots, k)$ subject to (7.3). Since the solution is independent of the particular function Φ, the τ_1^i are determined by the $\bar{x}_1^i$ as in the one-dimensional problem $(i = 1, 2, \cdots, k)$.

This remark is appropriate in particular when $n = 2, X_{01}$ is normal with mean 0 and standard deviation 1, and $X_{02} \equiv X_{01}^2$ (the superscript here indicates the square). The distribution of the exponential fumily generated by X_0, corresponding to the parameter point $\tau = (\tau_1, \tau_2)$ is normal with mean $\tau_1/(1 - 2\tau_2)$ and variance $1/(1 - 2\tau_2)$. Thus if the parameters τ_j^i, $i = 1, 2, \cdots, k, j = 1, 2$ are to be estimated by the maximum likelihood method subject to a partial ordering of the means $\mu_i \equiv_D \tau_1^i/(1 - 2\tau_2^i)$ and subject to the condition that τ_2^i is independent of i, then the μ_i are determined by the sample means as in the one-dimensional problem. This result appears in [7] and in [1].

A final remark is that the inequality (1.2) for the conditional expectation of a random variable can be used in a modification of the proof of the Rao-Blackwell theorem on sufficient sub-σ-fields. Let f be a statistic. Let $\mathscr{T}$ be a sufficient sub-σ-field, i.e., $g = E(f \mid \mathscr{T})$ is independent of the measure μ in the the class of measures considered. Let θ_0 denote the expectation of f. By (1.2),

$$J_\Phi(f, \theta_0) \geqq J_\Phi(f, g) + J_\Phi(g, \theta_0) .$$

Hence

$$J_\Phi(g, \theta_0) \leqq J_\Phi(f, \theta_0) , \tag{7.4}$$

For $\Phi(u) \equiv_D u^2/2$, (7.4) states that g has smaller variance than f. Further, let $L(u, v)$ represent the loss which occurs if the estimate of the parameter $E(f)$ is u when the true value is v. Suppose $L(u, v)$ is convex in u for fixed v. Set $\Phi(u) \equiv_D L(u, \theta_0)$ for constant θ_0—the true parameter value. From (7.4) it is then immediate that the risk is smaller for g than for f, whatever the true value θ_0.

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