ASYMPTOTIC ESTIMATES FOR LIMIT CIRCLE PROBLEMS

C. A. SWANSON

1. Preliminaries. Characteristic value problems will be considered for the second order, ordinary, linear differential operator L defined by

$$Lx = \frac{1}{k(s)}\left\{-\frac{d}{ds}\left[p(s)\frac{dx}{ds}\right] + q(s)x\right\} \tag{1.1}$$

on the open interval $\omega_- < s < \omega_+$, where k, p, q are real-valued functions on this interval with the properties that

(i) p is differentiable;

(ii) k and q are piecewise continuous; and

(iii) k and p are positive-valued. The points ω_- and ω_+ are in general singularities of L; the possibility that they are $\pm\infty$ is not excluded. It will be convenient to use the notations

$$(x, y)_s^t = \int_s^t x(u)\bar{y}(u)k(u)du, \qquad \omega_- \leqq s < t \leqq \omega_+ , \tag{1.2}$$

$$[xy](s) = p(s)[x(s)\bar{y}'(s) - x'(s)\bar{y}(s)] . \tag{1.3}$$

Then Green's symmetric formula for L has the form

$$(Lx, y)_s^t - (x, Ly)_s^t = [xy](t) - [xy](s) . \tag{1.4}$$

The symbols $[xy](\pm)$ will be used as abbreviations for the limits of $[xy](s)$ as $s \to \omega_\pm$, and (x, y) will be used for the left member of (1.2) when s, t have been replaced by ω_-, ω_+. Let $\mathfrak{H}$, $\mathfrak{H}_{ab}$ denote the Hilbert spaces which are the Lebesgue spaces with respective inner products (x, y), $(x, y)_a^b$ and norms $||x|| = (x, x)^{1/2}$, $||x||_a^b = [(x, x)_a^b]^{1/2}$, $\omega_- \leqq a < b \leqq \omega_+$.

Let a_0 and b_0 be fixed numbers satisfying $\omega_- < a_0 < b_0 < \omega_+$ and let R_0 be the rectangle in the $a-b$-plane described by the inequalities $\omega_- < a \leqq a_0$, $b_0 \leqq b < \omega_+$. Every closed, bounded subinterval $[a, b]$ of the basic interval (ω_-, ω_+) can be associated in a one-to-one manner with a point in R_0. For every such $[a, b]$ we shall consider the regular Sturm-Liouville problem

$$Ly = \mu y, \qquad U_a y = U_b y = 0 \tag{1.5}$$

on $[a, b]$, where U_a, U_b are the linear boundary operators

$$\begin{aligned} U_a y &= \alpha_0(a)y(a) + \alpha_1(a)y'(a) \\ U_b y &= \beta_0(b)y(b) + \beta_1(b)y'(b) , \end{aligned} \tag{1.6}$$

with α_0, α_1 real-valued functions not both 0 for any value of a on $(\omega_-, a_0]$,

Received January 5, 1961.

and with β_0, β_1 real-valued and not both 0 on $[b_0, \omega_+]$. Our problem is to obtain estimates for each characteristic value $\mu = \mu_{ab}$ of (1.5) for a, b near ω_-, ω_+ under hypotheses that will ensure that the limits of μ_{ab} as $a, b \to \omega_-, \omega_+$ will exist. Also, we shall obtain estimates for the corresponding characteristic functions $y = y_{ab} = y_{ab}(s)$ on $a \leqq s \leqq b$. Results like this for differential operators having a singularity at one endpoint were obtained previously by an integral equations approach [8], [9]. The present paper contains extensions of some of these results to operators (1.1) which have singularities at both endpoints. Furthermore, the present approach to the problem will be different; the estimates will now be obtained by means of projection mappings on suitable Hilbert spaces. The method arises from an idea communicated by Professor H. F. Bohnenblust, and affords an elegant and abstract approach to the type of perturbation problem at hand [1]. Also, the present method is powerful enough to handle a variety of domain-perturbed problems that arise in the study of elliptic partial differential equations. Some of these have been considered already [10] and the author has several others in preparation.

Here the method will be illustrated in the case that both of the singularities $\omega_\pm$ of the operator (1.1) are *limit circle* singularities in the well-known classification of H. Weyl [2, p. 225]. In another paper we shall consider the limit point cases (and mixed cases) in which some additional hypotheses are needed on the growth of the coefficient functions in (1.1) as $s \to \omega_\pm$ to ensure the existence of isolated characteristic values λ of L on (ω_-, ω_+); however, very general boundary operators U_a, U_b will then permit convergence of μ_{ab} to λ. For additional details, see [8]. In the limit circle case herein under consideration, no special assumptions will be imposed on the nature of L at $\omega_\pm$, but the generality of the boundary operators must be sacrificed in order to ensure the convergence of μ_{ab}. Our purpose here is to obtain asymptotic estimates rather than asymptotic expansions for the characteristic values and functions as $a, b \to \omega_-, \omega_+$. Asymptotic formulae and expansions will be published elsewhere.

2. Basic and perturbed problems. Rather than general spectral theory, we are interested in cases that the limits of μ_{ab} as $a, b \to \omega_-, \omega_+$ exist in the elementary sense. Thus, characteristic values of suitable singular boundary value problems for L on (ω_-, ω_+) are supposed to exist. These singular problems are described differently according as the points $\omega_\pm$ are in the limit point or limit circle categories. The description is made as follows when both are limit circle singularities [2], [6]: choose a complex number l_0 with $\operatorname{Im} l_0 \neq 0$, and let L_0 be the differential operator $L - l_0$. A theorem of Weyl [6] states that there exist linearly independent solutions $\varphi_\pm \in \mathfrak{H}$ of $L_0\varphi = 0$ such that

$$(2.1) \qquad [\varphi_-\varphi_-](-) = [\varphi_+\varphi_+](+) = 0, \qquad [\varphi_+\bar{\varphi}_-](s) = 1 .$$

Let $\mathfrak{D}$ denote the domain consisting of all $x \in \mathfrak{H}$ which have the following properties:

(a) x is differentiable on (ω_-, ω_+) and x' is absolutely continuous on every closed subinterval of this interval:

(b) $Lx \in \mathfrak{H}$

(c) x satisfies the end conditions

$$[x\varphi_-](-) = [x\varphi_+](+) = 0 . \tag{2.2}$$

Then L on $\mathfrak{D}$ is real and essentially self-adjoint [6]. The *basic* characteristic problem

$$Lx = \lambda x, \qquad x \in \mathfrak{D} \tag{2.3}$$

is known to have a denumerable set of characteristic values λ_n and corresponding characteristic functions x_n which are orthonormal and complete in $\mathfrak{H}$ $(n = 1, 2, \cdots)$.

Two classes of perturbation problems (1.5) will be considered. The limiting behaviour of *class* 1 *boundary operators* U_a, U_b as $a, b \to \omega_-, \omega_+$ is rather arbitrary (see §5) while the limiting behaviour of *class* 2 *operators* (§§2, 3, 4) is restricted as follows:

$$\begin{aligned} U_a y &= [y\varphi_-](a)[1 + o(1)] \qquad \text{as } a \to \omega_- \\ U_b y &= [y\varphi_+](b)[1 + o(1)] \qquad \text{as } b \to \omega_+ \end{aligned} \tag{2.4}$$

for every differentiable function y. A *perturbed domain* $\mathfrak{D}_{ab}$ is defined for each $[a, b] \in R_0$ to be the set of all y in the subspace $\mathfrak{F}_{ab}$ of $\mathfrak{H}$ which satisfy the following conditions:

(a) y is differentiable and y' is absolutely continuous on $[a, b]$;

(b) $Ly \in \mathfrak{F}_{ab}$

(c) y satisfies the homogeneous boundary conditions (1.5) where the boundary operators U_a, U_b have the limiting behaviour (2.4).

The perturbed characteristic value problem that corresponds to this domain is the regular Sturm-Liouville problem

$$Ly = \mu y, \qquad y \in \mathfrak{D}_{ab} . \tag{2.5}$$

In addition, we define a domain $\mathfrak{D}_a$ for each a on $(\omega_-, a_0]$ to be the set of all $z \in \mathfrak{F}_{a\omega_+}$ which satisfy the following:

(a) z is differentiable and z' is absolutely continuous on every closed subinterval of $[a, \omega_+)$;

(b) $Lz \in \mathfrak{F}_{a\omega_+}$

(c) z satisfies the conditions

$$U_a z = 0, \quad [z\varphi_+](+) = 0 . \tag{2.6}$$

The characteristic value problem

$$Lz = \nu z, \qquad z \in \mathfrak{D}_a \tag{2.7}$$

on the half-open interval $[a, \omega_+)$ may be regarded as intermediate between (2.3) and (2.5), and will be called a semi-perturbed problem.

In order to obtain estimates for the difference between the characteristic values and functions of (2.5) and (2.3), we shall proceed in two steps: (i) the comparison of (2.5) with (2.7), and (ii) the comparison of (2.7) with (2.3). The details of (i) and (ii) are included in §§3 and 4 respectively. Each comparison has independent interest because it is typical for a boundary variational problem when only one endpoint is varied and the unchanged endpoint is (i) an ordinary point; (ii) a singular point of the differential operator. Type (ii) variational problems arise for example in the theory of enclosed quantum mechanical systems[4], [5].

3. Comparison of the y and z problems. The characteristic value problems (2.5) and (2.7) will be compared, with (2.7) regarded as basic and (2.5) regarded as a perturbation on (2.7). In this case, the singular boundary condition $[z\varphi_+](+) = 0$ is replaced by the regular condition $U_b z = 0$ at the point b. We are going to estimate the variation of characteristic values and functions under this perturbation, and show that this variation has the limit 0 as $b \to \omega_+$. The ordinary endpoint a remains fixed in this section.

Let $G_{ab}(s, t)$ be the Green's function for the operator kL_0 associated with the boundary conditions (1.5), and let G_{ab} be the linear transformation on $\mathfrak{F}_{ab}$ defined by the equation

$$G_{ab}y = \int_a^b G_{ab}(s, t)y(t)k(t)dt, \qquad y \in \mathfrak{F}_{ab}\,. \tag{3.1}$$

Let $\nu = \nu_a$ be a characteristic value for (2.7) and let z_a be the corresponding characteristic function. Define a function f on $[a, b]$ by[1]

$$f = z_a - \gamma_a G_{ab} z_a \qquad \text{where } \gamma_a = \nu_a - l_0\,. \tag{3.2}$$

It is easily verified because of the linearity of all the operators involved that f is a solution of the boundary value problem

$$L_0 f = 0, \qquad U_a f = 0, \qquad U_b f = U_b z_a\,. \tag{3.3}$$

The solution ψ_a of $L_0 y = 0$ that is given by

$$\psi_a(s) = \varphi_-(s)U_a\varphi_+ - \varphi_+(s)U_a\varphi_- \tag{3.4}$$

satisfies the boundary condition $U_a y = 0$. Hence the unique solution of (3.3) is

[1] The function on $[a, b]$ which coincides with z_a on this interval will also be denoted by z_a.

$$f(s) = (U_b z_a / U_b \psi_a)\psi_a(s), \qquad a \leqq s \leqq b . \tag{3.5}$$

In fact, if g is any solution of (3.3), then the function $h = g - f$ satisfies $L_0 h = 0$, $U_a h = U_b h = 0$. This implies that h is the zero function, or $g = f$.

It follows from (2.1) that $[\varphi_+\varphi_+](b) \to 0$ as $b \to \omega_+$ and $[\varphi_-\varphi_-](a) \to 0$ as $a \to \omega_-$. The identity

$$[\varphi_+\varphi_+](t) - [\varphi_+\varphi_+](s) = (l_0 - \bar{l}_0)(\|\varphi_+\|_s^t)^2$$

is a consequence of (1.4), and since $\varphi_+ \in \mathfrak{H}$, the limit $[\varphi_+\varphi_+](-)$ exists. Similarly $[\varphi_-\varphi_-](+)$ exists. From (2.1) and the identity [6]

$$|[\varphi_+\bar{\varphi}_-](a)|^2 = [\varphi_-\varphi_-](a)[\varphi_+\varphi_+](a) + |[\varphi_+\varphi_-](a)|^2$$

we deduce that $|[\varphi_+\varphi_-](a)| \to 1$ as $a \to \omega_-$. Similarly $|[\varphi_+\varphi_-](b)| \to 1$ as $b \to \omega_+$. It has then been established that

$$\begin{aligned} &U_a\varphi_- \to 0, \quad |U_a\varphi_+| \to 1 \quad \text{as } a \to \omega_- ; \\ &U_b\varphi_+ \to 0, \quad |U_b\varphi_-| \to 1 \quad \text{as } b \to \omega_+ \end{aligned} \tag{3.6}$$

where (2.4) has been used. Since $\varphi_\pm \in \mathfrak{H}$, it follows from (3.4) that $\|\psi_a\|_a^b$ is uniformly bounded for $[a, b] \in R_0$. We obtain from (3.4) that

$$U_b\psi_a = U_b\varphi_- U_a\varphi_+ - U_b\varphi_+ U_a\varphi_-$$

and hence there are numbers a_0, b_0 (we may suppose that they coincide with the original choices of a_0, b_0) such that $U_b\psi_a$ is bounded away from zero on $a \leqq a_0$, $b_0 \leqq b$. These considerations enable us to deduce from (3.2), (3.5) that there exists a constant[2] C on R_0 such that

$$\|z_a - \gamma_a G_{ab} z_a\|_a^b \leqq C|U_b z_a| \, \|z_a\|_a^b, \qquad [a, b] \in R_0 . \tag{3.7}$$

Let $\mu^i = \mu^i_{ab}$ denote the ith characteristic value of the regular problem (2.5), $\mu^1 < \mu^2 < \cdots$, and let y^i denote the corresponding characteristic function, chosen so that $\{y^i\}$ is an orthonormal basis in $\mathfrak{F}_{ab}$. The following fundamental lemma was obtained by H. F. Bohnenblust in [1] by applying the Parseval completeness relation to the set $\{y^i\}$. An outline of the proof is reproduced below.

LEMMA. *Let $P(\delta)$ be the projection mapping from the Hilbert space $\mathfrak{F}_{ab}$ onto its subspace $\mathfrak{F}_{ab}(\delta)$ $(\delta > 0)$ spanned by all characteristic functions y^i of* (2.5) *such that their corresponding μ^i satisfy $|\mu^i - \nu_a| \leqq \delta$. Then for any $w \in \mathfrak{F}_{ab}$,*

$$\|w - P(\delta)w\|_a^b \leqq (1 + |\gamma_a|/\delta)\,\|w - \gamma_a G_{ab} w\|_a^b .$$

[2] The letter C will be used throughout as a generic notation for the image of a constant function from R_0 into the positive numbers.

Proof. The subscripts a, b will be omitted in this proof. Let $\alpha_i = (Gw, y^i)$. It is easily verified that $(w - \gamma Gw, y^i) = (\mu^i - \nu)\alpha_i$, and hence by the Parseval identity,

$$||w - \gamma Gw||^2 = \sum_i |\mu^i - \nu|^2 |\alpha_i|^2 \geqq \delta^2 \sum_i {}^* |\alpha_i|^2 ,$$

where the * denotes summation over only those indices i such that $|\mu^i - \nu| > \delta$. Then

$$||Gw - P(\delta)Gw||^2 = \sum{}^* |\alpha_i|^2 \leqq \delta^{-2} ||w - \gamma Gw||^2 ,$$

and the conclusion of the lemma follows easily from the Minkowski inequality.

The notation $\rho_b = C|\gamma_a U_b z_a|$ will be used. It follows from (2.4) and (2.6) that $\rho_b \to 0$ as $b \to \omega_+$ for each fixed a. With the choice $\delta = 2\rho_b$, we apply the lemma to $w = z_a$ (see footnote 1) and use (3.7) to obtain

$$||z_a - P(2\rho_b)z_a||_a^b \leqq (C|U_b z_a| + \tfrac{1}{2}) ||z_a||_a^b .$$

We may suppose that b_0 has been selected so that $C|U_b z_a| \leqq 1/4$ on $b_0 \leqq b < \omega_+$. Hence $P(2\rho_b)z_a = 0$ implies that $z_a = 0$ on $[a, b]$, and therefore $\mathfrak{F}_{ab}(2\rho_b)$ has dimension $\geqq 1$. Hence there exists at least one characteristic value $\mu = \mu_{ab}$ of (2.5) which satisfies

$$|\mu_{ab} - \nu_a| \leqq 2\rho_b, \qquad [a, b] \in R_0 . \tag{3.8}$$

To prove that there is exactly one, we conclude from the maximum-minimum principle for characteristic values [3], [7] that the absolute value of the ith characteristic value ν_a^i of (2.7) cannot decrease when a boundary condition at b is adjoined, and hence $|\nu_a^i| \leqq |\mu_{ab}^i|$ $(i = 1, 2, \cdots)$. Since the numbers ν_a^i do not accumulate and since $\rho_b \to 0$ as $b \to \omega_+$, there is a constant b_0 such that $2\rho_b$ is less than the minimum of all the differences $|\nu_a^j - \nu_a^i|$, $(i, j = 1, 2, \cdots; i \neq j)$ whenever $b \geqq b_0$. If $0 < \nu_a^1 < \nu_a^2$, it follows from (3.8) that exactly one characteristic value μ_{ab} of (2.5) lies in the interval $[\nu_a^1, \nu_a^1 + 2\rho_b]$. A similar statement applies to the case that one or both of ν_a^1, ν_a^2 are negative.

In order to prove by induction that there is exactly one μ_{ab}^i which satisfies $|\mu_{ab}^i - \nu_a^i| \leqq 2\rho_b$ $(i = 1, 2, \cdots)$, assume that this is true for each integer $i \leqq n$. In the case that $|\nu_a^{n+1}| < |\nu_a^{n+2}|$ there are at most $n + 1$ characteristic values μ_{ab}^i which satisfy $|\mu_{ab}^i| \leqq |\nu_a^{n+1}| + 2\rho_b$ since $|\mu_{ab}^i| \geqq |\nu_a^i|$ for each i. It then follows from the induction assumption that there is at most one characteristic value μ_{ab}^{n+1} satisfying $|\mu_{ab}^{n+1} - \nu_a^{n+1}| \leqq 2\rho_b$, and hence exactly one by (3.8). In the other case $\nu_a^{n+2} = -\nu_a^{n+1}$, it follows similarly that there are at most two μ_{ab}^i satisfying $|\nu_a^{n+1}| < |\mu_{ab}^i| \leqq |\nu_a^{n+1}| + 2\rho_b$, and again by (3.8) there is exactly one μ_{ab}^i near each of ν_a^{n+1}, ν_a^{n+2}.

THEOREM 1. *If the singularity ω_+ of* (1.1) *is the limit circle type, then for every characteristic value ν_a of* (2.7) *there exists a rectangle R_0 and a constant C on R_0 such that*[3] *a unique characteristic value μ_{ab} of the perturbed problem* (2.5) *lies in the interval $|\mu_{ab} - \nu_a| \leqq C|U_b z_a|$ whenever $[a, b] \in R_0$.*

This shows in particular that for each fixed a, there is a unique μ_{ab} of (2.5) such that $\mu_{ab} \to \nu_a$ as $b \to \omega_+$. In addition, the estimate of the theorem is valid uniformly on $\omega_- < a \leqq a_0$. One also finds for the characteristic functions y_{ab} and z_a associated with μ_{ab} and ν_a respectively that the estimate

$$||y_{ab} - z_a||_a^b \leqq C|U_b z_a|, \quad ||y_{ab}||_a^b = ||z_a||_a = 1$$

is valid on R_0.

4. Comparison of the z and x problems. The characteristic value problems (2.7) and (2.3) will now be compared, with (2.7) regarded as a perturbation of the basic problem (2.3). The perturbation arises from the singular end condition $[x\varphi_-](-) = 0$ being replaced by a homogeneous boundary condition at the point a. The novelty of this section is due to the singular nature of the unchanged endpoint ω_+.

Let λ be a characteristic value of (2.3) and let x be the corresponding normalized characteristic function. Let G_a be the linear integral operator on $\mathfrak{F}_{a\omega_+}$ whose kernel is the Green's function for kL_0 associated with the boundary conditions (2.6). This operator is defined similarly to the operator G_{ab} in (3.1) [6]. Let a function g on $[a, \omega_+)$ be defined by[4]

$$g = x - \gamma G_a x \qquad \text{where } \gamma = \lambda - l_0 . \tag{4.1}$$

The analogue of (3.5) is

$$g(s) = (U_a x / U_a \varphi_+)\, \varphi_+(s), \qquad a \leqq s < \omega_+ . \tag{4.2}$$

It follows from the postulated boundary conditions (2.2) at ω_- that $[x\varphi_-](a) \to 0$ as $a \to \omega_-$, and hence by (2.4) that $U_a x \to 0$ as $a \to \omega_-$. It was proved above (3.6) that $|U_a \varphi_+| \to 1$ as $a \to \omega_-$, and since $\varphi_+ \in \mathfrak{H}$, we obtain the inequality

$$||x - \gamma G_a x||_a \leqq C|U_a x|\, ||x||_a \tag{4.3}$$

for some constant C. The analogue of the lemma in §3 with $\mathfrak{F}_{ab}$ replaced by $\mathfrak{F}_{a\omega_+}$ leads to

$$\begin{aligned} ||x - P(\delta)x||_a &\leqq (1 + |\gamma|/\delta)\, ||x - \gamma G_a x||_a \\ &\leqq (1 + |\gamma|/\delta)\, C|U_a x|\, ||x||_a , \end{aligned}$$

[3] See footnote 2.

[4] The function on $[a, \omega_+)$ which coincides with x on this interval will also be denoted by x.

and the following theorem is obtained.

THEOREM 2. *If the singularities* $\omega_\pm$ *of* (1.1) *are both of the limit circle type, then for every characteristic value* λ *of the basic problem* (2.3) *there exist constants* a_0 *and* C *such that a unique characteristic value* ν_a *of* (2.7) *lies in the interval* $|\nu_a - \lambda| \leqq C|U_a x|$ *whenever* a *satisfies* $\omega_- < a \leqq a_0$. *If* x, z_a *are characteristic functions corresponding to* λ, ν_a *respectively with norms* $||x|| = ||z_a||_a = 1$, *then*

$$||z_a - x||_a \leqq C|U_a x|, \quad \omega_- < a \leqq a_0 , \tag{4.4}$$

and in particular $||z_a - x||_a \to 0$ *as* $a \to \omega_-$.

We shall next prove the following consequence of (4.4):

$$U_b z_a = U_b x + (|U_a x| + |U_b x|)o(1) , \tag{4.5}$$

the order symbol being valid as $b \to \omega_+$ uniformly on $\omega_- < a \leqq a_0$. We use formula (1.4) to obtain

$$[z_a\varphi_+](+) - [z_a\varphi_+](b) = (\nu_a - \bar{l}_0)(z_a, \varphi_+)_b ,$$
$$[x\varphi_+](+) - [x\varphi_+](b) = (\lambda - \bar{l}_0)(x, \varphi_+)_b .$$

Since $[x\varphi_+](+) = [z_a\varphi_+](+) = 0$ by (2.2), (2.6), we deduce from the Schwarz inequality on $\mathfrak{J}_{b\omega_+}$ that

$$\begin{aligned} |[z_a\varphi_+](b) - [x\varphi_+](b)| &\leqq |(\nu_a - \bar{l}_0)(z_a - x, \varphi_+)_b| + |(\nu_a - \lambda)(x, \varphi_+)_b| \\ &\leqq |\nu_a - \bar{l}_0|\, ||z_a - x||_b\, ||\varphi_+||_b \\ &\quad + |\nu_a - \lambda|\, ||x||\, ||\varphi_+||_b . \end{aligned}$$

The desired conclusion (4.5) then follows from Theorem 2 and (2.4). The following abbreviation will be used:

$$\rho_{ab} = |U_a x| + |U_b x| . \tag{4.6}$$

THEOREM 3. *If both singularities* $\omega_\pm$ *are of the limit circle type, then for every characteristic value* λ *of* (2.3) *there exists a rectangle* R_0 *and a constant* C *on* R_0 *such that exactly one characteristic value* μ_{ab} *of the perturbed problem* (2.5) *lies in the interval* $|\mu_{ab} - \lambda| \leqq C\rho_{ab}$ *for every* $[a, b] \in R_0$. *For the characteristic functions* x, y_{ab} *associated with* λ, μ_{ab} *respectively, normalized by* $||x|| = ||y_{ab}||_a^b = 1$, *the estimate* $||y_{ab} - x||_a^b \leqq C\rho_{ab}$ *is valid.*

Proof. It follows from Theorems 1 and 2 that

$$\begin{aligned} |\mu_{ab} - \lambda| &\leqq |\mu_{ab} - \nu_a| + |\nu_a - \lambda| \\ &\leqq C(|U_b z_a| + |U_a x|) . \end{aligned}$$

The first statement of the theorem is then a consequence of (4.5) and (4.6). The proof of the second statement is similar and will be omitted.

Finally, we shall obtain uniform estimates for the difference $y_{ab}(s) - x(s)$ on $a \leqq s \leqq b$. We remark in passing that the asymptotic result $y_{ab}(s) = x(s)[1 + o(1)]$ as $a, b \to \omega_-, \omega_+$ cannot be valid for s near the boundaries a, b nor can it be valid near any zeros of $x(s)$. Uniform estimates will now be derived by the same technique that proves useful in certain domain-perturbed problems concerning elliptic partial differential equations [1], when $\varphi_\pm(s)$ are bounded on (ω_-, ω_+).

First it will be shown that $(\lambda - l_0)G_{ab}x(s)$ gives a uniform estimate for $y_{ab}(s)$ on $a \leqq s \leqq b$. Let $\psi_a(s)$ be the function (3.4) and let $\psi_b(s)$ be defined by

$$\psi_b(s) = \varphi_-(s)U_b\varphi_+ - \varphi_+(s)U_b\varphi_- .$$

Then

$$\begin{aligned} G_{ab}(s, t) &= \sigma^{-1}\psi_a(t)\,\psi_b(s) \quad \text{if } a \leqq t \leqq s \leqq b\ , \\ &= \sigma^{-1}\psi_a(s)\,\psi_b(t) \quad \text{if } a \leqq s \leqq t \leqq b\ , \end{aligned}$$

where

$$\sigma = U_a\varphi_-\, U_b\varphi_+ - U_a\varphi_+\, U_b\varphi_- .$$

Then $|\sigma| \to 1$ as $a, b \to \omega_-, \omega_+$, and the function defined by

$$(\|G_{ab}\|_a^b)^2 = \int_a^b |G_{ab}(s, t)|^2 k(t)dt$$

is a bounded function of s, a, and b. Hence the inequality

$$\begin{aligned} |y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| &= |G_{ab}[(\mu_{ab} - l_0)y_{ab}(s) - (\lambda - l_0)x(s)]| \\ &\leqq \|G_{ab}\|_a^b(|\mu_{ab} - l_0|\,\|y_{ab} - x\|_a^b + |\mu_{ab} - \lambda|\,\|x\|)\,, \end{aligned}$$

and Theorem 3 show that there exists a constant C such that

$$(4.6)\qquad |y_{ab}(s) - (\lambda - l_0)G_{ab}x(s)| \leqq C\rho_{ab} \quad a \leqq s \leqq b\ .$$

Let h be the uniquely determined solution of the boundary value problem

$$L_0h = 0, \quad U_ah = U_ax, \quad U_bh = U_bx \qquad \text{on } a \leqq s \leqq b\ .$$

Let the function f on $[a, b]$ be defined by

$$f(s) = (\lambda - l_0)G_{ab}x(s) - x(s) + h(s)\ .$$

Since f satisfies $L_0f = 0$, $U_af = U_bf = 0$, f is identically zero. The following uniform estimate is then a direct consequence of (4.6):

$$(4.7)\qquad y_{ab}(s) = x(s) - h(s) + O(\rho_{ab}), \quad a \leqq s \leqq b\ .$$

It can be verified without much difficulty that $h(s) = O(\rho_{ab})$ on a fixed closed subinterval I_0 of $[a, b]$, valid for $[a, b] \in R_0$. The following uniform result on I_0 is therefore a special case of (4.7):

$$y_{ab}(s) = x(s) + O(\rho_{ab}) \qquad [a, b] \in R_0 .$$

5. Class 1 boundary operators. Instead of the restrictive limiting behaviour (2.4) of the boundary operators U_a, U_b, the limiting behaviour of class 1 boundary operators is essentially arbitrary. In regard to the perturbation $a \to \omega_-$, a class 1 boundary operator U_a is defined as follows. Let φ_+ be the function defined in §2 and let x be a characteristic function of the basic problem (2.3) corresponding to the characteristic value λ. Class 1 perturbation problems are possible when the singularity ω_- is not an accumulation point of the zeros of φ_+ and

$$x(s)/\varphi_+(s) = o(1) \qquad \text{as } s \to \omega_- . \tag{5.1}$$

In this event, U_a is said to be a *class* 1 *boundary operator* on $(\omega_-, a_0]$ whenever the ratio $\varphi_+(a) U_a x/x(a) U_a \varphi_+$ is bounded on this interval. This rather mild restriction on U_a implies that

$$\varepsilon_a = | U_a x / U_a \varphi_+ | = o(1) \qquad \text{as } a \to \omega_- . \tag{5.2}$$

An example is given in [8, pages 838–840] when $\omega_- = 0$ is a regular singularity of L, with $p(s) = 1$. In this event, a sufficient condition that the boundedness requirement above (5.2) be satisfied is that the limit $\sigma = \lim_{a \to 0} [a\alpha_0(a)/\alpha_1(a)]$ exists (finite or ∞) and $\sigma \neq -\rho$, where ρ denotes the smaller of two real, distinct exponents at the singularity 0.

Let g be defined by (4.1). Then (4.2) is valid but under the assumptions of this section, (4.3) is replaced by

$$||x - \gamma G_a x||_a \leqq C\varepsilon_a \, ||x||_a \tag{5.3}$$

where ε_a is defined by (5.2). In the notation of §§2, 3,

$$||x - P(\delta)x||_a \leqq (1 + |\gamma|/\delta)\, C\varepsilon_a \, ||x||_a .$$

Since $\varepsilon_a = o(1)$ as $a \to \omega_-$, Theorems 2 and 3 are valid with the replacement ε_a instead of $| U_a x|$. A similar statement is appropriate in the event that U_b is a class 1 boundary operator.

In the example of a regular singularity $\omega_- = 0$ with real exponents ρ_1, ρ_2, it turns out that $\varepsilon_a = O(a^{\rho_1 - \rho_2})$ if $\rho_1 > \rho_2$ and $\varepsilon_a = O(1/ln\, a)$ if $\rho_1 = \rho_2\, (0 < a \leqq a_0)$.

REFERENCES

1. H. F. Bohnenblust et al, *The Laplacian operator on perturbed domains*, Unpublished manuscript.

2. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill (1955).

3. R. Courant and D. Hilbert, *Methods of mathematical physics* I, Interscience (1953).

4. R. B. Dingle, *The solution of the Schrödinger equation for finite systems, etc.*, Proc. Camb. Philos. Soc., **49** (1953), 102-114.

5. T. E. Hull and R. S. Julius, *Enclosed quantum mechanical systems*, Canadian J. Phys., **34** (1956), 914-919. Additional references.

6. F. Rellich, *Spectral theory of a second order ordinary differential operator*, New York University (1953).

7. F. Riesz and B. Sz-Nagy. *Functional Analysis*, Blackie and Son (1956).

8. C. A. Swanson, *Differential operators with perturbed domains*, J. of Rat. Mech. Anal. **6** (1957), 823-846.

9. ———, *Asymptotic perturbation series for characteristic value problems*, Pacific J. Math., **9** (1959), 591-608.

10. ———, *Linear transformations with perturbed domains*, to appear in 1962.

THE UNIVERSITY OF BRITISH COLUMBIA