A THEOREM ON REGULAR MATRICES

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In this paper it will be proved that if any nonnegative, square matrix P of order r is such that $P^m > 0$ for some positive integer m, then $P^{r^2-2r+2} > 0$. This result has already appeared in the literature, [2], but the following is a complete and elementary proof given in detail except for one theorem of I. Schur in [1] which is stated without proof. The term regular is taken from Markov chain theory[1] in which a regular chain is one whose transition matrix has the above property.

A *graph* G_P associated with any nonnegative, square matrix P of order r is a collection of r distinct points $S = \{s_1, s_2, \cdots, s_r\}$, some or all of which are connected by directed lines. There is a directed line (indicated pictorially by an arrow) from s_i to s_j in the graph G_P if and only if $p_{ij} > 0$ in the matrix $P = (p_{ij})$. A *path sequence* or *path* in G_P is any finite sequence of points of S (not necessarily distinct) such that there is a directed line in G_P from every point in the sequence to its immediate successor. The *length* of a path is one less than the number of occurrences of points in its sequence. A *cycle* is any path that begins and ends with the same point and a *simple cycle* is a cycle in which no point occurs twice except, of course, for the first (and last). Two cycles are *distinct* if their sequences are not cyclic permutations of each other. A nonnegative, square matrix P is *regular* if $P^m > 0$ for some positive integer m. Likewise, a graph G_P associated with a nonnegative, square matrix P is *regular* if there exists a positive integer m such that an infinite set of paths $A_0, A_1, \cdots, A_n, \cdots$ can be found, the length of each path being $L_n = m + n$, $n = 0, 1, 2, \cdots$. The usual notation $p_{ij}^{(m)}$ is used to denote the ijth entry of the matrix P^m. In all that follows we shall consider only regular matrices P and their associated graphs G_P.

Some immediate consequences of these definitions and the definition of matrix multiplication are the following:

(1) There is a path $s_{k_1} \cdots s_{k_{m+1}}$ in G_P if and only if $p_{k_1 k_{m+1}}^{(m)} > 0$ in P^m.
(2) P is regular if and only if G_P is regular.
(3) There exists some path from any point in G_P to any point in G_P.
(4) For any given i and j there exists some m such that $p_{ij}^{(m)} > 0$.
(5) If $P^m > 0$ then $P^{m+n} > 0$, $n = 0, 1, 2, \cdots$.

Let $C = \{C_1, C_2, \cdots, C_t\}$ be all the distinct simple cycles of G_P and $\{c_1, c_2, \cdots, c_t\}$ be the corresponding lengths.

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[1] This is as treated by Kemeny and Snell in [3].

LEMMA 1. *The length of any cycle C^* is always of the form $c^* = \sum_{i=1}^t a_i c_i$, where a_i is some nonnegative integer.*

Proof. Let any cycle $C^* = s_{k_1}, s_{k_2}, \cdots, s_{k_m}$ be given ($k_1 = k_m$). Let $C^* = C_1^*$ and form C_{i+1}^* in the following manner from C_i^*: Wherever simple cycle C_i occurs in cycle C_i^* delete it except for its last point, thus forming the new cycle C_{i+1}^*. It is clear that after the tth step there will remain only a single point of the original C^*, which has of course zero length. If we let a_i be the number of times simple cycle C_i occurred in cycle C_i^* then the lemma follows.

THEOREM 1. *If G_P is any regular graph then it must contain a set of simple cycles whose lengths are relatively prime.*

Proof. By the regularity assumption and (1) there exists a positive integer m such that cycles of lengths $L_n = m + n$, $n = 0, 1, 2, \cdots$ can be found in G_P. Also, from Lemma 1, $L_n = \sum_{i=1}^t a_i c_i$ for $n = 0, 1, 2, \cdots$, and suitable a_i. Let d be the common factor of the simple cycle lengths c_i. Then

$$\sum_{1=1}^{t} a_i c_i = d \sum_{i=1}^{t} a_i c_i'$$

which could never equal $m + n$, $n = 0, 1, 2, \cdots$ unless $d = 1$.

We would like to find a *least* integer M such that for arbitrary points s_i and s_j there are paths beginning at s_i and ending at s_j and whose lengths are $L_n = M + n$, $n = 0, 1, 2, \cdots$. If we can do this, then, by (1), we shall have also found a least integer M such that $P^M > 0$ where P is the regular matrix associated with G_P.

Let us say that a path *touches* a given set of points if there is some point belonging to both the path and the set. Then we have

LEMMA 2. *Let G_P be a regular graph with r points, let S be a subset containing r_k distinct points of the graph, and let g be any point of G_P. Then there always exists a path from g which touches S whose length is less than or equal to $r - r_k$.*

Proof. If $g \in S$ then the lemma is trivial. Suppose $g \notin S$. By (3) there is at least one path which starts at g and touches the set S. Let $p = g_0, g_1, \cdots, s$ be such a path of shortest length. Obviously no point of S can precede the final point s in this path sequence p. Furthermore, there can be no repeated points in p, for the deletion of any cycle (except for its last point) would produce a path from g to S shorter than path p, contrary to the choice of p. Therefore, p can have at most $r - r_k$ points.

We shall say that a *minimal set* of relatively prime integers is a set of relatively prime integers such that if one of the integers is deleted the remaining integers are no longer relatively prime. A *step* along a path in G_P is a pair of consecutive points of the path sequence.

THEOREM 2. *If $R = \{R_1, R_2, \cdots, R_k\}$ is a set of simple cycles of graph G_P whose lengths $\{r_1, r_2, \cdots, r_k\}$ form a minimal set of relatively prime integers and if s_i and s_j are arbitrary points of G_P, then there is always a path which starts at s_i, ends at s_j, touches each cycle of R and whose length $L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1$.*

Proof. Note that the set of distinct points belonging to a simple cycle contains a number of points exactly equal to the length of the cycle. Hence, by Lemma 2 there is a path from an arbitrary point s_i which touches a particular cycle R_p and whose length is less than or equal to $r - r_p$. Thus, we have the following:

from		*to*		*greatest number of steps needed*
arb. pt.	s_i	cycle	R_1	$r - r_1$
cycle	R_1	″	R_2	$r - r_2$
⋮		⋮		⋮
cycle	R_{k-1}	cycle	R_k	$r - r_k$
″	R_k	arb. pt.	s_j	$r - 1$
		TOTAL		$L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1.$

We shall now state without proof I. Schur's theorem cited above and use it in our final theorem.

THEOREM 3. (Schur) *If $\{a_1, a_2, \cdots, a_n\}$ is a set of relatively prime integers with a_1 the least and a_n the greatest, then $B = \sum_{i=1}^{n} x_i a_i$ has solutions in nonnegative integers x_i for any $B \geqq (a_1 - 1)(a_n - 1)$. This is a best bound for $n = 2$.*

THEOREM 4. *If M is the least integer such that paths between any two points of G_P can be found whose lengths are $L_n = M + n$, $n = 0, 1, 2, \cdots$, then $M \leqq r^2 - 2r + 2$.*

Proof. Given any two points s_i and s_j of G_P we know by Theorem 2 that there is a path from s_i to s_j touching each of the cycles $\{R_1, R_2, \cdots, R_k\}$ and whose length is

$$L \leqq (k+1)r - \sum_{i=1}^{k} r_i - 1 .$$

We can, then, interject into this path the simple cycles $\{R_1, R_2, \cdots, R_k\}$ at the touching points, interjecting cycle R_i say x_i times. The length L of the original path has now been increased to $L + \sum_{i=1}^{k} x_i r_i = L + B$, the second part of which, by Schur's theorem, can be made to take on any integral value B where $B \geqq (r_s - 1)(r_g - 1)$, and $r_s = \min(r_1, r_2, \cdots, r_k)$, $r_g = \max(r_1, r_2, \cdots, r_k)$. Therefore, we have:

$$M \leqq L + B = (k+1)r - \sum_{i=1}^{k} r_i - r_s - r_g + r_s r_g \tag{7}$$

Case I. Suppose $k = 2$. Then $M \leqq 3r - (r_s + r_g) - r_s - r_g + r_s r_g = 3r - 2r_s - 2r_g + r_s r_g = 3r + (r_g - 2)(r_s - 2) - 4$. The right side of this inequality is obviously maximum when r_s and r_g are as large as possible. Recall that $r_g \leqq r$ and $r_s \leqq r - 1$. Therefore we have:

$$M \leqq 3r + (r-2)(r-3) - 4 = r^2 - 2r + 2 \, . \tag{8}$$

Case II. Suppose $k \geqq 3$. The reader may wish to skip the following formidable looking, though straightforward calculations. They result in a proof that the integer M with the desired property is in fact smaller when the arbitrary graph contains a larger set of these cycles.

Since the lengths of these cycles are a minimal set of relatively prime integers, it is certainly true that

$$\begin{aligned} \sum_{i=1}^{k} r_i &\geqq r_s + [r_s + 2] + [r_s + 4] + \cdots + [r_s + 2(k-2)] + r_g \\ &= (k-1)r_s + (k-1)(k-2) + r_g \, . \end{aligned}$$

Thus, with (7) we have:

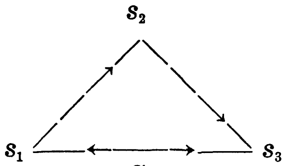
$$\begin{aligned} M &\leqq (k+1)r - [(k-1)r_s + (k-1)(k-2) + r_g] - r_s - r_g + r_s r_g \\ &= (k+1)r - kr_s - 2r_g + r_s r_g - (k-1)(k-2) \\ &= (k+1)r + (r_s - 2)(r_g - k) - 2k - (k-1)(k-2) \, . \end{aligned}$$

Since r_g must be larger than k, the right side again is maximum when r_g and r_s are as large as possible. But $r_g \leqq$ and $r_s \leqq r - k + 2$. So

$$\begin{aligned} M &\leqq (k+1)r + (r-k)(r-k) - k^2 + k - 2 \\ &= r^2 + (1-k)r + k - 2 \, . \end{aligned}$$

This is easily seen to be less than $r^2 - 2r + 2$ of Case I, if $r > 1$. So in any case $M \leqq r^2 - 2r + 2$.

To see that $r^2 - 2r + 2$ is the least value for an arbitrary graph of r points and thus for an arbitrary matrix of order r, we need only consider the following example in which $r = 3$ and $M = 5$.



G_P

$$\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & 0 & \times & \times \\ s_2 & 0 & 0 & \times \\ s_3 & \times & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & \times & 0 & \times \\ s_2 & \times & 0 & 0 \\ s_3 & 0 & \times & \times \end{array} \qquad \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & \times & \times & \times \\ s_2 & \times & 0 & \times \\ s_3 & \times & \times & \times \end{array}$$

$$P \qquad\qquad P^2 \qquad\qquad P^4$$

As a matter of fact it can be shown for *any* regular matrix P of order r whose graph G_P contains only two cycles, one of length r and one of length $r-1$, that P^{r^2-2r+1} is not positive. We have, therefore, established the claim of the paper as stated in the opening paragraph.

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