CATEGORY METHODS IN RECURSION THEORY[1,2]

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The heavy symbolism used in the theory of recursive functions has perhaps succeeded in alienating some mathematicians from this field, and also in making mathematicians who are in this field too embroiled in the details of thier notation to form as clear an overall picture of their work as is desirable.[3] In particular the study of degrees of recursive unsolvability by Kleene, Post, and their successors[4] has suffered greatly from this defect, so that there is considerable uncertainty even in the minds of those whose speciality is recursion theory as to what is superficial and what is deep in this area.[5] In this note we shall examine one particular theorem (namely the Kleene-Post theorem asserting the existence of incomparable degrees[6]) and show that it is a special case of a very easy and well-known theorem of set-theory. Exposition will be such as to require (except in a few footnotes) no preliminary acquaintance with recursive matters. It is to be hoped that some mathematicians in other areas may be stimulated by this exposition to try their hand at some open questions about recursive functions: it is to be hoped also that they will not carry away the impression that all of recursion theory is as trivial as this paper will show the Kleene-Post theorem to be.

First let me describe in an informal way what relative recursiveness is. The only properties of it which we shall need will be apparent from this informal discussion.

Denote by ε the set of all nonnegative integers. A *function* shall mean a number-theoretic function $f: \varepsilon \rightarrow \varepsilon$. A function is called *recursive* if it can be computed in an effective (mechanical) manner: we shall not need the details of the definition.[7] Sometimes two functions f and g are so related that the function f can be calculated in an effective

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[2] Category methods have also been used by the author in **[12]**, and form the basis of the entire treatment of degrees in **[3]**.

[3] A related (but much deeper) contribution to the methodology of recursion theory has made by Addison, e.g., in **[1]**.

[4] See, e.g., **[7]**, **[14]**, **[15]**, **[19]**. A sadly neglected paper in the same area which completely avoids these unnecessary complications is Lacombe **[10]**.

[5] The principal result of Spector **[19]** (minimal non-recursive degrees) is probably 'deep' in this sense, as is likewise the Friedberg-Mučnik proof (**[4]**, **[11]**) of the existence of incomparable degrees *of recursively enumerable sets.*

[6] Strictly speaking, the Kleene-Post theorem (**[7]**, p. 390) gives more information than our version, since it gives incomparable degrees $<\mathbf{0}'$. But this result too can be obtained by a category argument, as I shall show in a later publication.

[7] Cf., e.g., Davis **[2]**, p. 41.

(mechanical) way apart from requiring, for the computation of each particular function-value $f(n_0)$, a finite amount of information concerning values of the function g: in this case we say that f is *recursive in* or *relative to* g. The simplest way to envisage this relation is probably in terms of Turing machines.[8] We say that f is recursive in g if there exists a Turing machine with input and output tapes such that if the values $g(0), g(1), g(2), \ldots$ are fed in that order into the input, then for every nonnegative integer n the unique true statement of the form $f(n) = m$ will appear after a finite time on the output tape (and no false statement of that form will ever appear). Another characterization which may also aid the intuition is the following: f is recursive in g if there is a formal system[9] Σ such that every true statement of the form $f(n) = m$, and no false statement of that form, is deducible in Σ from a finite number of true statements of the form $g(x) = y$. The exact definitions of Turing machine and formal system are quite irrelevant for our purposes: all that matters is that

(1) only finitely many values of g are used to compute any value of f and

(2) the total number of Turing machines or formal systems is countable.

In both cases (2) is a consequence of the fact that the process of computation of one function from another can be described by a finite description using only symbols belonging to a finite alphabet fixed in advance; the same will be true if we characterize relative recursiveness in some way other than by Turing machines or formal systems.[10]

To every Turing machine or formal system corresponds uniquely a mapping Φ from functions to functions, called a *partial recursive operator*. It is important to notice that certain such Φ may not be defined for all functions as arguments. It may well be that a certain Turing machine $\boldsymbol{T}$, on being supplied with the values of a certain function g, will print statements of the form $f(m) = n$ on its tape only for certain m. In that case we say that $\boldsymbol{T}$ computes only a partial function from g. We regard the operator Φ as defined on the family of all those g from which $\boldsymbol{T}$ computes a full (everywhere defined) function. For example, suppose we consider the mapping which assigns to every function f the function $< \Phi f >$ such that

$$< \Phi f > (x) = (\mu y)\,(f(y) = 0);^{11}$$

[8] Davis [**2**], Ch. 1-2.

[9] For 'formal system' see Davis [**2**], Ch. 6 and 8, Smullyan [**17**] passim. The first use of formal systems to define partial reqursive functionals seems to date from Myhill-Shepherdson [**13**], p. 315, where we followed a suggestion of Marian Boykan (now Pour-El).

[10] E.g., by systems of recursion equations (Kleene [**5**], pp. 326-327).

[11] $(\mu y)\,(\ldots y \ldots)$ denotes the least y satisfying the condition $\ldots y \ldots$ if such exist, and otherwise is meaningless.

then Φ is a partial recursive operator whose domain of definition is the family of all functions which vanish for at least one value of the argument (and whose range is the family of all constant functions).

We denote by $\mathscr{T}$ the family of all functions, and we topologize it as the product of countably many replicas of the integers each with the discrete topology. This corresponds to the metric

$$\rho(f, g) = \frac{1}{(\mu x)(f(x) \neq g(x)) + 1}$$

or 0 if $f = g$. It is well-known[12] that this is a complete metric space, hence of second category on itself. This is the basic fact that we shall use in what follows.

By a *finite function* we mean a mapping of a finite subset of ε into ε; if f_0 is such a function, we define, $\mathscr{N}(f_0)$ as the family of all (full) functions which extend f_0. We can take as a (countable) basis for $\mathscr{T}$ the collection of all families $\mathscr{N}(f_0)$. $\Phi : \mathscr{F} \to \mathscr{T}$ with $\mathscr{F} \subseteq \mathscr{T}$ is continuous (in the induced topology on $\mathscr{F}$)[13] just in case

$$f \in \mathscr{F}, < \Phi f > (x) = y \to (\exists f_0)\,(f \in \mathscr{N}(f_0) \text{ and } (\forall f')$$
$$(f' \in \mathscr{N}(f_0), f' \in \mathscr{F} \to < \Phi f' > (x) = y))\ ,$$

i.e., if and only if any value $< \Phi f >$ is determined by finitely many values of f. In view of what was said above it follows that *all partial recursive operators are continuous*[14] (on their domain). For use later on we observe also that the domain of definition of such an operator is a $G\delta$ set; this too is an immediate consequence of the preceding informal remarks.

We write $f \leqq g$ if f is recursive in $g, f < g$ if $f \leqq g$ but not $g \leqq f$. The relation $f \leqq g$ is a pre-order; hence its symmetrization $f \equiv g$ (i.e., $f \leqq g$ and $g \leqq f$) is an equivalence relation. The equivalence classes into which it divides $\mathscr{T}$ are called *degrees*; we call one degree $\mathscr{D}$ *lower than* another degree $\mathscr{D}^*$ and write $\mathscr{D} < \mathscr{D}^*$ if $f < g$ for all (equivalently, for some) $f \in \mathscr{D}, g \in \mathscr{D}^*$.

Now we can prove the existence of incomparable degrees. Observe first the there are exactly c degrees, since there are c functions and at

[12] Sierpinski [**16**], p. 191.

[13] A partial recursive operator defined on a dense subset of $\mathscr{T}$ need not have a continuous extension to the whole space (Kleene [**5**], p. 685); and even when it does this extension need not be partial recursive (Lacombe [**10**], p. 155, Theorem XIX). Hence it will not suffice for our purposes to consider only everywhere defined operators.

[14] This observation is essentially Kleene's (cf. the proofs of Theorems XXIa and XXVI in [**5**], pp. 339, 348-349); that the property in question amounted to continuity was observed apparently independently by Lacombe (in a series of papers in *Comptes Rendus* going back at least to 1953) and later by Trahtenbrot [**20**]. Davis ([**2**], pp. 164 seqq.) oddly uses the word 'compact' to mean 'continuous'.

most (in fact, exactly, but we shall not need this) $\aleph_0$ functions belonging to any given degree. Observe also that there are at most $\aleph_0$ degrees lower than a given degree. For let $\mathscr{D}^*$ be a degree; then if f belongs to a degree lower than $\mathscr{D}^*$ it must be of the form $<\Phi g>$ where $g \in \mathscr{D}^*$ and Φ is partial recursive. But there are only countably many g's in $\mathscr{D}^*$ and only countably many Φ's; hence there are only countably many functions of degree $<\mathscr{D}^*$ and a fortiori only countably many degrees $<\mathscr{D}^*$.[15] This gives a plausibility argument for the existence of incomparable degrees, for if every two degrees were comparable we would have a simply ordered set of the power of the continuum in which each element had only a (finite or) countable number of predecessors; and this is easily seen[16] to imply the continuum hypothesis.

The continuum hypothesis is equivalent[17] to the assertion that the plane is the union of countable many curves (where a curve is the set of all points $(x, f(x))$ or of all points $(f(x), x)$ for some (not necessarily everywhere defined) real function f). We know also that the plane is not the union of countably many *continuous* curves,[18] since each such curve is nowhere dense and the plane is of second category on itself. These considerations yield at once the existence of incomparable degrees. If every two degrees were comparable the space $\mathscr{T}^2$ would be the union of all curves $\{(f, <\Phi f>)\}$ and $\{(<\Phi f>, f)\}$ with Φ partial recursive. But this is impossible because as we have seen each of these curves is continuous and hence by a classical argument nowhere dense,[19] and because $\mathscr{T}^2$, like $\mathscr{T}$, is a complete metric space and hence of second category on itself, q.e.d.

Now we use the same method to establish a stronger statement which answers a question rather recently raised (and still more recently settled) by Shoenfield.[20] Do there exist uncountably many degrees any two of which are incomparable? We shall obtain an affirmative answer to this question using only the hypotheses that $\mathscr{T}$ is a complete metric space and hence of second category on itself, and that there are only countably many partial recursive operators each of which is continuous

[15] For the lowest degree (that to which recursive functions belong) there are of course *no* degrees lower. There are also degrees than which only a finite nonzero number of degrees are lower (Spector [**19**], Theorem 4).

[16] Sierpinski [**16**], p. 23.

[17] Sierpinski [**16**], p. 11.

[18] Nor of countably many *measurable* curves (i.e., Lebesgue measurable in the plane); this is the foundation of Spector's proof in [**18**] of the existence of incomparable hyperdegrees. (Measure arguments have to replace category arguments in the study of hyperdegrees because hyperarithmetic operators are in general discontinuous.)

[19] The only hypothesis needed is that $\mathscr{T}$ is a Hausdorff space with no isolated points.

[20] Raised in [**15**], settled in [**14**]. More recently Sacks has obtained (unpublished) a *continuum* number of pairwise incomparable degrees and Lacombe and Nerode (unpublished) have obtained a continuum number of *independent* (and minimal non-recursive) degrees (see [**7**], p. 383 for the definition of independence).

in the topology induced on its domain.

Given any basic open set $\mathscr{N}(f_0)$ and any partial recursive operator Φ, it may or may not be the case that $< \Phi f >$ has the same value for all $f \in \mathscr{N}(f_0)$ for which it is defined. If this happens for some $\mathscr{N}(f_0)$ we call $< \Phi f >$ a *singular* function; in symbols

$$g \text{ singular} \leftrightarrow (\exists f_0)\,(\exists \Phi)\,(\Phi \text{ partial recursive and}$$
$$(\forall f \in \mathscr{N}(f_0))\,(< \Phi f > \text{defined} \rightarrow < \Phi f > = g))\ .$$

A function which is not singular we call *regular*. Clearly there are c regular and at most $\aleph_0$ singular functions.[21]

We wish to exhibit an uncountable collection of pairwise incomparable degrees, or, what comes to the same thing, an uncountable family of functions none of which is recursive in any other. We prove this by establishing successively the following propositions.

A. If f is regular and Φ partial recursive, then $\Phi^{-1}(f)$ is nowhere dense.

B. If f is regular, then the family of all functions of degree $\geqq$ the degree of f is of first category.

C. If f is regular, then the family of all functions of degree comparable with the degree of f is of first category.

D. If $\mathscr{F}$ is a (finite or) countable family of regular functions, then the family of all functions which are either singular or of degree comparable with that of some function belonging to $\mathscr{F}$ is of first category.

E. If $\mathscr{F}$ is a (finite or) countable family of regular functions, there exists a regular function of degree incomparable with the degree of every function in $\mathscr{F}$.

F. There exists an uncountable family of pairwise incomparable degrees.

Clearly $\mathrm{A} \rightarrow \mathrm{B} \rightarrow \mathrm{C} \rightarrow \mathrm{D} \rightarrow \mathrm{E} \rightarrow \mathrm{F}$, so we have only to prove A. Let then f be regular, $\mathscr{N}$ a basic open set, Φ a partial recursive operator. We seek a subneighborhood $\mathscr{N}_0$ of $\mathscr{N}$ such that for all $g \in \mathscr{N}_0$, Φg is undefined or $\neq f$. If $< \Phi g >$ is undefined for all $g \in \mathscr{N}$, take $\mathscr{N}_0 = \mathscr{N}$. If on the other hand $< \Phi g >$ is defined for some $g \in \mathscr{N}$, then there exists (since f is regular) such a g for which $< \Phi g > \neq f$. Let $\mathscr{F}$ be

[21] The singular functions are precisely the functions f for which the relation $f(x) = y$ is hyperarithmetic (see Davis **[2]**, p. 192 for the definition of hyperarithmetic). The proof is essentially contained in **[8]**.

the domain of Φ. Then $\{g \mid <\Phi g> \neq f\} = \mathscr{N}_1 \cap \mathscr{F}$ for some open $\mathscr{N}_1$. Consequently we can take $\mathscr{N}_0 = \mathscr{N} \cap \mathscr{N}_1$ and $\Phi^{-1}(f)$ is nowhere dense, q.e.d.

It must be stressed that some existence thorems in the literature of degrees apparently cannot be reduced to category arguments, at least not in the topology which we used.[22] Also Shoenfield's proof of the existence of $\aleph_1$ pairwise incomparable degrees is essentially different from the above, and yields the further information that *given any countable family of* non-recursive *functions* (i.e., not of the lowest degree, not effectively calculable) *there is a function of degree incomparable with all of them.* We only obtain the statement (E above) reading 'regular' for 'non-recursive'; and this is weaker as we have seen. If possible we seek a category argument which will yield this stronger result. However we cannot do this without more structure on $\mathscr{T}$. For we can exhibit a countable family of continuous operators

$$\Phi : \mathscr{F} \to \mathscr{T}$$

with the following four properties:

I. They are closed under composition whenever possible.

II. They contain the identity.

III. The domain of each is a $G\delta$.

IV. There exists a minimum in the induced ordering $f \leqq g$

such that it is *false* that given any countable family of functions none of which is minimal in the sense of IV, then there is a functions incomparable with them all.

The following additional assumption however, which is true for partial recursive operators, yields enough additional structure for us to obtain Shoenfield's result by essentially his method.

V. If the domain of Φ is dense on an open set, its intersection with that set contains a minimal (i.e., recursive) point.

It is obviously enough (in view of the earlier part of this paper) to prove that $\Phi^{-1}(f)$ is nowhere dense for each *non-recursive* f. For this, consider such an f and let $\mathscr{N}$ be a basic open set and Φ a partial recursive operator. We seek again a subneighborhood $\mathscr{N}_0$ of $\mathscr{N}$ disjoint from $\Phi^{-1}(f)$. If the domain $\mathscr{F}$ of Φ is not dense on $\mathscr{N}$, this is trivial;

[22] Spector's proof in **[19]** of the existence of minimal non-recursive degrees has been made into a category argument by Lacombe (unpublished); but the topology used is highly artificial.

so assume it is dense. By V, its intersection with $\mathscr{N}$ contains a recursive point g. If $< \Phi g > = f$, f would be recursive, contradicting the hypothesis. Hence $< \Phi g > \neq f$ and as alove we can take $\mathscr{N}_0 = \mathscr{N}_1 \cap \mathscr{N}$ where $\mathscr{N}_1$ is an open set such that $\{g \mid < \Phi g > \neq f\} = \mathscr{N}_1 \cap \mathscr{F}$, q.e.d.

The proof of V however seems to require essential use of (non-topological) properties of recursive *functions* as distinguished from operators, specifically their closure under a certain iterative procedure. We conclude that Shoenfield's result (and a fortiori the results of Sacks and Nerode mentioned in footnote 20) probably do not, like some of the other theorems on degrees mentioned in this note, rest solely on elementary set-theoretic considerations. However, the distinction between those which do and those which do not require more advanced and specialized means (i.e., between those which are truly 'recursive' and those which are merely set-theoretic) seems worth making, if only because it throws some light on aspects of the methodology of the whole domain which the present treatment in the literature leaves almost completely in the dark.

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